ACTA ARITHMETICA
V (1958)

On a question of additive number theory

by

P. ERDÖS (Saskatoon) and P. SCHERK (Saskatoon)

1. Let $A = \{a\}$, $B = \{b\}, \ldots$ denote sets of non-negative integers containing the number zero;

$$\sum_1^k A_\lambda = \Big\{\sum_1^k a_\lambda\Big\} \quad (a_\lambda \epsilon A_\lambda,\ \lambda = 1, 2, \ldots, k).$$

Thus $\sum A_\lambda$ consists of all the numbers $a_1 + a_2 + \ldots + a_k$ where each a_λ lies in the corresponding A_λ. For a given integer n let $[A]$ denote the number of positive elements of A up to and including n. $\bar{A}$ denotes the set of the integers $\leqslant n$ which do not belong to A.

It is well known and easy to see that $n \notin A + B$ implies $[A] + [B] \leqslant n-1$. The corresponding problem for three or more sets does not lead to anything new. For then

$$n \notin \sum_1^k A_\lambda \tag{1}$$

implies $n \notin A_\lambda + A_\mu$ and thus $[A_\lambda] + [A_\mu] \leqslant n-1$; $1 \leqslant \lambda < \mu \leqslant k$. Adding these $\frac{1}{2}k(k-1)$ inequalities we readily obtain

$$\sum_1^k [A_\lambda] \leqslant \tfrac{1}{2}k(n-1). \tag{2}$$

That (2) cannot be improved can be seen by taking $A_1 = A_2 \ldots = A_k =$ = set of integers between $[\frac{1}{2}n]+1$ and $n-1$ together with 0.

This question becomes more interesting if we require n to be the smallest number not in $\sum A_\lambda$. For $k = 3$ and $n < 15$ one can show(1) that

$$[A_1] + [A_2] + [A_3] \leqslant n-1.$$

(1) Written communication from Professor H. B. Mann.

However this estimate becomes false if $n \geqslant 15$.

Surprisingly enough, (2) is asymptotically correct. Put

$$(3) \qquad f_k(n) = \max \sum_1^k [A_\lambda]$$

where $A_1, \ldots, A_k$ range through those sets which satisfy (1) and

$$(4) \qquad \{1, 2, \ldots, n-1\} \subset \sum A_\lambda.$$

Thus $f_2(n) = n-1$. In the present paper we shall prove the existence of two positive constants $\alpha = \alpha_k$ and $\gamma = \gamma_k$ such that

$$(5) \qquad \tfrac{1}{2}kn - \alpha n^{(k-1)/k} < f_k(n) < \tfrac{1}{2}kn - \gamma n^{(k-1)/k}$$

for every $k > 2$. The first half of (5) will be proved in § 2, the second in § 3.

It would be of interest to obtain an explicit formula for $f_k(n)$ if $k > 2$. In particular it may be true that

$$(6) \qquad f_k(n) = \tfrac{1}{2}kn + \big(\beta + o(1)\big) n^{(k-1)/k}$$

for some positive constant $\beta = \beta_k$. But we are unable to prove (6), still less to determine β.

2. Let $B_\lambda = \{b_\lambda\}$ denote the set of all integers requiring only the digits 0 and 2^λ in the number system with the basis 2^k; $\lambda = 0, 1, \ldots, k-1$. Thus every integer x permits a unique representation

$$(1) \qquad x = \sum_0^{k-1} b_\lambda.$$

Suppose that n has the representation

$$(2) \qquad n = \sum_0^{k-1} b_\lambda^0, \quad b_\lambda^0 \in B_\lambda.$$

Obviously one of the b_λ^0's must be greater than $\frac{1}{2}n$. Renumbering the B_λ's if necessary, we may assume

$$(3) \qquad b_0^0 > \tfrac{1}{2}n.$$

We obtain the set C_0 by omitting the number b_0^0 from B_0. Thus

$$n \notin C_0 + \sum_1^{k-1} B_\lambda$$

and every number lies in $C_0+\sum_{1}^{k-1} B_\lambda$ except the numbers

$$b_0^0+\sum_{1}^{k-1} b_\lambda.$$

We now define

$$(4)\qquad C_h = B_h \cup \{b_0^0+b_1^0+\ldots+b_{h-1}^0+b_h\}, \quad b_h \neq b_h^0; \quad h = 1, 2, \ldots, k-1.$$

