On Taylor Series of Functions Regular in Gaier Regions

To Alexander Ostrowski for his 60th birthday

By Paul Erdös in Los Angeles, California, Fritz Herzog in East Lansing, Michigan, and George Piranian in Ann Arbor, Michigan

1. Introduction

This paper deals with the Taylor coefficients of functions that are regular inside the unit circle C and have only one singularity on C. Much of its motivation comes from the following lemma of Gaier [3, pp. 327, 328]:

If for some $a > 0$ the function $f(z) = \Sigma a_n z^n$ is regular and bounded in the disc $|z + a| < 1 + a$, then $a_n = O(n^{-1/2})$.

We shall apply the term Gaier *region* to any open region which contains the unit disc $|z| < 1$ and whose boundary does not meet the unit circle C except at $z = 1$. In particular, if a Gaier region is one of the circular discs in the lemma above, we shall call it a Gaier *disc*.

In § 2, we show that Gaier's lemma cannot be improved, in the sense that the O cannot be replaced by o, and we state our Theorem 2, of which Gaier's lemma is a special case. Three sections are devoted to the proof of this theorem.

While Theorem 2 provides bounds for individual Taylor coefficients of functions satisfying certain restrictions in a Gaier disc, Theorem 3 (§ 6) gives a bound on the sum of the moduli of coefficients in certain blocks of coefficients. On the one hand, this bound cannot be deduced from Theorem 2; on the other hand, certain results of Fejér show that the bound is the best possible.

§ 7 uses the technique of § 6 to obtain a theorem on the series $\Sigma n\,|a_n|^2$; § 8 deals with the convergence of Σa_n and with the uniform convergence on the unit circle of $\Sigma a_n z^n$; and § 9 is devoted to a partial analogue of Theorem 2 for Gaier regions other than Gaier discs.

2. On Gaier's lemma

Since Gaier's lemma is proved by means of the equation

$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{n+1}}\, dz\,,$$

where Γ is a suitable contour, it is reasonably regarded as a generalization of Cauchy's inequality $|a_n| \leqq M$ on the Taylor coefficients of a bounded function. Since Cauchy's inequality can be sharpened (for large n) to the relation $a_n = o(1)$ by Gutzmer's relation

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\Theta})|^2 d\Theta,$$

the question arises whether Gaier's lemma can be improved in the same way. If the hypotheses are slightly strengthened, this is indeed the case: in a private communication, Gaier has pointed out that if $f(z)$ is continuous in the closure of a Gaier disc, then $a_n = o(n^{-1/2})$. But Gutzmer's proof of his theorem [5] is based on the relation $\cos(\Theta + \pi) = -\cos\Theta$; and the customary modern proof of his theorem relies heavily on the fact that the set of functions $\{z^n\}$ $(n = 0, 1, 2, \ldots)$ form an orthogonal set on C. In the case of Gaier's lemma, the trigonometric relation cannot be used, and the functions $\{z^n\}$ do not form an orthogonal set on Gaier's contour; therefore the following result should not come as a surprise.

Theorem 1. *There exists a function* $f(z) = \Sigma a_n z^n$, *regular and bounded in a* Gaier *disc, for which* $\limsup(|a_n| n^{1/2}) > 0$.

To prove this theorem, we show first that for any fixed number $b (0 < b < 1)$ the function

$$g(z) = \sum_{j=0}^{\infty} (-1)^j z^{m_j}$$

is bounded in the disc $|z - b| < 1 - b$, provided the sequence of positive integers m_j increases fast enough.

We denote by C_b the circle $|z - b| = 1 - b$, and we choose a sequence $\{\varepsilon_j\}$ with $\varepsilon_j > 0$ and $\Sigma \varepsilon_j < \infty$. The integers m_0 and m_1 can be chosen arbitrarily. There then exists an open arc A_0 on C_b, containing the point $z = 1$, and such that

$$|z^{m_0} - z^{m_1}| < \varepsilon_0$$

on A_0. We choose m_2 and m_3 large enough so that

$$|z|^{m_3} < |z|^{m_2} < \varepsilon_1$$

on the complement of A_0 relative to C_b. The arc A_0 has an open subarc A_1, containing $z = 1$, such that

$$|z^{m_2} - z^{m_3}| < \varepsilon_1$$

on A_1. We choose m_4 and m_5 large enough so that

$$|z|^{m_5} < |z|^{m_4} < \varepsilon_2$$

on the complement of A_1. If the construction is continued in this manner, then

$$\left|\sum_{j=0}^{J} (z^{m_{2j}} - z^{m_{2j+1}})\right| < 2 + 2\sum_{j=0}^{J} \varepsilon_j$$

on C_b and consequently $g(z)$ is bounded inside of C_b.

