MATHEMATICS

SOME LINEAR AND SOME QUADRATIC RECURSION FORMULAS II [1])

BY

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(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of February 23, 1952)

§ 5. *Case* 1

Throughout this section we assume that we are in case 1, that is

$$f(1)=1; \qquad f(n)=\sum_{k=1}^{n-1} c_k f(n-k) \qquad (n=2,3,\ldots) \tag{5.1}$$

$$c_k>0 \qquad (k=1,2,3,\ldots), \tag{5.2}$$

$$\limsup_{k\to\infty} c_k^{1/k}=\infty. \tag{5.3}$$

For, (5. 3) is equivalent to the condition that $c_1x+c_2x^2+\ldots$ diverges when $|x|>0$.

It was proved in § 3 that

$$\lim_{n\to\infty} (f(n))^{1/n}=\infty; \tag{5.4}$$

but apart from this the preceding sections give little information about the behaviour of $f(n)$ in case 1.

(5. 4) implies that

$$\alpha=\liminf \frac{f(n)}{f(n+1)}=0, \tag{5.5}$$

but on the other hand

$$\beta=\limsup \frac{f(n)}{f(n+1)} \tag{5.6}$$

can be positive (see example 1 below). Anyway β is finite, by (2. 6). ($\beta \leqslant c_1^{-1}$).

Theorem 13. We have, if $n\to\infty$,

$$\limsup c_n/f(n)=\infty, \tag{5.7}$$

$$\limsup c_n/f(n+1)=1. \tag{5.8}$$

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[1]) The first part appeared in this journal: Kon. Ned. Akad. v. Wetensch. (A) **54**, 374–382 (1951) = Indagationes Mathematicae **13**, 374–382 (1951).

Proof. (5. 7) follows from theorem 18 (§ 6). For assume $c_n < Mf(n)$, M not depending on n; then theorem 18 gives $f(n) = O\{(4M)^n\}$, which contradicts (5. 4).

In order to prove (5. 8), we first remark that $f(n+1) > c_n$ for all $n > 1$ (see (5. 1)). So if (5. 8) were false, then there would exist a constant λ $(0 < \lambda < 1)$ such that

$$c_n < \lambda f(n+1) \qquad (n = 2, 3, \dots). \tag{5. 9}$$

It follows, by (5. 1), that

$$\begin{cases} (1-\lambda) f(n+1) < f(2) f(n) + f(3) f(n-1) + \dots + \\ \qquad\qquad + f(n) f(2) \qquad (n = 2, 3, \dots). \end{cases} \tag{5. 10}$$

The sequence $\{\varrho_n\}$, defined by

$$\varrho_n = (n+1)^2 \left\{ \frac{1}{2^2 \cdot n^2} + \frac{1}{3^2 (n-1)^2} + \dots + \frac{1}{n^2 \cdot 2^2} \right\} \qquad (n = 2, 3, \dots)$$

is convergent; hence a constant C exists such that $\varrho_n < C$ $(n = 2, 3, \dots)$. It is now easily deduced from (5. 10) that

$$f(n) \leqslant \frac{4 f(2)}{n^2} \cdot \left(\frac{4 f(2) C}{1-\lambda} \right)^{n-2} \qquad (n = 2, 3, \dots).$$

This contradicts (5. 4), and so (5. 8) is proved.

The set of equations (5. 1) can be solved explicitly:

$$f(n) = \sum_{h=1}^{n-1} \sum c_{i(1)} c_{i(2)} \dots c_{i(h)}, \qquad (n = 2, 3, \dots) \tag{5. 11}$$

where, in the second sum, the summation variables $i(1), \dots, i(h)$ are subjected to the conditions

$$i(1) > 0, \dots, i(h) > 0; \qquad i(1) + \dots + i(h) = n - 1.$$

We can obtain (5. 11) from the formal expansion (see (1. 4))

$$F(x) = x + \sum_{h=1}^{\infty} (C(x))^h.$$

The summands of (5. 11) correspond one-to-one to the ordered partitions of $n-1$; hence the total number of terms equals 2^{n-2}.