Let $x \neq n$; cf. (1) and (2). If $b_0 \neq b_0^0$,

$$x \in C_0+\sum_{1}^{k-1} B_\lambda \subset C_0+\sum_{1}^{k-1} C_\lambda = \sum_{0}^{k-1} C_\lambda.$$

If $b_0 = b_0^0$, there is an $h \geqslant 1$ such that

$$x = \sum_{0}^{h-1} b_\lambda^0+\sum_{h}^{k-1} b_\lambda, \quad b_h \neq b_h^0.$$

Hence

$$x \in C_h+\sum_{h+1}^{k-1} B_\lambda \subset C_h+\sum_{h+1}^{k-1} C_\lambda \subset \sum_{0}^{k-1} C_\lambda.$$

Thus every number $\neq n$ lies in $\sum_{0}^{k-1} C_\lambda$.

We next show

$$(5)\qquad n \notin \sum_{0}^{k-1} C_\lambda.$$

Suppose

$$(6)\qquad n = \sum_{0}^{k-1} c_\lambda, \quad c_\lambda \in C_\lambda.$$

Then for each $h > 0$ either $c_h = b_h \in B_h$ or

$$(7)\qquad c_h = \sum_{0}^{h-1} b_\lambda^0+b_h, \quad b_h \neq b_h^0.$$

Since the representation (2) of n was unique and since $b_0^0 \notin C_0$, the first alternative cannot occur for all $h > 0$. On the other hand (3) shows that (7) cannot occur more than once. Thus (7) will hold for exactly one index $h > 0$. This leads to

$$(8)\qquad n = \sum_{0}^{h-1} b_\lambda+\Big(\sum_{0}^{h-1} b_\lambda^0+b_h\Big)+\sum_{h+1}^{k-1} b_\lambda, \quad b_h \neq b_h^0.$$

Comparing (8) with (2) we obtain

$$\sum_{h}^{k-1} b_\lambda^0 = \sum_{0}^{h-1} b_\lambda + b_h + \sum_{h+1}^{k-1} b_\lambda, \qquad b_h \neq b_h^0. \tag{9}$$

The representation of the number (9) being unique, we obtain in particular $b_h^0 = b_h$, a contradiction. This proves (5).

Define

$$D_h = \sum_{\substack{0 \\ \lambda \neq h}}^{k-1} C_\lambda, \qquad h = 0, 1, \ldots, k-1 \tag{10}$$

and let A_λ be the union of C_λ with the set of all the numbers

$$n - \bar{d}_\lambda > \tfrac{1}{2} n, \qquad \bar{d}_\lambda \in \bar{D}_\lambda.$$

Then

$$n \notin \sum_{0}^{k-1} A_\lambda.$$

Thus n remains the only number not in $\sum_{0}^{k-1} A_\lambda$.

It remains to estimate $\sum_{0}^{k-1} [A_\lambda]$. Let $2^{km} < n \leqslant 2^{k(m+1)}$. Then

$$[B_\lambda] < 2^{m+1} = 2 \cdot 2^m < 2n^{1/k}, \qquad \lambda = 0, 1, \ldots, k-1.$$

Therefore

$$[C_0] < 2n^{1/k}; \qquad [C_\lambda] < 4n^{1/k} \quad \text{if} \quad 0 < \lambda \leqslant k-1.$$

Thus

$$\Big[\sum_{1}^{k-1} C_\lambda\Big] \leqslant \prod_{1}^{k-1} [C_\lambda] < 4^{k-1} n^{(k-1)/k}$$

and

$$\Big[\sum_{\substack{0 \\ \lambda \neq h}}^{k-1} C_\lambda\Big] \leqslant \prod_{\substack{0 \\ \lambda \neq h}}^{k-1} [C_\lambda] < \tfrac{1}{2} \cdot 4^{k-1} n^{(k-1)/k}, \qquad h = 1, \ldots, k-1.$$

Hence

$$[A_0] > \tfrac{1}{2} n - 4^{k-1} n^{(k-1)/k}, \qquad [A_h] > \tfrac{1}{2} n - \tfrac{1}{2} \cdot 4^{k-1} n^{(k-1)/k}, \qquad h = 1, \ldots, k-1,$$

and

$$\sum_{0}^{k-1} [A_\lambda] > \tfrac{1}{2} kn - (k+1) 2^{2k-3} n^{(k-1)/k}.$$

This proves the first part of our result with $\alpha = (k+1) 2^{2k-3}$.