We turn now to the function

$$f(z) = g\left(\frac{z+1}{2}\right) = \sum_{j=0}^{\infty} (-1)^j \left(\frac{z+1}{2}\right)^{m_j} = \sum_{n=0}^{\infty} a_n z^n .$$

If $m_j \to \infty$ fast enough, $f(z)$ is bounded in the Gaier disc $|z + 1/2| < 3/2$. Also, a slight computation shows that if $m_j \to \infty$ fast enough, then

$$|a_n| \sim (\pi n)^{-1/2}$$

for $n = [m_j/2]$, $j = 0, 1, 2, \ldots$. This proves the theorem.

The following theorem differs from Gaier's lemma in that it replaces the boundedness of $f(z)$ by the boundedness of $(1-z)^k f(z)$, where k is a real constant.

Theorem 2. *Let $f(z) = \Sigma a_n z^n$ be regular in some* Gaier *disc $|z + a| < 1 + a$, and let k be a real number such that $(1-z)^k f(z)$ is bounded in this disc. Then*

$$a_n = O(n^{k-1}) \qquad \text{if } k > 1,$$
$$a_n = O(\log n) \qquad \text{if } k = 1,$$
$$a_n = O(n^{(k-1)/2}) \qquad \text{if } k < 1.$$

In this estimate, the O cannot generally be replaced by o; the replacement is permissible if $k \leqq 1$ and $(1-z)^k f(z)$ approaches a limit whenever $z \to 1$ from the interior of the Gaier *disc.*

3. The case $k > 1$

Here the estimate is well known. It can be obtained from Cauchy's formula by integration along the circle $|z| = 1 - 1/n$. That the O cannot be replaced by o is seen from the example $f(z) = (1-z)^{-k}$. It is noteworthy that, in the case $k > 1$, the hypothesis that $(1-z)^k f(z)$ is regular in a Gaier disc and continuous on the closure of this disc does not yield a better estimate on a_n than does the hypothesis that $(1-z)^k f(z)$ is regular and bounded in the unit disc.

4. The case $k = 1$

Again, integration along the circle $|z| = 1 - 1/n$ gives the estimate $a_n = O(\log n)$. For a precise discussion of the situation where $(1-z) f(z)$ is merely assumed to be regular and bounded in the unit disc, the reader is referred to Neder [7]. Here we shall only show that

i) $a_n = o(\log n)$ if $(1-z) f(z)$ is regular and bounded in the unit disc and approaches a limit as $z \to 1$ from the interior of the unit disc;

ii) regularity and boundedness of $(1-z) f(z)$ in a Gaier disc does not imply that $a_n = o(\log n)$.

To prove the first of these propositions, suppose that $(1-z)\,f(z)$ is regular and bounded in $|z|<1$, and that $\lim\limits_{z\to 1}(1-z)\,f(z)=A$. Then

$$f(z)=\frac{A}{1-z}+\frac{\Phi(z)}{1-z},$$

where $\Phi(z)$ is bounded in $|z|<1$ and $\Phi(z)\to 0$ as $z\to 1$ in the unit disc. The TAYLOR coefficients of $A/(1-z)$ are all equal to A and cause no trouble. Let

$$\frac{\Phi(z)}{1-z}=\sum_{n=0}^{\infty}b_n z^n .$$

Then

$$|b_n|\leqq\frac{1}{2\pi}\int\limits_{\Gamma_n}\frac{|\Phi(z)|\,|dz|}{|z|^{n+1}\,|1-z|},$$

where Γ_n is the contour $|z|=1-1/n$. Since the value of $|z|^{n+1}$ on Γ_n approaches $1/e$ as $n\to\infty$, it suffices to prove that

$$\int\limits_{\Gamma_n}\frac{|\Phi(z)|\,|dz|}{|1-z|}=o(\log n),$$

and geometrical considerations reduce the problem to the task of showing that, with $z=(1-1/n)\,e^{i\Theta}$,

$$\int\limits_{-\pi}^{\pi}\frac{|\Phi(z)|\,d\Theta}{\sqrt{\Theta^2+n^{-2}}}=o(\log n).$$

Now, if $0<\Theta_n\leqq\pi$ and $M_n=\max|\Phi(z)|$ for $z=(1-1/n)\,e^{i\Theta}$, $-\Theta_n\leqq\Theta\leqq\Theta_n$, then

$$\int\limits_{-\Theta_n}^{\Theta_n}\frac{|\Phi(z)|\,d\Theta}{\sqrt{\Theta^2+n^{-2}}}\leqq 2\,M_n\int\limits_{0}^{\Theta_n}\frac{d\Theta}{\sqrt{\Theta^2+n^{-2}}}=2\,M_n\log\left(n\,\Theta_n+\sqrt{n^2\,\Theta_n^2+1}\right)$$

$$<2\,M_n\log(1+2n\,\Theta_n)\leqq 2\,M_n\log(1+2\pi n).$$

On the other hand,

$$\int\limits_{\Theta_n}^{2\pi-\Theta_n}\frac{|\Phi(z)|\,d\Theta}{\sqrt{\Theta^2+n^{-2}}}\leqq 2\,M\int\limits_{\Theta_n}^{\pi}\frac{d\Theta}{\Theta}=2\,M\log(\pi/\Theta_n),$$

where M is a bound for $|\Phi(z)|$ in $|z|<1$. If we choose $\Theta_n=1/\log n$, then $M_n\to 0$, and it follows that $b_n=o(\log n)$.