Let I_n denote the largest one of these 2^{n-2} summands. Then clearly

$$1 \leqslant \{ f(n)/I_n \}^{1/n} \leqslant 2 \qquad (n = 2, 3, \dots). \tag{5. 12}$$

We can even show

Theorem 14. $\lim_{n \to \infty} \{ f(n)/I_n \}^{1/n} = 1.$

Proof. Let ε be a positive number, and N an integer > 1. Write

$$f(n) = \Sigma_1 + \Sigma_2,$$

where Σ_2 is the sum of all summands $c_{i(1)} \ldots c_{i(h)}$ which satisfy

$$(5.13) \qquad \sum_{1 \leqslant j \leqslant h,\ i(j) < N} i(j) < \varepsilon n.$$

If ε and N are fixed, then Σ_2 can be roughly described as corresponding to those ordered partitions of $n-1$ where the contribution of the small integers is small. We shall show that the number of summands in Σ_2 is relatively small, if ε and N^{-1} are small but fixed. If a partition $n-1 = i(1) + \ldots + i(h)$ satisfies (5.13) then we have

$$(5.14) \qquad h \leqslant \eta n \qquad (\eta = \varepsilon + N^{-1}).$$

For, the number of i's $< N$ is at most εn, since each i is $\geqslant 1$. And, the number of i's $\geqslant N$ is at most $N^{-1}n$, their sum being $\leqslant n-1$.

The number of partitions satisfying (5.14) equals

$$(5.15) \qquad \sum_{1 \leqslant h \leqslant \eta n} \binom{n-2}{h-1} < \exp\{n(\eta - \eta \log \eta)\}.$$

For we have generally, if $0 < u < 1$, $n-1 \geqslant m \geqslant 0$,

$$u^m \sum_{h \leqslant m} \binom{n-1}{h} \leqslant \sum_{h \leqslant m} u^h \binom{n-1}{h} \leqslant (1+u)^n.$$

Taking $m = [\eta n]$, $u = \eta$ we obtain (5.15).

We can show that, if n is large, the largest summand I_n of (5.11) occurs in Σ_2, and we even have $I_n > \Sigma_1$. To this end we prove that to each summand t of Σ_1 a second summand t' (of either Σ_1 or Σ_2) can be found such that $t' > 2^n t$.

In virtue of (5.3) we can choose $k = k(N, \varepsilon)$ such that

$$(5.16) \qquad c_k^{1/k} > c_1, \ c_k^{1/k} > 4^{1/\varepsilon} c_1^{-1} \mu^2 \qquad (\mu = \max_{1 \leqslant j < N} c_j^{1/j}).$$

We put $n_0 = n_0(N, \varepsilon) = 2k\varepsilon^{-1}$; henceforth assume $n > n_0$.

Let the term t of Σ_1 correspond to the partition $n-1 = i(1) + \ldots + i(h)$, then we have, say,

$$i(1) + \ldots + i(r) = s > \varepsilon n, \ i(1) < N, \ldots, i(r) < N.$$

Now we obtain t' from t by replacing the factors

$$c_{i(1)} \ldots c_{i(r)} \quad \text{by} \quad c_k^{[s/k]} c_1^{s - k[s/k]}.$$

Then we have

$$t'/t = (c_k/c_1^k)^{[s/k]} c_1^s \mu^{-s}.$$

We have $s > \varepsilon n > 2k$ and so $[s/k] > \frac{1}{2} s/k$. Therefore, by (5.16),

$$(c_k/c_1^k)^{[s/k]} > (c_k/c_1^k)^{\frac{1}{2} s/k} > 4^{\frac{1}{2} s/\varepsilon} c_1^{-\frac{1}{2} s} \cdot c_1^{-\frac{1}{2} s} \mu^s,$$

and so

$$t'/t > 2^{s/\varepsilon} > 2^n.$$

This shows that each term of Σ_1 is less than $2^{-n} I_n$; therefore $\Sigma_1 < I_n$. The number of terms of Σ_2 is bounded above by the right-hand-side of (5. 15); hence

$$\Sigma_2 / I_n < \exp \{ n (\eta - \eta \log \eta) \}.$$

It now follows from $f(n) = \Sigma_1 + \Sigma_2$ that

$$\limsup_{n \to \infty} \{ f(n)/I_n \}^{1/n} \leqslant \exp (\eta - \eta \log \eta).$$

As $\eta = \varepsilon + N^{-1}$, the theorem now follows by making $\varepsilon \to 0$, $N \to \infty$.

The following theorem was already announced in § 4 (theorem 9).

Theorem 15. If $c_{n+1}/c_n \to \infty$, then $f(n+1)/f(n) \to \infty$.

Proof. We first prove: if $A > 0$, there exists a number $B = B(A) > 0$, such that $f(n+1) > B\, f(n)$ implies $f(n+2) > A\, f(n+1)$.

