ON RANDOM INTERPOLATION

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In a recent paper Salem and Zygmund [1] proved the following result: Put

$$a_\nu = \alpha_\nu^{(n)} = \frac{2\pi\nu}{2n+1} \quad (\nu = 0, 1, \ldots, 2n)$$

and denote the $\varphi_\nu(t)$ the ν-th Rademacher function. Denote by $L_n(t, \theta)$ the unique trigonometric polynomial (in θ) of degree not exceeding n for which

$$L_n(t, \alpha_\nu) = \varphi_\nu(t) \quad (\nu = 0, 1, \ldots, 2n).$$

Denote $M_n(t) = \max\limits_{0 \leqq \theta < 2\pi} | L_n(t, \theta)|$. Then for almost all t

$$\overline{\lim_{n=\infty}} \frac{M_n(t)}{(\log n)^{\frac{1}{2}}} \leqq 2.$$

I am going to prove the following sharper

THEOREM 1. For almost all t

$$\underline{\lim_{n=\infty}} \frac{M_n(t)}{\log \log n} = \overline{\lim_{n=\infty}} \frac{M_n(t)}{\log \log n} = \frac{2}{\pi}.$$

Instead of Theorem 1 we shall prove the following stronger (throughout this paper $c_1, c_2, \ldots$ will denote suitable positive constants)

THEOREM 2. To every c_1 there exists a constant $c_2 = c_2(c_1)$ so that for $n > n_0(c_1, c_2)$ the measure of the set in t for which

$$\frac{2}{\pi} \log \log n - c_2 < M_n(t) < \frac{2}{\pi} \log \log n + c_2$$

is not satisfied, is less than $1/n^{c_1}$.

Theorem 1 follows immediately from Theorem 2 by the Borel-Cantelli Lemma. Thus we only have to prove Theorem 2.

First we need two simple combinatorial lemmas. Let m be a sufficiently large integer, we define for $1 \leqq i < m$ (for the purpose of these lemmas)

$$\varphi_{m+i}(t) = \varphi_i(t), \ \varphi_{-i}(t) = \varphi_{m-i}(t).$$

Lemma 1. Let $m > m_0(c_1)$. Then neglecting a set in t of measure less than $1/2m^{c_1}$ there exists for every t a k, $0 \leqq k \leqq m$ satisfying

(1) $$\varphi_{k+l}(t) = \varphi_{k-1-l}(t) = (-1)^l \text{ for all } 0 \leqq l < \tfrac{1}{2}\log m.$$

The measure of the set in t for which $k = [r \log m]$ satisfies (1) is clearly equal to

$$2^{-2[(\frac{\log m}{2}]+1)} < 2^{-\log m}.$$

But there are clearly $[m/\log m] + 1$ possible choices of r (i.e. r can take all the values $0 \leqq r < m/\log m$). Thus by an obvious independence argument the measure of the set in t for which none of the possible choices of r satisfies (1) is less than

$$(1 - 2^{-\log m})^{m/\log m} < \tfrac{1}{2}m^{-c_1}$$

for every c_1 if $m > m_0(c_1)$, which proves Lemma 1.

Lemma 2. To every c_1 there exists a c_3 so that for $m > m_0(c_1, c_3)$ neglecting a set (in t) of measure less than $\frac{1}{2}m - c_1$ we have for every t, r, $(0 \leqq r \leqq m)$ and ν, $(-m/2 < \nu < m/2)$

(2) $$s_{\nu,k}(t) = |\sum_{i=0}^{\nu} (-1)^i \varphi_{k+i}(t)| < c_3 \nu^{\frac{1}{2}} (\log m)^{\frac{1}{2}}.$$

It is well known that the measure of the set in t for which

$$|\sum_{i=0}^{\nu} (-1)^i \varphi_{k+i}(t)| \geqq c_3 \nu^{\frac{1}{2}} (\log m)^{\frac{1}{2}}$$

holds is less than

(3) $$c_4 e^{-\frac{1}{2}c_3^2 \log m} < \tfrac{1}{2}m^{-c_1-2}$$

for sufficiently large c_3. In (3) there are fewer than m^2 possible choices for r and ν, thus Lemma 2 clearly follows from (3).

Now we are ready to prove our Theorem. (Define for $0 < \nu \leqq n$ $\alpha_{-\nu} = \alpha_{2n-\nu}, \alpha_{2n+\nu} = \alpha_\nu$). It is well known that

(4) $$L_n(t, \theta) = \frac{1}{2n+1} \sum_{\nu=0}^{2n} \varphi_\nu(t) D_n(\theta - \alpha_\nu)$$

where $D_n(\theta) = \sin(n+\frac{1}{2})\theta / \sin \frac{1}{2}\theta$ is the Dirichlet kernel. Let $\alpha_k \leqq \theta < \alpha_{k+1}$. We have

(5) $$L_n(t, \theta) = \frac{1}{2n+1} \left(\sum_{\nu=0}^{n} \varphi_\nu(t) D_n(\theta - \alpha_{k+\nu}) + \sum_{\nu=1}^{n} \varphi_\nu(t) D_n(\theta - \alpha_{k-\nu})\right) = \Sigma_1 + \Sigma_2.$$

Now we consider only the t which satisfy Lemmas 1 and 2, (put $m = 2n$), by our Lemmas we thus neglect a set in t of measure less than n^{-c_1}. Put