To prove that regularity and boundedness of $(1-z)\,f(z)$ in a GAIER disc does not imply that $a_n=o(\log n)$, we use the polynomials

$$P_n(z)=\frac{1}{n}+\frac{z}{n-1}+\cdots+\frac{z^{n-1}}{1}-\frac{z^n}{1}-\frac{z^{n+1}}{2}-\cdots-\frac{z^{2n-1}}{n}.$$

FEJÉR [2, pp. 74—76] proved that on the unit circle C these polynomials have a bound which is independent of n. We believe that the following new proof of this proposition is of interest because of its simple and elementary character.

At $z = e^{i\Theta}$, the sum of the $2r$ middle terms of $P_n(z)$ has modulus

$$\left| \sum_{j=1}^{r} \frac{z^{n-j}(1-z^{2j-1})}{j} \right| \leqq |\Theta| \sum_{j=1}^{r} \frac{2j-1}{j} \leqq 2r|\Theta|.$$

Also, by ABEL's summation, the sum of the first $n-r$ terms is

$$\sum_{j=r+1}^{n} \frac{z^{n-j}}{j} = \frac{1}{r+1} \sum_{h=0}^{n-r-1} z^h - \sum_{j=r+1}^{n-1} \frac{1}{j(j+1)} \sum_{h=0}^{n-j-1} z^h,$$

and, for $0 < |\Theta| \leqq \pi$, this has modulus less than $2\pi/(r+1)|\Theta|$. The modulus of the last $n-r$ terms has the same bound, and therefore

$$|P_n(e^{i\Theta})| \leqq 2r|\Theta| + \frac{4\pi}{(r+1)|\Theta|}$$

for $0 < |\Theta| \leqq \pi$. The choice $r = \min(n, [\pi/|\Theta|])$ then gives the desired result.

We now write

$$Q_n(z) = z^{n^3} P_n(z^{n^2})$$
$$= \frac{z^{n^2}}{n} + \frac{z^{2n^2}}{n-1} + \dots + \frac{z^{n^3}}{1} - \frac{z^{n^3+n^2}}{1} - \frac{z^{n^3+2n^2}}{2} - \dots - \frac{z^{2n^3}}{n},$$

and we form the function

$$F(z) = \sum_{i=1}^{\infty} Q_{n_i}(z).$$

We assume that the sequence $\{n_i\}$ is chosen in such a way that $2n_i^3 < n_{i+1}^2$ for $i = 1, 2, \dots$. Since the TAYLOR series of $F(z)$ has infinitely many terms with coefficient one, $F(z)$ is not bounded in $|z| < 1$. However, it follows from considerations similar to those in § 2 that $F(z)$ is bounded in the disc $|z - 1/4| < 3/4$, provided $n_i \to \infty$ fast enough.

Let

$$G(z) = F\left(\frac{z+1}{2}\right) = \sum_{n=0}^{\infty} b_n z^n.$$

Then $G(z)$ is bounded in the GAIER disc $|z + 1/2| < 3/2$. On the other hand, it is easily verified that, if $n_i \to \infty$ fast enough,

$$\sum_{j=0}^{[n_i^3/2]} b_j > \tfrac{1}{2} \log n_i - O(n_i^{-1}),$$

where the second term on the right-hand side is obtained by a slight modification of Problem 145 of Pólya and Szegö [10, vol. I, pp. 66 and 230]. It follows that, if

$$f(z) = \frac{G(z)}{1-z} = \sum_{n=0}^{\infty} a_n z^n,$$

then $(1-z) f(z)$ is bounded in a Gaier disc, while $a_n/\log n$ remains greater than a positive constant for $n = [n_i^3/2]$.