Let K be such that $c_{k+1} > A\, c_k$ for all $k \geqslant K$, and take

$$B = A \{ 1 + c_2 c_1^{-2} + c_3 c_1^{-3} + \ldots + c_K c_1^{-K} \}.$$

Now assume $f(n+1) > B\, f(n)$, and put $L = \min (n, K-1)$. Then we have (empty sums are zero)

$$f(n+1) = c_1 f(n) + c_2 f(n-1) + \ldots + c_L f(n+1-L) + \sum_{k=L+1}^{n} c_k f(n+1-k),$$

$$f(n+2) \geqslant c_1 f(n+1) + \sum_{k=L+1}^{n} c_{k+1} f(n+1-k).$$

We have $f(m+1) \geqslant c_1 f(m)$ for all m, and so

$$c_1 f(n) + \ldots + c_L f(n+1-L) \leqslant f(n) \{ c_1 + c_2 c_1^{-1} + \ldots + c_L c_1^{1-L} \} \leqslant$$
$$\leqslant c_1 f(n)\, B/A < c_1 f(n+1)/A.$$

It follows that $f(n+2) > A\, f(n+1)$.

By iteration of this result we find: If $A > 0$, $k > 0$, there exists a positive number $C(A, k)$ such that

(5. 17) $$f(n+1) > C(A, k)\, f(n)$$

implies

(5. 18) $$f(n+j+1) > A\, f(n+j) \qquad (j = 0, 1, \ldots, k).$$

We can now show that $f(n+1)/f(n) \to \infty$. Let A be an arbitrary positive number, and choose K such that $c_{k+1} > A\, c_k$ for all $k \geqslant K$.

We have $\limsup f(n+1)/f(n) = \infty$ (see (5. 5)); therefore we can take N such that $N > K$, $f(N+1) > C(A, K)\, f(N)$. We can show that

(5. 19) $$f(N+j+1) > A\, f(N+j) \qquad (j = 0, 1, 2, \ldots).$$

By (5. 17) and (5. 18) we know that (5. 19) holds if $j = 0, 1, \ldots, K$.

We proceed by induction. Assume (5. 19) to be true for $j < K + m$, where m is a positive integer. Then we have

$$f(N+K+m+1) < \sum_{j=1}^{K} c_j f(N+K+m+1-j) + \sum_{j=K+1}^{N+K+m-1} c_{j+1} f(N+K+m-j) >$$

$$> A \sum_{j=1}^{K} c_j f(N+K+m-j) + A \sum_{j=K+1}^{N+K+m-1} c_j f(N+K+m-j) = A\, f(N+K+m).$$

This proves (5. 19). Since A is arbitrary, we obtain $f(m+1)/f(m) \to \infty$.

Lemma. For $n = 1,2,3,\ldots$ we have

$$\frac{f(n+1)}{f(n)} \leqslant c_1 + \max_{1 \leqslant j < n} \frac{c_{j+1}}{c_j}.$$

Proof. Denoting the right-hand-side by $c_1 + \mu$, we have

$$f(n+1) = c_1 f(n) + \sum_{1}^{n-1} c_{k+1} f(n-k) \leqslant c_1 f(n) + \mu \sum_{1}^{n-1} c_k f(n-k) = (c_1 + \mu) f(n).$$

Theorem 16. If

$$\frac{c_2}{c_1} \leqslant \frac{c_3}{c_2} \leqslant \frac{c_4}{c_3} \leqslant \cdots \tag{5. 20}$$

then we have

$$\frac{f(2)}{f(1)} \leqslant \frac{f(3)}{f(2)} \leqslant \frac{f(4)}{f(3)} \leqslant \ldots, \tag{5. 21}$$

$$\limsup_{n\to\infty} \frac{f(n+1)}{f(n)} \cdot \frac{c_{n-1}}{c_n} = 1, \tag{5. 22}$$

$$\lim_{n\to\infty} \{ f(n)/c_{n-1} \}^{1/n} = 1. \tag{5. 23}$$

Proof. For (5. 21) see the proof of theorem 12, § 4.

As to (5. 22), the lemma shows that the lim sup is at most 1. For, (5. 3) and (5. 20) imply that $c_n/c_{n-1} \uparrow \infty$. On the other hand, the lim sup cannot be less than 1, since

$$\prod_{k=1}^{n} \left\{ \frac{f(k+1)}{f(k)} \cdot \frac{c_{k-1}}{c_k} \right\} = \frac{f(n+1)}{c_n} c_1 \geqslant c_1.$$

Finally, (5. 23) follows from (5. 22) and from the fact that $f(n+1) \geqslant c_n$ for all n.

We can deduce (5. 23) from theorem 14 also: Without loss of generality we may assume $c_1 = 1$ (see the transformation (1. 5)). Then (5. 20) implies that $\{ c_k^{1/k} \}$ is a non-decreasing sequence, whence $I_n = c_{n-1}$ for all n.