3. Let $n > 0$ and $k > 2$ be fixed. Let

$$n \notin \sum_{1}^{k} A_\lambda, \tag{1}$$

$$\{1, 2, \ldots, n-1\} \subset \sum_{1}^{k} A_\lambda. \tag{2}$$

In this section we construct an absolute positive constant γ_k such that

$$\sum_{1}^{k} [A_\lambda] \leqslant \tfrac{1}{2}kn - \gamma_k n^{(k-1)/k}. \tag{3}$$

Without loss of generality we may assume

$$[A_1] \geqslant [A_2] \geqslant \ldots \geqslant [A_k]. \tag{4}$$

Let $\gamma > 0$ be given. From now on we assume

$$\sum_{1}^{k} [A_\lambda] > \tfrac{1}{2}kn - \gamma n^{(k-1)/k}. \tag{5}$$

LEMMA 1.

$$[A_1] < \frac{n}{2} + \frac{\gamma}{k-2} n^{(k-1)/k}, \tag{6}$$

$$[A_{\lambda-1}] \geqslant [A_\lambda] > \frac{n}{2} - \frac{k-3+\lambda}{(k-2)(k-\lambda+1)} \gamma n^{(k-1)/k}, \quad \lambda = 2, \ldots, k. \tag{7}$$

Proof. Since $n \notin A_1 + A_\lambda$, we have $[A_\lambda] < n - [A_1]$. Thus (5) implies

$$\tfrac{1}{2}kn - \gamma n^{(k-1)/k} < [A_1] + (k-1)(n - [A_1]).$$

This yields (6). Also by (4), (5) and (6)

$$\begin{aligned} \tfrac{1}{2}kn - \gamma n^{(k-1)/k} &< (\lambda-1)[A_1] + (k-\lambda+1)[A_\lambda] \\ &< (\lambda-1)\left(\frac{n}{2} + \frac{\gamma}{k-2} n^{(k-1)/k}\right) + (k-\lambda+1)[A_\lambda]. \end{aligned}$$

This implies (7).

We now define

$$B_i = \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{k} A_\lambda, \quad i = 1, 2, \ldots, k. \tag{8}$$

Thus

$$\sum_{1}^{k} A_\lambda = A_i + B_i, \quad i = 1, 2, \ldots, k. \tag{9}$$

LEMMA 2.

(10)
$$\frac{n}{2}-\frac{\gamma}{k-2}n^{(k-1)/k}<[B_i]<\begin{cases}\dfrac{n}{2}+\dfrac{\gamma}{k-2}n^{(k-1)/k} & \text{if } i=1,\\[2mm] \dfrac{n}{2}+\dfrac{k+i-3}{(k-2)(k-i+1)}\gamma n^{(k-1)/k} & \text{if } 1<i\leqslant k.\end{cases}$$

Proof. B_i contains either A_1, or A_2. Thus the first estimate follows immediately from (7) with $\lambda=2$.

By (9), $n\notin A_i+B_i$. Henee $[B_i]<n-[A_i]$ and (7) also yields the second inequality.

LEMMA 3.

$$\left.\begin{matrix}[B_1\cap\bar{A}_\mu]\\ [B_\mu\cap\bar{A}_1]\end{matrix}\right\}<\frac{1}{k-2}\left(1+\frac{k+\mu-3}{k-\mu+1}\right)\gamma n^{(k-1)/k};\qquad \mu=2,\ldots,k. \tag{11}$$

Proof. If $\lambda\neq\mu$, $A_\mu\subset B_\lambda$. Thus $[B_\lambda\cap\bar{A}_\mu]=[B_\lambda]-[A_\mu]$ and (11) is a corollary of Lemmas 1 and 2.