5. The case $k < 1$

In proving that $a_n = O(n^{(k-1)/2})$, we shall essentially follow Gaier and estimate a_n by Cauchy's formula, with the contour of integration composed of the arc

$$\Gamma: \quad z = (1 + c_1 \Phi^2) e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi,$$

where c_1 is a positive constant small enough so that the curve Γ lies in the Gaier disc $|z + a| < 1 + a$, except for the point $z = 1$. (A moment's consideration shows that this path of integration is permissible even when $0 < k < 1$.) Then

$$2\pi |a_n| = \left| \int_\Gamma \frac{f(z)}{z^{n+1}} dz \right| \leqq c_2 \int_0^\pi \frac{1}{\Phi^k (1 + c_1 \Phi^2)^n} \left(1 + \frac{2 c_1 \Phi}{1 + c_1 \Phi^2}\right) d\Phi$$

$$\leqq c_3 \int_0^\pi \frac{d\Phi}{\Phi^k (1 + c_1 \Phi^2)^n} = c_3 (c_1 n)^{(k-1)/2} \int_0^{\pi\sqrt{c_1 n}} \frac{du}{u^k (1 + u^2/n)^n}.$$

Now, when $u \geqq 0$ and p is a positive integer,

$$\left(1 + \frac{u^2}{n}\right)^n \geqq 1 + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{p-1}{n}\right) \frac{u^{2p}}{p!};$$

any choice of p greater than $(1-k)/2$ shows that the last integral has a bound independent of n, and it follows that $a_n = O(n^{(k-1)/2})$.

If the function $(1-z)^k f(z)$ is not only bounded in a Gaier disc but is continuous in the closure of a Gaier disc, then

$$f(z) = \frac{A}{(1-z)^k} + \frac{\Phi(z)}{(1-z)^k},$$

where $\Phi(z)$ is bounded in the Gaier disc and $\Phi(z) \to 0$, as $z \to 1$. The contribution to a_n from the n-th Taylor coefficient of the first function on the right is $O(n^{k-1}) = o(n^{(k-1)/2})$. The contribution from the second term on the right can be treated much as in the discussion of the case $k = 1$ (see § 4); we omit the details.

We will now show that for every $k < 1$ there exists a function $f(z) = \Sigma a_n z^n$ such that $(1-z)^k f(z)$ is regular and bounded in the Gaier disc $|z + 1/2| < 3/2$, and such that $\limsup (|a_n| n^{(1-k)/2}) > 0$.

For any fixed k, we choose an integer h such that $h + k > 0$. We begin with the function

$$G(z) = \sum_{j=1}^{\infty} g(z, p_j),$$

where $\{p_j\}$ is an increasing sequence of positive integers and

$$g(z,p) = p^k z^{2p^2} (1 - z^p)^h.$$

We choose a positive constant c_4 small enough so that the curve

$$K: \qquad z = (1 - c_4 \Phi^2) e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi$$

encloses the disc $|z - 1/4| < 3/4$. On K,

$$|z|^{2p^2} = (1 - c_4 \Phi^2)^{2p^2} \leqq e^{-2c_4 p^2 \Phi^2}.$$

Also on K,

$$|1 - z^p| = |1 - (1 - c_4 \Phi^2)^p e^{ip\Phi}|$$
$$= \left\{ [1 - (1 - c_4 \Phi^2)^p]^2 + 4(1 - c_4 \Phi^2)^p \sin^2 \frac{p\Phi}{2} \right\}^{1/2}.$$

Since $1 - x^p \leqq p(1 - x)$ for $0 \leqq x \leqq 1$, the first term in the braces is not greater than $(c_4 p \Phi^2)^2$, and it follows that $|1 - z^p| \leqq c_5 |p \Phi|$. Therefore, for z on K,

$$|(1 - z)^k g(z, p)| \leqq c_6 |p \Phi|^{h+k} e^{-2c_4 (p\Phi)^2},$$

and since $h + k > 0$, this has an upper bound independent of p and Φ. Moreover, $(1 - z)^k g(z, p) \to 0$, as $z \to 1$ along K; and on any closed arc of K that does not pass through $z = 1$, the function $(1 - z)^k g(z, p)$ can be made arbitrarily small by choosing the integer p sufficiently large. It follows that we can apply the method used in the proof of Theorem 1 to choose the sequence $\{p_j\}$ in such a way that $(1 - z)^k G(z)$ is regular and bounded in the interior of K.

Finally, we consider the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = G\left(\frac{z+1}{2}\right) = \sum_{j=1}^{\infty} g\left(\frac{z+1}{2}, p_j\right).$$

From our discussion of $G(z)$ it follows that $(1 - z)^k f(z)$ is regular and bounded in the GAIER disc $|z + 1/2| < 3/2$. It remains only to examine the coefficients a_n.

The coefficient of z^{p^2} in the polynomial $g\left(\frac{z+1}{2}, p\right)$ is

$$p^k 2^{-2p^2} \sum_{\lambda=0}^{h} (-1)^\lambda \binom{h}{\lambda} 2^{-\lambda p} \binom{2p^2 + \lambda p}{p^2} = p^k 2^{-2p^2} \binom{2p^2}{p^2} \left[1 - \sum_{\lambda=1}^{h} \prod_{r=1}^{\lambda p} \frac{2p^2 + r}{2p^2 + 2r}\right].$$

Each product in the last expression is of the form

$$\prod_{r=1}^{\lambda p} \left(1 - \frac{r}{2p^2 + 2r}\right),$$

and its logarithm approaches $-\lambda^2/4$ as $p \to \infty$.