Theorem 17. Let C and α be positive constants, and $\psi(k) = C\, k^{\alpha}$. Then if

$$\frac{c_{k+1}}{c_k} \uparrow \infty, \quad \frac{c_{k+1}}{c_k} > \psi(k) \qquad (k = 1, 2, 3, \ldots),$$

we have

$$f(n+1)/c_n \to 1 \qquad (n \to \infty).$$

Proof. We have, by (5. 1),

$$0 \leqslant \frac{f(n+1)}{c_n} - 1 = \sum_1^n c_j f(n+1-j)/c_n \leqslant \sum_1^n \frac{c_j c_{n-j}}{c_n} \max_{1 \leqslant k < n} \frac{f(k+1)}{c_k}.$$

Since c_{k+1}/c_k is increasing, $c_j c_{n-j}$ decreases with increasing j in the interval $1 \leqslant j < \frac{1}{2}n$.

Let q be an integer $> \alpha^{-1}$, and assume $n > 3q$. Then we have, when $q \leqslant j < \frac{1}{2}n$,

$$\frac{c_j c_{n-j}}{c_n} \leqslant \frac{c_q c_{n-q}}{c_n} \leqslant c_q \prod_{n-q}^{n-1} \frac{c_k}{c_{k+1}}.$$

Hence we obtain, for q fixed,

$$0 \leqslant \frac{f(n+1)}{c_n} - 1 \leqslant \tfrac{1}{2} n \cdot \max_{1 \leqslant k < n} \frac{f(k+1)}{c_k} \cdot O(n^{-\alpha q}) = o(1) \cdot \max_{1 \leqslant k < n} \frac{f(k+1)}{c_k}.$$

Now the theorem easily follows.

Theorems 13—17 seem to be comparatively weak. We shall, however, give some examples which show that not very much more can be obtained.

Example 1. $f(n+1)/f(n)$ need not tend to infinity if c_{n+1}/c_n does not tend to infinity. Define c_n, $f(n)$ by (5. 1) and by $(n = 1, 2, 3, \ldots)$

$$c_{2n} = n \left(\sum_1^{2n} f(k)\right) \cdot \left(\sum_1^{2n-1} c_k\right), \quad c_1 = 1, \quad c_{2n+1} = c_{2n}.$$

Clearly $c_{2n}/c_{2n-1} \to \infty$; hence we are in case 1. We have, if $n > 2$,

$$\begin{aligned} f(2n+2) = c_1 f(2n+1) + \sum_2^{2n-1} c_k f(2n+2-k) + c_{2n} f(2) + c_{2n+1} f(1) < \\ < c_1 f(2n+1) + c_{2n} + c_{2n} f(2) + c_{2n} f(1) < \\ < f(2n+1)\{c_1 + 1 + f(2) + f(1)\}. \end{aligned}$$

Therefore $f(2n+2)/f(2n+1) = O(1)$. The sequence $f(2n+1)/f(2n)$ is not bounded, of course (see (5. 5)).

Example 2. The expressions $c_n/f(n)$ and $c_n/f(n+1)$, whose upper limit was established in theorem 13, can have lower limit zero, even if $c_{n+1}/c_n \to \infty$.

Let $\{\varphi(n)\}$ be any positive sequence, then we can find a sequence $\{c_n\}$ satisfying $c_{n+1}/c_n \to \infty$, such that

(5. 24) $\qquad f(n) > \varphi(n)\, c_n, \; f(n) > \varphi(n)\, c_{n-1}$ infinitely often.

To this end we take

$$c_n = \{\psi(k)\}^{n+2} \qquad (4^{k-1} \leqslant n < 4^k, \; k = 1, 2, \ldots),$$

where $\psi(k)$ is the maximum of $\varphi(n)$ in the range $4^{k-1} \leqslant n < 4^k$. We may assume, of course, that $\varphi(n) \to \infty$.

For any k we have, if $m = 2 . 4^{k-1} - 1$,

$$f(2m+1) = \sum_1^{2m} c_j f(2m+1-j) > c_m f(m+1) > c_m^2$$

and

$$c_m^2 = \{\psi(k)\}^{2m+4} = c_{2m+1} \psi(k).$$

Therefore

$$f(2m+1) > \varphi(2m+1) c_{2m+1}.$$

This proves the first part of (5. 24). The second part is a direct consequence, since $c_{n+1}/c_n \to \infty$.

Example 3. If c_{n+1}/c_n tends to infinity monotonically, then (5. 24) cannot be true if $n^{-1} \log \varphi(n)$ has a positive upper limit (see 5. 23). But, if $\eta(n)$ is an arbitrary positive function satisfying $\eta(n) \to 0 (n \to \infty)$, then a sequence $\{c_n\}$ can be found such that $(c_{n+1}/c_n) \uparrow \infty$, and

(5. 25) $\qquad f(n)/c_{n-1} > e^{n\eta(n)}$ for infinitely many values of n [1]).