LEMMA 4.

$$[B_1\cup B_2\cup\ldots\cup B_k]<\tfrac{1}{2}n+3k\gamma n^{(k-1)/k}. \tag{12}$$

Proof. If x lies in $B_1\cup B_2\cup\ldots\cup B_k$, $n-x$ lies in $\bar{A}_1\cup\ldots\cup\bar{A}_k$. Hence

$$\begin{aligned}[B_1\cup B_2\cup\ldots\cup B_k]&\leqslant[\bar{A}_1\cup\bar{A}_2\cup\ldots\cup\bar{A}_k]\\ &=[\bar{A}_k]+[A_k\cap(\bar{A}_1\cup\bar{A}_2\cup\ldots\cup\bar{A}_{k-1})]\\ &\leqslant[\bar{A}_k]+[A_k\cap\bar{A}_1]+\sum_2^{k-1}[A_k\cap\bar{A}_\mu]\\ &\leqslant[\bar{A}_k]+[B_2\cap\bar{A}_1]+\sum_2^{k-1}[B_1\cap\bar{A}_\mu].\end{aligned} \tag{13}$$

Now by (7) and (11)

$$[\bar{A}_k]=n-[A_k]<\frac{n}{2}+\frac{2k-3}{k-2}\gamma n^{(k-1)/k}\leqslant\frac{n}{2}+3\gamma n^{(k-1)/k},$$

$$[B_2\cap\bar{A}_1]<\frac{2}{k-2}\gamma n^{(k-1)/k}\leqslant 2\gamma n^{(k-1)/k},$$

and

$$[B_1\cap\bar{A}_\mu]<\frac{1}{k-2}\left(1+\frac{2k-4}{2}\right)\gamma n^{(k-1)/k}\leqslant 2\gamma n^{(k-1)/k}$$

if $2\leqslant\mu\leqslant k-1$. Thus (13) yields (12).

Let C denote the set of those elements of $\sum_1^k A_\lambda$ which lie in none of the B_λ. Lemma 4 implies

LEMMA 5.

$$[C] > \tfrac{1}{2}n - 3k\gamma n^{(k-1)/k}. \tag{14}$$

For each $c \in C$ we choose a canonical representation

$$c = \sum_1^k a_\lambda, \quad a_\lambda \in A_\lambda, \tag{15}$$

in the following way: First a_1 is chosen maximally among all the representations of c. If $a_1, \ldots, a_\lambda$ have been fixed, $a_{\lambda+1}$ will be maximal among all the representations of c which use $a_1 + a_2 + \ldots + a_\lambda$.

LEMMA 6. *Let*

$$c' = \sum a'_\lambda \in C, \quad a'_\lambda \in A_\lambda \tag{16}$$

be the canonical representation of c'. *Let*

$$1 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_h \leqslant k$$

and suppose

$$\sum_1^h a_{\lambda_\mu} = \sum_1^h a'_{\lambda_\mu}. \tag{17}$$

Then

$$a_{\lambda_\mu} = a'_{\lambda_\mu}, \quad \mu = 1, 2, \ldots, h. \tag{18}$$

Proof. Substituting (17) in (15) we obtain another representation of c. Since a_{λ_1} was maximal, we have $a_{\lambda_1} \geqslant a'_{\lambda_1}$. Similarly, (17) and (16) imply $a'_{\lambda_1} \geqslant a_{\lambda_1}$. Thus $a_{\lambda_1} = a'_{\lambda_1}$ and (18) follows by induction.

LEMMA 7. *Let* $1 \leqslant l \leqslant k$. *The number of elements* b_i *occuring in the representation of elements* $c = a_i + b_i$ *of* C *is less than*

$$2\frac{k-1}{k-2}\gamma n^{(k-1)/k} \leqslant 4\gamma n^{(k-1)/k}.$$

This remark is obvious. If b_i occurs in the representation of numbers of C, b_i cannot occur in any A_μ with $\mu \neq i$. Hence the number of these b_i's is $\leqslant [B_i \cap \bar{A}_\mu]$. Choosing $\mu = 1$ if $i > 1$ and μ arbitrarily if $i = 1$, we obtain our estimate from (11).

We now construct a sequence of subsets

$$C = D_0 \supset D_1 \supset D_2 \supset \ldots \supset D_{k-1}$$

of C in the following fashion: Let $\delta > 0$ be given. D_h consists of those elements

$$c^* = \sum_1^k a_\lambda^* = b_\mu^* + a_\mu^* \quad (a_\lambda^* \in A_\lambda,\ \lambda = 1, \ldots, k) \tag{19}$$

of D_{h-1} such that for every $i > h$ there are not less than $\delta 2^{1-h} n^{1/k}$ elements of D_{h-1} of the form $b_i^* + a_i$ $(h = 1, \ldots, k)$.