It follows that the coefficient of z^{p^2} in $g\left(\frac{z+1}{2}, p\right)$ is asymptotically equal to

$$p^k (\pi p^2)^{-1/2} \sum_{\lambda=0}^{h} (-1)^\lambda \binom{h}{\lambda} e^{-\lambda^2/4}.$$

Since $e^{-1/4}$ is a transcendental number, the sum in the last expression is different from zero. Now let $n = p_i^2$. Because the contributions to a_n from the terms $g\left(\frac{z+1}{2}, p_j\right)$ are zero for $j < i$ and $o(p_i^{k-1})$ for $j > i$, provided $p_j \to \infty$ fast enough, it follows that, as $n \to \infty$ through the values p_i^2,

$$a_n \sim \beta p_i^{k-1} = \beta n^{(k-1)/2},$$

where β is a constant different from zero. This concludes the proof of Theorem 2.

6. On blocks of terms from the series $\Sigma |a_j|$

We now turn our attention from individual coefficients to sums of the form

$$S_n = \sum_{j=n}^{2n} |a_j|.$$

If $f(z)$ satisfies the conditions of Theorem 2, then

$$S_n = O(n^k) \quad \text{if } k > 1,$$
$$S_n = O(n \log n) \quad \text{if } k = 1,$$
$$S_n = O(n^{(k+1)/2}) \quad \text{if } k < 1.$$

For the case $k > 1$, this estimate cannot be improved, as is shown by the example $f(z) = (1-z)^{-k}$. The following theorem improves the estimate for the remaining values of k.

Theorem 3. *If $f(z)$ satisfies the condition of Theorem 2, then*

$$S_n = O(n^k) \quad \text{if } k > 1/2,$$
$$S_n = O\left(\sqrt{n \log n}\right) \quad \text{if } k = 1/2,$$
$$S_n = O(n^{k/2+1/4}) \quad \text{if } k < 1/2;$$

if $k \neq 1/2$ the O in this estimate cannot be replaced by o.

In the proof we will need the following auxiliary result.

Lemma. *Let $f(z) = \Sigma a_j z^j$ be regular in $|z| < 1$, and let m be a non-negative integer such that*

$$\int_{\Gamma_n} |f^{(m)}(z)|^2 |dz| < n^\alpha, \qquad n = 2, 3, \ldots,$$

where Γ_n is the circle $|z| = 1 - 1/n$. Then

$$S_n < c\, n^{(\alpha - 2m + 1)/2},$$

where c depends only on m and on α.

By GUTZMER's relation (see § 2), the hypothesis of the lemma implies that (with $z = (1-1/n)\, e^{i\Theta}$)

$$\sum_{j=n}^{2n} [j(j-1)\ldots(j-m+1)\,|\,a_j\,|]^2 \left(1-\frac{1}{n}\right)^{2j-2m} \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\,f^{(m)}(z)\,|^2\, d\Theta < n^\alpha .$$

But for $j \leqq 2\,n$,

$$\left(1-\frac{1}{n}\right)^{2j-2m} \geqq \left(1-\frac{1}{n}\right)^{4n-2m} > c_1 > 0\,,$$

where c_1 is independent of n; also, for $j > m$,

$$j(j-1)\ldots(j-m+1) > c_2\, j^m\,,$$

where $c_2 > 0$. Therefore, the last inequality implies that

$$S_n^2 \leqq \sum_{j=n}^{2n} j^{2m}\,|\,a_j\,|^2 \cdot \sum_{j=n}^{2n} j^{-2m} < c_3\, n^\alpha\, n^{-2m+1}\,,$$

and the lemma follows.

We now begin with the proof of the theorem for the case $k > 1/2$. Here we will use our lemma with $m = 0$. To estimate $|\,f(z)\,|$ at the point $z = (1-1/n)\, e^{i\Theta}$, we apply CAUCHY's formula, with the circular contour $|\,\zeta - z\,| = 1/2n$. On this contour, $|\,1-\zeta\,| > c_4(|\,\Theta\,| + n^{-1})$; therefore, $|\,f(z)\,| < c_5(|\,\Theta\,| + n^{-1})^{-k}$; and consequently

$$\int_{\Gamma_n} |\,f(z)\,|^2\,|\,dz\,| < 2\,c_5 \int_0^{\pi} (\Theta + n^{-1})^{-2k}\, d\Theta < c_6\, n^{2k-1}\,.$$

The desired estimate now follows from the lemma. That the O cannot be replaced by o is seen at once from the example $f(z) = (1-z)^{-k}$.

The case $k = 1/2$ is treated in the same way. The only difference is this, that

$$\int_{\Gamma_n} |\,f(z)\,|^2\,|\,dz\,| < 2\,c_5 \int_0^{\pi} (\Theta + n^{-1})^{-1}\, d\Theta < c_7 \log n\,.$$

A slight modification in the computations in the proof of the lemma gives the estimate $S_n = O\left(\sqrt{n \log n}\right)$.