Let the sequence $\{t_n\}$ satisfy $1 < t_1 < t_2 < \ldots$, $\lim t_n = \infty$. Let $\{c_n\}$ be defined by

$$c_1 = 1; \quad c_{n+1}/c_n = t_k \quad (N_k \leqslant n < N_{k+1},\ k = 1, 2, 3, \ldots),$$

where the integers N_i $(1 = N_1 < N_2 < N_3 < \ldots)$ will be chosen such that (5. 25) holds infinitely often. To this end we prove: If $N_1, \ldots, N_{K+1}$ have been fixed, then N_K can be found such that (5. 25) holds for $n = N_K$.

Let $\{c_n^*\}$ be defined by

$$c_1^* = 1; \quad c_{n+1}^*/c_n^* = t_k \quad (N_k \leqslant n < N_{k+1},\ k = 1, \ldots, K-2),$$
$$c_{n+1}^*/c_n^* = t_{K-1} \quad (n \geqslant N_{K-1}).$$

The sequence $\{c_n^*\}$ belongs to case 2 (see § 2), and we have [2])

$$R^{-1} = t_{K-1}, \quad 0 < \gamma < R, \quad f^*(n)\gamma^n \to C \quad (C > 0), \quad (c_n^*)^{1/n} \to R^{-1}.$$

It follows that $\lim (f^*(n)/c_{n-1}^*)^{1/n} = R/\gamma > 1$. Since $\eta(n) \to 0$, we can find a number $N_K > N_{K-1}$ such that

$$f^*(n) > e^{n\eta(n)} c_{n-1}^* \qquad (n = N_K).$$

Now N_K has been fixed, and obviously $c_n = c_n^*$, $f(n) = f^*(n)$ $(n \leqslant N_K)$. Hence (5. 25) holds for $n = N_K$.

Example 4. There exists a sequence c_n with $(c_{n+1}/c_n) \uparrow \infty$, such that

$$\liminf \frac{f(n+1)}{f(n)} \cdot \frac{c_{n-1}}{c_n} = 0.$$

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[1]) The same thing can be obtained for $f(n)/c_n$, without much extra trouble.

[2]) $\{f^*(n)\}$ is the sequence corresponding to $\{c_n^*\}$ by the analogue of (5. 1).

This shows that, in (5. 22), "lim sup" may not be replaced by "lim".

An example can be obtained along the same lines as above; we only take care that $t_{k+1}/t_k \to \infty$.
Further, N_K has to be determined such that

$$f^*(n+1)\, c^*_{n-1} < 2\, f^*(n)\, c^*_n \qquad (n = N_K + 1),$$
$$c^*_n / f^*(n+1) < K^{-1} \qquad (n = N_K + 1).$$

Then it is easily verified that

$$f(n+1)\, c_{n-1}/(f(n)\, c_n) < 2\,\{(t_K/t_{K-1})^{-1} + K^{-1}\} \qquad (n = N_K + 1).$$

Example 5. The following example shows that, in theorem 17, the condition $c_{k+1}/c_k \uparrow \infty$ is essential. It shows that no function $\psi(k)$ has the property that $c_{k+1}/c_k > \psi(k)$ implies $f(n+1)/c_n \to 1$. For take $c_1, c_2, \ldots$ such that $c_{k+1}/c_k > \psi(k)$ $(k = 1,\ 2,\ 3,\ \ldots)$ and such that $c^2_m/c_{2m} > 2$ for infinitely many m. Then obviously for these m we have

$$f(2m+1) = \sum_1^{2m} c_k\, f(2m+1-k) > c_m\, f(m+1) > c^2_m > 2\, c_m.$$

We finally remark that theorem 17 is best possible in the following sense: If the increasing function $\psi(k)$ has the property that

$$\frac{c_{k+1}}{c_k} \uparrow \infty,\ \frac{c_{k+1}}{c_k} > \psi(k) \quad (k = 1, 2, \ldots) \quad \text{imply} \quad f(n+1)/c_n \to 1,$$

then we have $\psi(k) > Ck^a$ for suitable positive constants C and a. We omit the proof.

§ 6. *The quadratic recursion formula*

Consider

$$(6.\ 1) \qquad f(1) = 1, \quad f(n) = \sum_{k=1}^{n-1} d_k\, f(k)\, f(n-k) \qquad (n = 2, 3, \ldots),$$

where $d_k > 0$ $(k = 1, 2, 3, \ldots)$. Consequently also $f(n) > 0$ $(n = 1, 2, 3, \ldots)$.

