POLYNOMIALS WHOSE ZEROS LIE ON THE UNIT CIRCLE

By P. Erdös, F. Herzog and G. Piranian

1. Introduction. Let

$$(1) \qquad P(z) = \prod_{j=1}^{n} (1 - z/\omega_j),$$

where the points ω_j lie on the unit circle C. It has been shown by Cohen[1] that, on some path Γ which joins the origin to C, the inequality $|P| < 1$ holds everywhere except at $z = 0$. In an oral communication, C. Loewner has established the existence of a polynomial (1) for which every radius of the unit disc passes through a point at which $|P| > 1$.

We will describe (see Theorem 1) a very simple example of a polynomial (1) with the property that on each radius of the unit disc there exist two points z' and z'' such that $|P(z')| < 1$ and $|P(z'')| > 1$.

In connection with Theorem 1, the following question might be asked: Does there exist a universal constant L such that for every polynomial (1) the inequality $|P| < 1$ holds on a path which connects the origin to C and has length at most L? This question has recently been answered in the negative by G. R. MacLane [2].

Section 3 deals with the polynomials (1) in the cases $n \leq 4$. In these cases there always exist two half-lines from the origin on which

$$(2) \qquad |P(z)| \leq |1 - |z|^n| \quad \text{and} \quad |P(z)| \geq 1 + |z|^n,$$

respectively. Here we point out the problem of determining the greatest degree n for which a polynomial (1) always satisfies the inequalities (2) on two appropriate radii of the unit disc or on two half-lines from the origin.

2. The example. The polynomial to be described is of the form

$$(3) \qquad P(z) = \prod_{j=1}^{q} [1 + (z/\omega_j)^{k_j}]^{k_j} \qquad (|\omega_j| = 1; j = 1, 2, \cdots, q).$$

Roughly speaking, each factor determines a set of directions θ, of total range slightly less than π, such that on every radius in one of these directions $P(z)$ takes values of modulus greater than 1. The crucial problem in the construction is this, to choose the integers k_j in such a way that each factor bears the sole responsibility, on some circular arc concentric with the unit circle, of determining the signum of $\log |P(z)|$.

Let Aj be the set of all ω on C for which

$$(4) \qquad -\pi/3 \leq \arg(\omega/\omega_j)^{k_j} \leq \pi/3, \qquad \text{modulo } 2\pi,$$

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and let B_j be the set of all ω on C for which

$$(5) \qquad 2\pi/3 \le \arg(\omega/\omega_j)^j \le 4\pi/3, \qquad \text{modulo } 2\pi.$$

(A_j is the union of j disjoint, closed arcs, each of length $2\pi/3j$; the same is true of B_j.)

We write

$$(6) \qquad \log\,[1 + (z/\omega_j)^j] = (z/\omega_j)^j \phi_j(z),$$

where $\phi_j(0) = 1$ and $\phi_j(z)$ is holomorphic and different from zero in $|z| < 1$. With each index j we associate a number r_j $(0 < r_j < 1)$, to be determined below. Then by (3) and (6) we have, for $|\omega| = 1$ and $p = 1, 2, \cdots, q$,

$$(7) \qquad \begin{aligned} \log P(r_p\omega) &= \sum_{j=1}^{q} k_j r_p^j (\omega/\omega_j)^j \phi_j(r_p\omega) \\ &= k_p r_p^p \left(\frac{\omega}{\omega_p}\right)^p \phi_p(r_p\omega) \left\{1 + \sum_{j\ne p} \frac{k_j}{k_p} r_p^{j-p} \frac{(\omega/\omega_j)^j}{(\omega/\omega_p)^p} \frac{\phi_j(r_p\omega)}{\phi_p(r_p\omega)}\right\}. \end{aligned}$$

If $\omega \in A_p$ or B_p, the factor $(\omega/\omega_p)^p$ satisfies the inequality (4) or (5), respectively; the factor $\phi_p(r_p\omega)$ is arbitrarily close to 1 if r_p is small enough; and, as we shall show below, the modulus of the sum $\sum_{j\ne p}$ in (7) can be made arbitrarily small by the proper choice of the k_j and the r_j. It follows then that, for $\omega \in A_p$,

$$-\pi/2 < \arg \log P(r_p\omega) < \pi/2, \qquad \text{that is,} \qquad |P(r_p\omega)| > 1;$$

and that, for $\omega \in B_p$,

$$\pi/2 < \arg \log P(r_p\omega) < 3\pi/2, \qquad \text{that is,} \qquad |P(r_p\omega)| < 1.$$

It remains to show that the sum $\sum_{j\ne p}$ in (7) can be made arbitrarily small. It will suffice to show that, if the k_j and the r_j are properly chosen, then the $q(q-1)$ quantities $(k_j/k_p)\, r_p^{j-p}$ $(j \ne p)$ are arbitrarily small.