LEMMA 8.

$$[D_0 \frown \overline{D}_1] < 4(k-1)\gamma\delta n. \tag{20}$$

Proof. Let C_i denote the set of those numbers (19) of D_0 such that there are fewer than $\delta n^{1/k}$ elements of D_0 of the form $b_i^* + a_i$ $(i = 2, \ldots, k)$. Thus

$$D_0 \frown \overline{D}_1 = \bigcup_2^k C_i.$$

Let $1 < i \leqslant k$ be fixed. By Lemma 7 there are less than $4\gamma n^{(k-1)/k}$ numbers b_i occuring in the representation of elements $c = a_i + b_i$ of C. In particular there are fewer than $4\gamma n^{(k-1)/k}$ numbers b_i^*. Each of them occurs in fewer than $\delta n^{1/k}$ elements of C_i and each $c^* \in C_i$ has a representation $c^* = b_i^* + a_i^*$. Hence

$$[C_i] < 4\gamma n^{(k-1)/k} \cdot \delta n^{1/k} = 4\gamma\delta n$$

and

$$[D_0 \frown \overline{D}_1] \leqslant \sum_2^k [C_i] < 4(k-1)\gamma\delta n.$$

LEMMA 9.

$$[D_h \frown \overline{D}_{h+1}] < (k-h-1)[D_{h-1} \frown \overline{D}_h], \quad h = 1, 2, \ldots, k-2. \tag{21}$$

Proof. Let C_i denote the set of those elements (19) of $D_h \frown \overline{D}_{h+1}$ such that there are fewer than $\delta 2^{-h} n^{1/k}$ elements of D_h of the form $b_i^* + a_i$ $(i = h+2, \ldots, k)$. Thus

$$D_h \frown \overline{D}_{h+1} = \bigcup_{h+2}^k C_i.$$

Let i be fixed; $h+1 < i \leqslant k$. If b_i^* occurs in the representation of some $c^* \in C_i$, there are not less than $\delta 2^{1-h} n^{1/k}$ elements of D_{h-1} of the form $b_i^* + a_i$ while fewer than $\delta 2^{-h} n^{1/k}$ of them belong to D_h. Hence more than $\delta 2^{-h} n^{1/k}$ of them will lie in $D_{h-1} \frown \overline{D}_h$. The number of these b_i^* is therefore less than

$$[D_{h-1} \frown \overline{D}_h]/(\delta 2^{-h} n^{1/k}).$$

Each of these b_i^*'s gives rise to less than $\delta 2^{-h} n^{1/k}$ elements of C_i. Conversely each element of C_i has a representation $c^* = b_i^* + a_i$. Hence

$$[C_i] < \delta 2^{-h} n^{1/k}\big([D_{h-1} \cap \bar{D}_h]/(\delta 2^{-h} n^{1/k})\big) = [D_{h-1} \cap \bar{D}_h].$$

This yields (21).

LEMMA 10. *Let* $0 < h \leqslant k-1$ *be given,*

$$c^* = \sum a_\lambda^* = b_i^* + a_i^* \in D_h. \tag{22}$$

Let $i_1, \ldots, i_h$ *be any h-tuple of distinct indices satisfying* $i_\lambda > \lambda$; $\lambda = 1, 2, \ldots, h$. *Then there are at least*

$$\delta^h 2^{-\binom{h}{2}} n^{h/k}$$

numbers

$$\Big(c^* - \sum_1^h a_{i_\lambda}^*\Big) + \sum_1^h a_{i_\lambda} \in C. \tag{23}$$

Proof. For $h = 1$ our assertion follows from the definition of D_1. Suppose it is proved for $h-1$ and assume (22). From the definition of D_h there are at least $\delta 2^{1-h} n^{1/k}$ numbers a_{i_h} such that $b_{i_h}^* + a_{i_h} \in D_{h-1}$. By induction assumption there are to each of them not less than

$$\delta^{h-1} 2^{-\binom{h-1}{2}} n^{(h-1)/k}$$

numbers

$$\Big(b_{i_h}^* + a_{i_h} - \sum_1^{h-1} a_{i_\lambda}^*\Big) + \sum_1^{h-1} a_{i_\lambda} = \Big(c^* - \sum_1^h a_{i_\lambda}^*\Big) + \sum_1^h a_{i_\lambda} \in C.$$

Altogether we have at least

$$(\delta 2^{1-h} n^{1/k})(\delta^{h-1} 2^{-\binom{h-1}{2}} n^{(h-1)/k}) = \delta^h 2^{-\binom{h}{2}} n^{h/k}$$

numbers (23). By Lemma 6 they are mutually distinct.