Putting $d_k f(k) = c_k$, we have $c_k > 0$ $(k = 1, 2, \ldots)$. Therefore, we can use the results and the division into 5 cases introduced in § 2.

In the first place it follows that $\{f(n)\}^{-1/n}$ always tends to a finite limit as $n \to \infty$. We have, however, no simple formula which relates its value γ to the numbers d_k.

If $d_k \to \infty$, then we have $\gamma = 0$ (case 1). For then, by $f(n+1) \geqslant c_n = d_n f(n)$, we have $f(n+1)/f(n) \to \infty$. On the other hand we have

Theorem 18. If $d_k = O(1)$, then $\gamma > 0$.

Proof. It is sufficient to show that $f(n) = O(P^n)$ for some P. Assume $d_n \leqslant M$ for all n. Then the sequence $\{f(n)\}$ is majorised by the sequence $\{g(n)\}$ satisfying

$$g(1) = 1, \ g(n) = M \textstyle\sum_1^{n-1} g(k)\, g(n-k) \qquad (n = 2, 3, \ldots).$$

The unique solution is obtained from the generating function $G(x)$ which satisfies

$$G(x) - x = M\,G^2(x), \qquad G(0) = 0,$$

whence

$$G(x) = (2M)^{-1}\{1-(1-4Mx)^{\frac{1}{2}}\},\ g(n) = \frac{1}{2n-1}\frac{(2n)!}{n!\,n!}M^n = O\{(4M)^n\}.$$

It follows that $f(n) = O\{(4M)^n\}$, and so $\gamma \geqslant (4M)^{-1}$.

If $\liminf d_k < \limsup d_k = \infty$, then we may have either case 1, or 4, or 5 (see the examples in the beginning of § 4 and example 1, § 5). If $0 < \liminf d_k \leqslant \limsup d_k < \infty$, then we are in case 5. For then we have $\gamma > 0$ and $\Sigma f(n)\,\gamma^n = O(\Sigma c_n\gamma^n) < \infty$, which is only possible in case 5 (see (2. 3)). An interesting example is obtained by taking $d_1 = d_3 = \ldots = a > 0, d_2 = d_4 = \ldots = b > 0$. It can be shown that $f(2n+1)/f(2n) \to A > 0, f(2n)/f(2n-1) \to B > 0$, where $A \neq B$ if $a \neq b$.

Theorem 19. Necessary and sufficient that we are in case 2 is that

$$\limsup_{n\to\infty} (d_n)^{1/n} < 1. \tag{6. 2}$$

Proof. In case 2 we have (2. 4), where δ is such that $C(x)$ is regular for $|x| \leqslant \delta$. Therefore

$$\limsup (c_n)^{1/n} \leqslant \delta^{-1}, \text{ and so } \limsup (d_n)^{1/n} \leqslant \gamma/\delta < 1.$$

If, on the other hand, (6. 2) holds, then we know by theorem 18 that the series $F(x)$ has a positive radius of convergence, and further, by (6. 2) and $c_n = d_n f(n)$, that the radius of convergence of $C(x)$ is larger than the one of $F(x)$, which equals the least positive root of $C(x) = 1$ (see (1. 4)). It follows that we are in case 2.

Theorem 20. Necessary and sufficient that we are in case 3 is that

$$\sum n d_n < \infty, \quad \limsup (d_n)^{1/n} = 1. \tag{6. 3}$$

Proof. In case 3 we have $\Sigma\, nc_n\gamma^{n-1} < \infty$, and $f(n)\ \gamma^n$ tends to a positive limit. Hence $\Sigma\, nd_n < \infty$. Consequently, the lim sup in (6. 3) cannot be > 1. It cannot be < 1 either, because of theorem 19.

If on the other hand (6. 3) holds, then case 1 is excluded by theorem 18, case 2 by theorem 19, and case 5 by theorem 3(§ 4). Furthermore, by (2. 5) we infer $C'(\gamma) = \Sigma\, nc_n\gamma^{n-1} < \infty$, which excludes case 4.

If $\Sigma\, d_n < \infty$, $\Sigma\, n\, d_n = \infty$ then we are in case 4 (the cases 1, 2, 3, 5 are excluded, respectively, by theorems 18, 19, 20, 3). Moreover we find that $f(n)/f(n+1) \to \gamma$ (theorem 11).

If $d_n \to 0$, $\Sigma\, d_n = \infty$ then we are either in case 4 or in case 5.

If we have $0 < \limsup d_n < \infty$, then we are again either in case 4, or 5 (see the examples in the beginning of § 4).