We point out that in the case where $k \geqq 1/2$, we have only used the fact that $(1-z)^k f(z)$ is bounded in the unit circle, rather than in a GAIER disc.

For $k < 1/2$, we apply our lemma, with the integer m chosen in such a way that $2\,m + k - 1/2 > 0$. To estimate $|\,f^{(m)}(z)\,|$ at the point $z = (1-1/n)\, e^{i\Theta}$, we use the circle

$$|\,\zeta - z\,| = 1/2n \qquad \text{if } |\,\Theta\,| \leqq n^{-1/2}\,,$$

and the circle

$$|\,\zeta - z\,| = c_8\,\Theta^2 \qquad \text{if } n^{-1/2} < |\,\Theta\,| \leqq \pi\,;$$

here c_8 denotes a positive constant small enough so that $(1-\zeta)^k f(\zeta)$ is regular and bounded inside of the curve $\zeta = (1 + c_8 \Phi^2) e^{i\Phi}$, $-\pi \leqq \Phi \leqq \pi$.

We observe that in either case the relations

$$c_9(|\Theta| + n^{-1}) < |1-\zeta| < c_{10}(|\Theta| + n^{-1})$$

hold on the circle associated with the point $z = (1-1/n) e^{i\Theta}$. It follows that, for $|\Theta| \leqq n^{-1/2}$,

$$|f^{(m)}(z)| < c_{11} n^m (|\Theta| + n^{-1})^{-k},$$

and

$$\int_{-n^{-1/2}}^{n^{-1/2}} |f^{(m)}|^2 d\Theta < c_{12} n^{2m+k-1/2}.$$

Similarly, for $n^{-1/2} < |\Theta| \leqq \pi$,

$$|f^{(m)}(z)| < c_{13} \Theta^{-2m} (|\Theta| + n^{-1})^{-k};$$

since $|\Theta| < |\Theta| + n^{-1} < 2|\Theta|$, this leads to the estimates

$$|f^{(m)}(z)| < c_{14} |\Theta|^{-2m-k},$$

$$\int_{-\pi}^{-n^{-1/2}} + \int_{n^{-1/2}}^{\pi} |f^{(m)}|^2 d\Theta < c_{15} n^{2m+k-1/2},$$

and it follows from the lemma that $S_n = O(n^{k/2+1/4})$.

Fejér [1] (see also Perron [8] and [9, § 5]) has shown that if $f(z) = \Sigma a_n z^n = (1-z)^{-k} e^{1/(z-1)}$, then as $n \to \infty$, while k is a fixed real number,

$$a_n = \frac{1}{\sqrt{\pi e}} n^{k/2-3/4} \left\{ \sin\left[2\sqrt{n} - \left(\frac{k}{2} - \frac{3}{4}\right)\pi\right] + O\left(\frac{1}{\sqrt{n}}\right) \right\}.$$

Consequently, the O in the estimate above cannot be replaced by o.

The following result is an immediate consequence of Theorem 3.

Theorem 4. *If $f(z) = \Sigma a_n z^n$ satisfies the condition of Theorem 2, with $k < -1/2$, then $\Sigma |a_n| < \infty$.*

Again, Fejér's example shows that the theorem becomes false for $k = -1/2$.

7. On the series $\sum_j j|a_j|^2$

If $f(z)$ satisfies the hypothesis of Theorem 2, with $k < -1$, then the conclusion of Theorem 2 implies that $\sum_j j|a_j|^2 < \infty$. Again this result can be improved by the method of the preceding section.

Theorem 5. *If $f(z) = \Sigma a_n z^n$ satisfies the hypothesis of Theorem 2, with $k < -1/2$, then $\sum_j j|a_j|^2 < \infty$.*

It suffices to deal with the case $-1 \leqq k < -1/2$, so that we may apply the procedure of § 6, with $m = 1$. The theorem then follows from the inequalities

$$\sum_{j=n}^{2n} j \mid a_j \mid^2 \leqq n^{-1} \sum_{j=n}^{2n} j^2 \mid a_j \mid^2 \leqq c_1 n^{-1} \int_{\Gamma_n} \mid f'(z) \mid^2 \mid dz \mid$$

and the fact that the integral in the last member is less than $c_2 n^{k+3/2}$.

If $k = -1/2$ the conclusion need not hold, as is easily seen from FEJÉR's example.

8. Convergence and uniform convergence on the unit circle

Theorem 6. *If $f(z) = \Sigma a_j z^j$ satisfies the condition of Theorem 2, with $k < 0$, then Σa_j converges.*

We apply Theorem 2 to the function

$$g(z) = \frac{f(z)}{1-z} = \sum_{j=0}^{\infty} s_j z^j,$$

where $s_j = a_0 + a_1 + \cdots + a_j$. Since $(1-z)^{k+1} g(z)$ is bounded in a GAIER disc, $s_n = O(n^{k/2})$, and the proof is complete.