With m a positive integer to be determined below, let

$$k_j = 2^{m(2q-j)(j-1)}, \qquad r_j = 2^{-m(2q-2j+1)} \qquad (j = 1, 2, \cdots, q),$$

so that $r_j \le 2^{-m}$. It is easily verified that, for $j \ne p$,

$$(k_j/k_p) r_p^{j-p} = 2^{-m(j-p)^2} \le 2^{-m}.$$

Thus we need only choose m sufficiently large, in order to accomplish our purpose.

Since $\sum_1^\infty 1/j = \infty$, it is possible to choose a finite q and a corresponding set of points ω_j $(j = 1, 2, \cdots, q)$ such that each of the sets $\cup A_j$ and $\cup B_j$ covers C. The following result is now immediate.

THEOREM 1. *There exists a polynomial* (1) *such that on every radius of the unit disc there exist points z' and z'' with $|P(z')| < 1$ and $|P(z'')| > 1$.*

3. Polynomials of degree at most four.

THEOREM 2. *Let* $P(z) = \prod_{\nu=1}^{n} (z - z_\nu)$, *with* $|z_\nu| = 1$. *If* $n \leq 4$, *there exist two values* θ' *and* θ'' *such that*

$$|P(re^{i\theta'})| \leq |1 - r^n| \quad \text{and} \quad |P(re^{i\theta''})| \geq 1 + r^n$$

for $0 \leq r < \infty$.

We omit the trivial cases $n = 1$ and $n = 2$. In the case $n = 3$, let α, β, γ denote the three angles formed by the radii $0z_\nu$, with $2\pi \geq \alpha \geq \beta \geq \gamma \geq 0$ and $\alpha + \beta + \gamma = 2\pi$.

To show the existence of θ', we write $z_1 = 1$, $z_2 = e^{i\beta}$, and $z_3 = e^{-i\gamma}$, and we prove that $|P(r)| \leq |1 - r^3|$ for $0 \leq r < \infty$. Since

$$|P(r)| = |1 - r|(1 - 2r\cos\beta + r^2)^{\frac{1}{2}}(1 - 2r\cos\gamma + r^2)^{\frac{1}{2}},$$

it will suffice to show that

$$\begin{aligned} \Delta' &= (1 - r^3)^2 - |P(r)|^2 \\ &= (1 - r)^2[2r(1 + r^2)(1 + \cos\beta + \cos\gamma) + r^2(1 - 4\cos\beta\cos\gamma)] \geq 0. \end{aligned}$$

Now

$$(8) \qquad 0 \leq (\beta - \gamma)/2 \leq \beta/2 \leq \pi/2 \qquad \text{and} \qquad 0 \leq (\beta + \gamma)/2 \leq 2\pi/3,$$

and therefore

$$\begin{aligned} 1 + \cos\beta + \cos\gamma &= 1 + 2\cos[(\beta + \gamma)/2]\cos[(\beta - \gamma)/2] \\ &\geq 1 - \cos[(\beta - \gamma)/2] \geq 0; \end{aligned}$$

and because $1 + r^2 \geq 2r$, it follows that

$$\begin{aligned} \Delta' &\geq r^2(1 - r)^2[4(1 + \cos\beta + \cos\gamma) + 1 - 4\cos\beta\cos\gamma] \\ &= r^2(1 - r)^2[9 - 4(1 - \cos\beta)(1 - \cos\gamma)] \\ &= r^2(1 - r)^2[9 - 16\sin^2(\beta/2)\sin^2(\gamma/2)]. \end{aligned}$$

But, by (8),

$$\begin{aligned} 0 &\leq 4\sin(\beta/2)\sin(\gamma/2) = 2\cos[(\beta - \gamma)/2] - 2\cos[(\beta + \gamma)/2] \\ &\leq 2 - 2(-\tfrac{1}{2}) = 3, \end{aligned}$$

and therefore the value $\theta' = 0$ has the required property.

To show the existence of θ'' in the case $n = 3$, let α, β, γ be the same as above. We write $z_1 = e^{i\alpha/2}$, $z_2 = e^{-i\alpha/2}$, and $z_3 = e^{i(\alpha/2+\beta)}$; and we will show that $|P(r)| \geq 1 + r^3$ for $0 \leq r < \infty$. If $\alpha \geq \pi$, then $|P(r)| \geq (1 + r^2)^{3/2} \geq 1 + r^3$. In what follows we therefore restrict α to the interval $2\pi/3 \leq \alpha < \pi$; and we write $\cos(\alpha/2) = t$, so that $\cos(3\alpha/2) = 4t^3 - 3t$ and $0 < t \leq 1/2$.

Since $\pi \leq \alpha/2 + \beta \leq 3\alpha/2$, we have $|r - z_3| \geq |r - e^{3i\alpha/2}|$, and therefore

$$|P(r)| \geq (1 - 2rt + r^2)[1 - 2r(4t^3 - 3t) + r^2]^{\frac{1}{2}};$$

it remains to show that the quantity

$$\Delta'' = (1 - 2rt + r^2)^2[1 - 2r(4t^3 - 3t) + r^2] - (1 + r^3)^2$$

is nonnegative for $0 \leq r < \infty$ and $0 < t \leq 1/2$.