We do not know whether the existence of lim d_n is or is not sufficient for the existence of lim $f(n)/f(n+1)$. A positive example is

Theorem 21. If $A > 0$, $0 < \eta < 1$ and $d_n = A + O(\eta^n)$, then we have $f(n)/f(n+1) \to \gamma$, and even

$$f(n) \sim Bn^{-3/2}\gamma^{-n} \qquad (B > 0).$$

Proof. We can exclude cases 1, 2 and 3, by theorems 18, 19, 20. Putting $\Sigma\,(d_n - A)\,f(n)x^n = \Phi(x)$, we have

$$F(x) - x = \{\,AF(x) + \Phi\,(x)\,\}\,F(x),$$

where $AF(x) + \Phi\,(x)$ has non-negative coefficients. $F(x)$ is regular for $|x| < \gamma$, and has a singularity at $x = \gamma$, its coefficients being non-negative. As $\eta < 1$, $\Phi\,(x)$ is regular for $|x| \leqslant \gamma$. Furthermore

$$\{\,AF(x) + \tfrac{1}{2}\Phi\,(x) - \tfrac{1}{2}\}^2 = \tfrac{1}{4}\,\{\,\Phi\,(x) - 1\,\}^2 - Ax.$$

Since $x = \gamma$ is a singularity of $F(x)$, we infer that $AF(\gamma) + \frac{1}{2}\,\Phi\,(\gamma) = \frac{1}{2}$. Further, $AF(x) + \frac{1}{2}\,\Phi\,(x) = \frac{1}{2}AF(x) + \frac{1}{2}\{\,AF(x) + \Phi\,(x)\,\}$ has non-negative coefficients, and so $|AF(x) + \frac{1}{2}\,\Phi\,(x)| < \frac{1}{2}$ if $|x| \leqslant \gamma, x \neq \gamma$. It also follows that the root of $AF(x) + \Phi(x) - \frac{1}{2}$ at $x = \gamma$ is a single one. Consequently $F(x)$ has no further singularities on the circle $|x| = \gamma$, and we have

$$AF(x) + \tfrac{1}{2}\,\Phi\,(x) = \tfrac{1}{2} - (\gamma - x)^{\frac{1}{2}}\,h(x),$$

where $h(x)$ is regular for $|x| \leqslant \gamma$, and $h(\gamma) \neq 0$. It can now be shown (e.g. by Cauchy's theorem) that

$$A\,f(n) \sim \tfrac{1}{2}\,\pi^{-1/2}\,n^{-3/2}\,h(\gamma)\,\gamma^{-n+1/2}.$$

We are in case 5, since

$$C(\gamma) = AF(\gamma) + \Phi(\gamma) = 2\{AF(\gamma) + \tfrac{1}{2}\Phi(\gamma)\} - AF(\gamma) = 1 - AF(\gamma) < 1.$$

§ 7. *A generalisation*

We shall consider, in theorem 24, a more general quadratic recursion formula. We first generalise the method of § 3, where we used the fact that for any sub-additive function $g(n)$ the limit of $g(n)/n$ exists (it may be $-\infty$.) We can prove a slightly better result:

Theorem 22. Let the sequence $g(n)$ $(n = 1, 2, \ldots)$ satisfy

(7.1) $\qquad g(n + m) \leqslant g(n) + g(m)$ whenever $\frac{1}{2}n \leqslant m \leqslant 2n$.

Then we have

(7. 2) $$\frac{g(n)}{n} \to L$$

for some L $(-\infty \leqslant L < \infty)$, and

(7. 3) $$\frac{g(n)}{n} \geqslant L \qquad (n = 1, 2, \ldots).$$

Proof. Put $g(n)/n = h(n)$. Clearly we have

$$(7.4)\qquad h(2^k n) \leqslant h(n) \qquad (n = 1, 2, 3, \ldots;\ k = 0, 1, 2, \ldots).$$

Further it is easily proved by induction that $h(n) \leqslant h(1)$ for all n (every integer $n > 1$ can be written as $a + b$, where $\frac{1}{2}a \leqslant b \leqslant 2a$).

Let u and v be positive integers, and $u \geqslant \frac{3}{2} v$. Let the integer k be determined by $\frac{1}{3} u < 2^k v \leqslant \frac{2}{3} u$. Put $2^k v = w$, $u - w = z$. Then we have $\frac{1}{2} \leqslant z/w < 2$, and so, by (7. 1) $u\, h(u) \leqslant z\, h(z) + w\, h(w)$.
By (7. 4) we have $h(w) \leqslant h(v)$; furthermore we have $w = u - z$, and $z < \frac{2}{3} u$. Therefore

$$(7.5)\qquad h(u) - h(v) \leqslant \frac{z}{u}\{h(z) - h(v)\} \leqslant \tfrac{2}{3}\{h(z) - h(v)\}.$$

Summarizing: if $u \geqslant \frac{3}{2} v$, then there is a number z $(\frac{1}{3} u \leqslant z < \frac{2}{3} u)$ such that (7. 5) holds. By iteration of (7. 5) we obtain

$$(7.6)\qquad h(u) - h(v) \leqslant \tfrac{3}{2}\left(\frac{3v}{2u}\right)^{\lambda}\{h(1) - h(v)\} \qquad (u \geqslant \tfrac{3}{2} v),$$

where $\lambda = (\log \frac{3}{2})/\log 3$.