It follows from a theorem of GAIER [3, Zusatz, p. 331] that mere continuity of $f(z)$ in the closure of a GAIER disc does not imply convergence of Σa_j. This is also seen from the following example:

Let

$$F(z) = \sum_{i=1}^{\infty} b_i Q_{n_i}(z),$$

where the $Q_{n_i}(z)$ are the functions constructed in § 4. If the sequence $\{n_i\}$ is chosen as in § 4, and if $b_i \to 0$, then $\Sigma b_i Q_{n_i}(z)$ converges uniformly in the disc $|z - 1/4| \leqq 3/4$ and thus $F(z)$ is continuous in this disc. Hence the function $f(z) = F\left(\frac{z+1}{2}\right)$ is continuous in the disc $| z + 1/2 | \leqq 3/2$. Since, for $n = [n_i^3/2]$,

$$\left| \sum_{j=0}^{n} a_j \right| > c \, | b_i | \log n_i,$$

where $c > 0$, the partial sums of the series Σa_j are not even bounded if $b_i \to 0$ slowly enough.

We turn now to the problem of uniform convergence. If $k < 1$ and $(1-z)^k f(z)$ is regular and bounded in a GAIER disc, the TAYLOR series $\Sigma a_n z^n$ of $f(z)$ converges uniformly on every arc of the unit circle that does not contain the point $z = 1$; this follows from Theorem 2 in conjunction with a well-known theorem of M. RIESZ [12] (see LANDAU [6, p. 73]). On the other hand, we know from Theorem 6 that the TAYLOR series converges at the point $z = 1$ if $k < 0$, and from Theorem 4 that the

Taylor series converges uniformly on the entire unit circle if $k < -1/2$. The question remains as to whether this last statement can be extended to the case $k < 0$. The answer is in the affirmative.

Theorem 7. *If $f(z) = \Sigma\, a_n z^n$ satisfies the condition of Theorem 2, with $k < 0$, then $\Sigma\, a_n z^n$ converges uniformly on $|z| = 1$.*

By the preceding remarks, it suffices to assume that $-1/2 \leqq k < 0$ and to prove uniform convergence of $\Sigma\, a_n e^{in\alpha}$ for $0 < |\alpha| < \pi/4$. For the sake of convenience, we shall restrict ourselves to the interval $0 < \alpha < \pi/4$; the proof for the interval $-\pi/4 < \alpha < 0$ is analogous.

As at the beginning of § 5, let Γ be the curve $z = (1 + c_1 \Phi^2)\, e^{i\Phi}$, $-\pi \leqq \Phi \leqq \pi$; and let $z_0 = e^{i\alpha}$, $0 < \alpha < \pi/4$. We note first that for any z on Γ,

$$|z - z_0| = |(1 + c_1 \Phi^2)\, e^{i\Phi} - e^{i\alpha}| > c_2(\Phi^2 + |\Phi - \alpha|),$$

where $c_2 > 0$. Hence we obtain, for positive integral n and p,

$$\left|\sum_{j=n+1}^{n+p} a_j z_0^j\right| = \left|\frac{1}{2\pi i}\int_\Gamma f(z)\, \frac{z_0^{n+1}}{z^{n+2}}\, \frac{1-(z_0/z)^p}{1-z_0/z}\, dz\right|$$

$$\leqq \frac{1}{\pi}\int_\Gamma \frac{|f(z)|\,|dz|}{|z|^{n+1}\,|z - z_0|} < c_3 \int_{-\pi}^{\pi} \frac{|\Phi|^{-k}\, d\Phi}{(1 + c_1\Phi^2)^n\,(\Phi^2 + |\Phi - \alpha|)}.$$

Let the parts of the last integral that correspond to the intervals $-\pi \leqq \Phi \leqq \alpha/2$, $\alpha/2 \leqq \Phi \leqq 2\,\alpha$, and $2\,\alpha \leqq \Phi \leqq \pi$ be denoted by I_1, I_2, and I_3, respectively.