A simple computation shows that $\Delta'' = A_1 r(1 + r^4) + A_2 r^2(1 + r^2) + A_3 r^3$, where

$$A_1 = 2t(1 - 4t^2) \geq 0,$$
$$A_2 = 3 - 20t^2 + 32t^4 = (1 - 4t^2)(3 - 8t^2) \geq 0,$$
$$A_3 = -2 + 4t + 8t^3 - 32t^5.$$

Since $1 + r^2 \geq 2r$,

$$\begin{aligned}\Delta'' &\geq r^3(2A_2 + A_3)\\ &= 4r^3(1 + t - 10t^2 + 2t^3 + 16t^4 - 8t^5)\\ &= 4r^3(1 - 2t)(1 - 2t^2)(1 + 3t - 2t^2) \geq 0.\end{aligned}$$

In the case $n = 4$, let $\alpha, \beta, \gamma, \delta$ denote, in cyclical order, the four nonnegative angles formed by the radii $0z_\nu$, with $\alpha + \beta + \gamma + \delta = 2\pi$.

In order to establish the existence of θ', we assume that the notation has been chosen in such a way that $\gamma + \delta \leq \pi$. We write $z_1 = 1$, $z_2 = e^{i\delta}$, $z_3 = e^{i(\delta+\alpha)} = e^{-i(\beta+\gamma)}$, $z_4 = e^{-i\gamma}$, and we will show that $|P(r)| \leq |1 - r^4|$ for $0 \leq r < \infty$.

We note that $|r - z_1| = |1 - r|$ and $|r - z_3| \leq 1 + r$. Since $0 \leq \delta \leq \pi - \gamma \leq \pi$, we have the inequality

$$|r - z_2| \leq |r - e^{i(\pi-\gamma)}| = |r + e^{-i\gamma}|,$$

and therefore

$$|(r - z_2)(r - z_4)| \leq |r^2 - e^{-2i\gamma}| \leq 1 + r^2.$$

The required result is now immediate.

To show the existence of θ'' in the case $n = 4$, let $\alpha, \beta, \gamma, \delta$ be the same as above; assume that the notation has been chosen in such a way that

$$\alpha + \beta \geq \pi \geq \gamma + \delta, \quad \alpha + \delta \geq \pi \geq \beta + \gamma, \quad \beta \geq \delta. \tag{9}$$

We write $z_1 = e^{-i\alpha/2}$, $z_2 = e^{i\alpha/2}$, $z_3 = e^{i(\alpha/2+\beta)}$, $z_4 = e^{-i(\alpha/2+\delta)}$. We note that (9) implies

$$\pi - \alpha/2 \leq \alpha/2 + \delta \leq \alpha/2 + \beta \leq \pi + \alpha/2. \tag{10}$$

Two cases arise. If $\alpha \geq \pi/2$, we shall show that $|P(r)| \geq 1 + r^4$ for $0 \leq r < \infty$. For $\alpha \geq \pi$, the matter is trivial. For $\pi/2 \leq \alpha < \pi$ and $\nu = 3, 4$, the inequalities (10) give

$$|r - z_\nu| \geq |r - e^{i(\pi-\alpha/2)}| = |r + z_1|;$$

therefore

$$\begin{aligned} | P(r) | &\geq | r - z_1 |^2 \, | r + z_1 |^2 = | r^2 - z_1^2 |^2 \\ &= 1 - 2r^2 \cos\alpha + r^4 \geq 1 + r^4. \end{aligned}$$

If $\alpha < \pi/2$, we shall show that $| P(ir) | \geq 1 + r^4$ for $0 \leq r < \infty$. By (10), $| ir - z_3 | \geq | ir - e^{i(\alpha/2+\delta)} | = | ir + z_4 |$, hence

$$\begin{aligned} | (ir - z_3)(ir - z_4) | &\geq | r^2 + z_4^2 | \\ &= [1 + 2r^2 \cos(\alpha + 2\delta) + r^4]^{\frac{1}{2}} \\ &\geq (1 + 2r^2 \cos\alpha + r^4)^{\frac{1}{2}}. \end{aligned}$$

Also,

$$| (ir - z_1)(ir - z_2) | = (1 + 2r^2 \cos\alpha + r^4)^{\frac{1}{2}},$$

so that

$$| P(ir) | \geq 1 + 2r^2 \cos\alpha + r^4 \geq 1 + r^4.$$

This completes the proof of Theorem .

In conclusion, we note that in the case $n = 4$ the inequality $| P(z) | \geq 1$ does not necessarily hold everywhere on the bisector of the greatest of the four angles involved. To see this, let $z_1 = e^{i\pi/3}$, $z_2 = -1$, $z_3 = z_4 = e^{-i\pi/3}$. Then $| P(1/2) | = (9/16)\, 3^{1/2} < 1$. Considerations of continuity show that even if $\alpha > \beta > \gamma > \delta > 0$, the inequality $| P(z) | \geq 1$ need not hold everywhere on the bisector of α.

References

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University of Notre Dame,
Michigan State College,
and
University of Michigan