From (7. 6) we infer $\limsup h(u) \leqslant \inf h(v)$, and the theorem follows.

It may be remarked that the inequality in (7. 1) cannot be replaced by $\mu^{-1} n \leqslant m \leqslant \mu n$ for any $\mu < 2$.

Theorem 23. Let $\varphi(t)$ be positive and increasing for $t > 0$, and assume

$$\int_1^\infty \varphi(t)\, t^{-2}\, dt < \infty.$$

Let the sequence $\{ g(n) \}$ satisfy

$$(7.7)\qquad g(n + m) \leqslant g(n) + g(m) + \varphi(n + m) \qquad (\tfrac{1}{2}n \leqslant m \leqslant 2n)$$

Then $g(n)/n \to L$ for some L $(-\infty \leqslant L < \infty)$.

Proof. Put

$$g(n) + 3n \int_n^\infty \varphi(3t)\, t^{-2}\, dt = G(n) \qquad (n = 1, 2, \ldots).$$

Then, we have, by (7. 7), if $\frac{1}{2}n \leqslant m \leqslant 2n$,

$$G(n+m) - G(n) - G(m) \leqslant \varphi(3n) + \varphi(3m) - 3n \int_n^{n+m} - 3m \int_m^{n+m} \leqslant$$

$$\leqslant \varphi(3n)\left\{1 - 3n\left(\frac{1}{n} - \frac{1}{n+m}\right)\right\} + \varphi(3m)\left\{1 - 3m\left(\frac{1}{m} - \frac{1}{n+m}\right)\right\}.$$

The latter expression is $\leqslant 0$, since we have $\frac{1}{2}n \leqslant m \leqslant 2n$. Therefore, theorem 22 can be applied to the function $G(n)$. Finally we have obviously $\{ G(n) - g(n) \}/n \to 0$.

Theorem 24. Let $\varphi(t)$ satisfy the conditions mentioned in theorem 23, and let the numbers $c_{k,n}$ satisfy

$$(7.8) \qquad c_{k,n} > 0 \quad (1 \leqslant k < n < \infty), \; c_{k,n} > e^{-\varphi(n)} \quad (\tfrac{1}{3} n \leqslant k \leqslant \tfrac{1}{2} n)$$

Let

$$(7.9) \qquad f(1) = 1, \; f(n) = \sum_{1}^{n-1} c_{k,n} f(k) f(n-k). \qquad (n = 2, 3, \ldots).$$

Then $\{ f(n) \}^{-1/n}$ tends to a finite limit. The limit is positive if we add the condition $c_{k,n} < M \quad (1 \leqslant k < n < \infty)$.

Proof. We have $f(n) \geqslant c_{k,n} f(k) f(n-k)$. Putting $g(n) = -\log f(n)$, we have (7. 7), and the result follows from theorem 23.

If $c_{k,n} < M$, then $f(n)$ is majorized by the solution of the according equation with $c_{k,n} \equiv M$, and theorem 18 gives $\gamma > 0$.

Wright [5] discussed an equation of the type (7. 9), viz. $c_{k,n} = (n-1)^{-1} e^{-(k-1)\alpha}$ $(\alpha > 0)$. He proved that $\{ f(n) \}^{-1/n}$ tends very slowly to infinity, and more precisely, that $-n^{-1} \log f(n)$ is of the order of $j(t)$, where $j(t)$ is defined by

$$j(t) = 0 \quad (1 \leqslant t < e), \; j(t) = j(\log t) + 1 \quad (t \geqslant e).$$

In fact his equation just escapes our theorem 24, since $\varphi(t)$ is of the order of t, and $\int_1^\infty t^{-1}\, dt = \infty$.

Cooper [2] considers, among others, the formula

$$n^r f(n) = \sum_{1}^{n-1} k^{-\alpha} f(k) f(n-k) \qquad (r > 0, \; \alpha > 0).$$

He showed that $\{ f(n) \}^{1/n}$ oscillates between finite positive limits. From our theorem 24 we immediately deduce its convergence.

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