In I_1 we use the estimate $\Phi^2 + |\Phi - \alpha| \geqq \alpha - \Phi \geqq |\Phi|$; in I_3 we use the estimate $\Phi^2 + |\Phi - \alpha| \geqq \Phi - \alpha \geqq \Phi/2$. From these and from the method used in § 5 we get the inequality

$$I_1 + I_3 \leqq \int_{-\pi}^{\pi} \frac{2\,|\Phi|^{-k-1}\, d\Phi}{(1 + c_1\Phi^2)^n} < c_4\, n^{k/2}.$$

In I_2, we use the estimate $\Phi^2 + |\Phi - \alpha| \geqq \alpha^2/4 + |\Phi - \alpha|$ and obtain

$$I_2 \leqq \frac{(2\,\alpha)^{-k}}{(1 + c_1\,\alpha^2/4)^n} \int_{\alpha/2}^{2\alpha} \frac{d\Phi}{\alpha^2/4 + |\Phi - \alpha|}.$$

The integral in the last member can be evaluated directly and is equal to

$$\log\frac{\alpha + 2}{\alpha} + \log\frac{\alpha + 4}{\alpha} < -c_5 \log\alpha < c_6\, \alpha^{k/2}.$$

Thus

$$I_2 < c_7 \frac{\alpha^{-k/2}}{1+c_1 n \alpha^2/4} = c_7 n^{k/4} \frac{(n\alpha^2)^{-k/4}}{1+c_1 n \alpha^2/4} < c_8 n^{k/4}$$

(note that $0 < -k/4 \leqq 1/8$, and that therefore $t^{-k/4}/(1+c_1 t/4) < M(k, c_1)$ for $t > 0$).

The preceding estimates lead to the inequality

$$\left| \sum_{j=n+1}^{n+p} a_j z_0^j \right| \leqq c_3 (I_1 + I_2 + I_3) < c_9 n^{k/4},$$

and the proof is complete.

9. Functions regular in Gaier regions

We shall call a Gaier region (see § 1) a Gaier *region of order* $p (p > 0)$ if it contains the interior of the curve

$$z(\Phi) = (1 + c |\Phi|^p) e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi,$$

for some positive value of c.

Theorem 8. *Let* $f(z) = \Sigma a_n z^n$ *be regular in some* Gaier *region of order* p $(p \geqq 1)$, *and let* k *be a real number* $(k < 1)$ *such that* $(1-z)^k f(z)$ *is bounded in this region. Then*

$$a_n = O(n^{(k-1)/p}).$$

The proof proceeds as in § 5. We choose an appropriate curve

$$\Gamma: \qquad z = (1 + c_1 |\Phi|^p) e^{i\Phi}, \qquad -\pi \leqq \Phi \leqq \pi,$$

and we use the fact that

$$2\pi |a_n| = \left| \int_\Gamma \frac{f(z)}{z^{n+1}} dz \right| \leqq c_2 \int_0^\pi \frac{1}{\Phi^k (1 + c_1 \Phi^p)^n} \left[1 + \frac{c_1 p \Phi^{p-1}}{1 + c_1 \Phi^p} \right] d\Phi$$

$$\leqq c_3 \int_0^\pi \frac{d\Phi}{\Phi^k (1 + c_1 \Phi^p)^n} < c_4 n^{(k-1)/p}.$$

We note that if $(1-z)^k f(z)$ is continuous in the closure of a Gaier region of order p, the O in the statement of Theorem 8 can be replaced by o. Moreover, for functions continuous in the closure of a Gaier region of order 1, our result reduces to a theorem of M. Riesz [11] (see also Landau [6, p. 64]). For a similar result closely related to Riesz's theorem, see Gaier [4, Theorems 1 and 2].

Bibliography

[1] L. FEJÉR, Asymptotikus értékek meghatározásáról. Math. és Termész. Értesitö **27**, 1—33 (1909).

[2] L. FEJÉR, Sur les singularités de la série de Fourier des fonctions continues. Annales Scientifiques de l'École Normale Supérieure (3) **28**, 63—103 (1911).

[3] D. GAIER, Über die Summierbarkeit beschränkter und stetiger Potenzreihen an der Konvergenzgrenze. Math. Z. **56**, 326—334 (1952).

[4] D. GAIER, Complex Tauberian theorems for power series. Trans. Amer. Math. Soc. **75**, 48—68 (1953).

[5] A. GUTZMER, Ein Satz über Potenzreihen. Math. Ann. **32**, 596—600 (1888).

[6] E. LANDAU, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. Berlin 1929.

[7] L. NEDER, Über die Koeffizientensumme einer beschränkten Potenzreihe. Math. Z. **11**, 115—123 (1921).

[8] O. PERRON, Über das infinitäre Verhalten der Koeffizienten einer gewissen Potenzreihe. Arch. d. Math. u. Phys. (3) **22**, 329—340 (1914).

[9] O. PERRON, Über das Verhalten einer ausgearteten hypergeometrischen Reihe bei unbegrenztem Wachstum eines Parameters. J. reine angew. Math. **151**, 63—78 (1920).

[10] G. PÓLYA und G. SZEGÖ, Aufgaben und Lehrsätze aus der Analysis. Berlin 1925.

[11] M. RIESZ, Sur un problème d'Abel. Rend. Circ. Mat. Palermo **30**, 339—345 (1910).

[12] M. RIESZ, Über einen Satz des Herrn Fatou. J. reine angew. Math. **140**, 89—99 (1911).

American University
Michigan State College
University of Michigan

Eingegangen am 29. 8. 1953