LEMMA 11. *Let*

$$\delta = \sqrt[k-1]{4\gamma}\, 2^{k/2-1}. \tag{24}$$

Then D_{k-1} *is empty.*

Proof. The case $h = k-1$ of Lemma 10 yields: If there is a number $c^* = \sum a_i^* \in D_{k-1}$, then there are at least

$$\delta^{k-1} 2^{-\binom{k-1}{2}} n^{(k-1)/k}$$

elements $a_1^* + b_1$ of C. By Lemma 7 fewer than $4\gamma n^{(k-1)/k}$ numbers b_1 can occur. Thus

$$\delta^{k-1} 2^{-\binom{k-1}{2}} n^{(k-1)/k} < 4\gamma n^{(k-1)/k}.$$

This contradicts (24).

LEMMA 12. *Let*

$$\gamma_k = \gamma = \frac{1}{2^{k/2+4}} \cdot \frac{1}{(k-1)!}. \tag{25}$$

Define δ *through* (24). *Then*

$$\left(1 - 8e(k-1)!\,\gamma\delta\right) n^{1/k} > 6k\gamma$$

for every n.

Proof. Since $\sqrt[k-1]{4\gamma} < 1$, we have

$$8e(k-1)!\,\gamma\delta + 6k\gamma < 8e(k-1)!\,2^{k/2-1}\gamma + 8(4-e)(k-1)!\,2^{k/2-1}\gamma$$
$$= 2^{k/2+4}(k-1)!\,\gamma = 1.$$

Hence

$$\left(1 - 8e(k-1)!\,\gamma\delta\right) n^{1/k} \geqslant 1 - 8e(k-1)!\,\gamma\delta > 6k\gamma.$$

We are now ready to show that the constant (25) satisfies (3).

Lemmas 8 and 9 imply by induction

$$[D_h \frown \bar{D}_{h+1}] < 4 \cdot \frac{(k-1)!}{(k-h-2)!}\,\gamma\delta n, \qquad h = 0, 1, \ldots, k-2.$$

Thus by Lemmas 5 and 11

$$\frac{1}{2} n - 3k\gamma n^{(k-1)/k} < [C] = \sum_0^{k-2} [D_h \frown \bar{D}_{h+1}]$$
$$< 4(k-1)!\,\gamma\delta n \sum_0^{k-2} \frac{1}{(k-h-2)!}$$
$$< 4e(k-1)!\,\gamma\delta n.$$

Hence

$$\left(1 - 8e(k-1)!\,\gamma\delta\right) n^{1/k} < 6k\gamma.$$

Thus Lemma 12 shows that our assumption (5) leads to a contradiction if γ is chosen according to (25).

4. If n is a given integer and if S and $C = \{c\}$ are sets of non-negative integers, the set $S - C$ consists of all the integers $x \geqslant 0$ such that $x + c \in S$ for every c with $x + c \leqslant n$.

Let $h > 1$,

$$n \notin S, \qquad 0 \in A_\lambda \qquad (\lambda = 1, 2, \ldots, h)$$

and let

$$S - \sum_1^h A_\lambda = \{0\} \qquad \left(\text{thus } \sum_1^h A_\lambda \subset S\right).$$

Then there are two positive constants $\gamma_1 = \gamma_1(h)$ and $\gamma_2 = \gamma_2(h)$ which are independent of $n, S, A_1, \ldots, A_h$ such that always

$$\sum_{1}^{h} [A_\lambda] < [S] + \tfrac{1}{2}(h-1)n - \gamma_1 n^{h/(h+1)}$$

and that for a suitable $(h+1)$-tuple $A_1, \ldots, A_h, S$

$$\sum_{1}^{h} [A_\lambda] > [S] + \tfrac{1}{2}(h-1)n - \gamma_2 n^{h/(h+1)}.$$

These results follow at once from the preceding sections if we put $h = k-1$ and choose for A_k the set of all the numbers of the form $n - \bar{s}$ where $0 \leqslant \bar{s} \leqslant n$, $\bar{s} \notin S$.

UNIVERSITY OF SASKATCHEWAN
ASKATOON, SASK., CANADA

Reçu par la Rédaction le 15. 3. 1958