(6) $$\Sigma_1 = \Sigma_1' + \Sigma_1''$$

where in Σ_1' $0 \leqq \nu \leqq [\log n]$ and in Σ_1'' $[\log n] \leqq \nu \leqq n$. We evidently have by $|D_n(\theta)| \leqq 2n+1$ and a simple computation

$$\Sigma_1' \leqq \frac{1}{2n+1} \sum_{0 \leqq \nu \leqq [\log n]} |D_n(\theta - \alpha_{k+\nu})| \tag{7}$$
$$\leqq 1 + \frac{1}{2n+1} \sum_{1 \leqq r \leqq \log n} \frac{1}{\sin \frac{r\pi}{2n+1}} < \frac{1}{\pi} \log\log n + c_4.$$

Further by partial summation and Lemma 2

$$\Sigma_1'' = \frac{1}{2n+1} \sum_{[\log n] < \nu \leqq n} \big(s_{\nu,k}(t) - s_{\nu-1,k}(t)\big)(-1)^\nu D_n(\theta - \alpha_{k+\nu})$$
$$= \frac{1}{2n+1} \sum_{\nu=[\log n]+1}^{n-1} s_{\nu,k}(t)\big((-1)^\nu D_n(\theta - \alpha_{k+\nu}) - (-1)^{\nu+1} D_n(\theta - \alpha_{k+\nu+1})\big)$$
$$- \frac{1}{2n+1} s_{[\log n],k}(t)(-1)^{[\log n]+1} D_n(\theta - \alpha_{k+[\log n]+1}) \tag{8}$$
$$+ \frac{1}{2n+1} s_{n,k}(t)(-1)^n D_n(\theta - \alpha_{k+n})$$
$$\leqq \frac{1}{2n+1} c_5 (\log n)^{\frac{1}{2}} \sum_{\nu > \log n} \nu^{-\frac{3}{2}} + c_6 < c_7$$

since a simple computation shows that for $\alpha_k \leqq \theta < \alpha_{k+1}$

$$|D_n(\theta - \alpha_{k+\nu})| < c_8 \frac{n}{\nu}$$

and

$$|D_n(\theta - \alpha_{k+\nu}) + D_n(\theta - \alpha_{k+\nu+1})| < c_8 \frac{n}{\nu^2}.$$

(6), (7) and (8) implies

$$\Sigma_1 < \frac{\log\log n}{\pi} + c_4 + c_7. \tag{9}$$

Similarly we can show

$$\Sigma_2 < \frac{\log\log n}{\pi} + c_9 \tag{10}$$

(9), (10) and (5) implies that for our t (i.e. for all t neglecting a set in t of measure $< n^{-c_1}$).

$$|L_n(t, \theta)| < \frac{2}{\pi} \log\log n + c_{10}. \tag{11}$$

Let now k satisfy Lemma 1 and put $\theta_0 = \pi(2k+1)/(2n+1)$. Then we have by (4) and the definition of k

$$L_n(t,\theta_0)=\frac{1}{2n+1}\sum_{\nu=0}^{n}\varphi_\nu(t)D_n(\theta_0-\alpha_\nu)$$

(12)
$$=\frac{1}{2n+1}\sum_{|\nu-k|<\frac{1}{2}\log n}|D_n(\theta_0-\alpha_\nu)|+\frac{1}{2n+1}\sum_{|\nu-k|\geq\frac{1}{2}\log n}\varphi_\nu(t)D_n(\theta_0-\alpha_\nu)$$
$$=\Sigma_1+\Sigma_2$$

Further clearly

$$\Sigma_1=\frac{1}{2n+1}\sum_{|r|<\frac{1}{2}\log n}\frac{2}{\sin\frac{(2r+1)\pi}{2(2n+1)}}$$

(13)
$$=\frac{1}{2n+1}\sum_{|r|<\frac{1}{2}\log n}\frac{2}{\frac{(2r+1)\pi}{2(2n+1)}+0\left(\frac{r^3}{n^3}\right)}>\frac{2}{\pi}\log\log n-c_{11}.$$

As in (8) we can show that

(14)
$$|\Sigma_2|<c_{12}.$$

(12), (13) and (14) implies

(15)
$$|L_n(t,\theta_0)|>\frac{2}{\pi}\log\log n-c_{11}-c_{12}.$$

(11) and (15) complete the proof of Theorem 2.

By more complicated arguments we could prove the following sharper,

THEOREM 3. There exists an absolute constant C so that neglecting a set in t whose measure tends to 0 as n tends to infinity we have

$$M_n(\theta)=\frac{2}{\pi}\log\log n+c+o(1).$$

(The exceptional set whose measure goes to 0 depends on n).

Using the methods of another paper by Salem and Zygmund [2] we can prove the following

THEOREM 4. There exists a distribution function ($\psi(\alpha)$ (i.e. $\psi(\alpha)$, $-\infty<\alpha<\infty$ is non decreasing, $\psi(-\infty)=0$, $\psi(+\infty)=1$), so that, neglecting a set in t whose measure tends to 0 as n tends to infinity, we have

$$m(\theta\colon L_n(t,\theta)<\alpha)\to\psi(\alpha).$$

In other words: If we neglect a set in t of measure tending to 0 (the exceptional set may depend on n) we have for a t not belonging to this exceptional set the following situation: The measure of the set in θ for which $L_n(t,\theta)<\alpha$ holds, equals $\psi(\alpha)+o(1)$.

We do not discuss in this paper the proofs of Theorem 3 and 4.

References

[1] Salem and Zygmund, Third Berkeley Symposium for probability and statistics, Vol. 2, 243–246.
[2] Salem and Zygmund, Acta Math. 91 (1954).

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