論培爾曼的幾個問題和羅曼諾夫的一個定理

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命

$$\sigma_s(n) = \sum_{d/n} d^s.$$

本文證明: 當 $f(x)$ 爲整係數多項式時

$$\sum_{n=1}^{x} \sigma_{-1}(f(a^n)) = Ax + o(x)$$

式中 A 爲常數, a 爲一固定的正數, 又當 s 爲充分小的正數時

$$\frac{1}{x} \sum_{n=1}^{x} \sigma_{-s}(a^n \pm 1) \longrightarrow \infty.$$

這解決了培爾曼所提出問題的一部份. 本文並證明形如 $p+f(a^n)$ 的整數的密度是正的. 這包括着羅曼諾夫的一個定理.

ON SOME PROBLEMS OF BELLMAN AND A THEOREM OF ROMANOFF

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Denote

$$\sigma_s(n) = \sum_{d/n} d^s$$

Bellman[1] proved that if $f(n)$ is any polynomial with integer coefficients and $s > 0$ then (c_1 depends on f)

1. Bellman, Duke Math. Journal 17 (1950), 159-168.

$$\sum_{n=1}^{x} \sigma_{-s}(f(n)) = c_1 x + o(x).$$

I proved[2] that if $f(n)$ is irreducible then ($\sigma_0(n) = d(n)$ = number of divisors of n)

$$c_2 x \log x < \sum_{n=1}^{x} d(f(n)) < c_3 x \log x.$$

Bellman[1] also raised the problem of investigating sums of the form

$$\sum_{n=1}^{x} \sigma_{-s}(a^n + 1) \quad \text{and} \quad \sum_{n=1}^{x} d(a^n + 1).$$

In the present paper we prove that

$$\sum_{n=1}^{x} \sigma_{-1}(f(a^n)) = c_3 x + o(x). \tag{1}$$

and that for s small enough

$$\frac{1}{x} \sum_{n=1}^{x} \sigma_{-s}(a^n - 1) \longrightarrow \infty \tag{2}$$

By a slightly more complicated argument we could also prove that

$$\frac{1}{x} \sum_{n=1}^{x} \sigma_{-s}(a^n + 1) \longrightarrow \infty. \tag{3}$$

We suppress the proof of (3). It seems likely that for any $s < 1$ and any polynomial $f(x)$

$$\frac{1}{x} \sum_{n=1}^{x} \sigma_{-s}[f(a^n)] \longrightarrow \infty.$$

Romanoff[3] proved that the density of integers of the form $p + a^n$ is positive. In this note we outline a proof of the result that the density of

2. London Math. Soc. Journal (1951).
3. Math. Annalen (1934).

integers of the form $p+f(a^n)$ is positive. One of his main lemmas was that the series

$$\sum_{k=1}^{\infty} \frac{1}{k\, l_a(k)} \tag{4}$$

converges, where $l_a(K)$ denotes the exponent of a (mod K) i.e. the smallest integer t so that $a^t \equiv 1$ (mod K). Romanoff's original proof was complicated. Later Turan and I[4] found a much simpler proof. In the present paper I give a perhaps still simpler proof and also prove several generalisations.

THEOREM 1. *Let $b_1 < b_2 < \cdots$ be a sequence of integers satisfying*

$$\sum_{k=1}^{\infty} \frac{\log\log b_k}{k^2} < \infty.$$

Denote by $l(d)$ the smallest index i so that $b_i \equiv 0$ (mod d). If no b is a multiple of d then $l(d) = \infty$

Then $\sum_{d=1}^{\infty} \frac{1}{d\, l(d)}$ converges. In fact

$$\sum_{d=1}^{\infty} \frac{1}{d\, l(d)} < c_4 \sum_{k=1}^{\infty} \frac{\log\log b_k}{k^2} + c_5. \tag{5}$$

Define

$$\varepsilon_r = \sum_{\substack{d/b_r \\ d \nmid b_i,\, 1 \le i < r}} \frac{1}{d}, \quad t_k = \sum_{r=1}^{k} \varepsilon_r. \tag{6}$$

By a well known result

$$\sigma_{-1}(y) = \sum_{d/y} \frac{1}{d} < c_6 \log\log y. \tag{7}$$

Thus

$$t_k < \sigma_{-1}(b_1 b_2 \cdots b_k) < c_6 (\log\log b_k^k) = c_6 (\log\log b_k + \log k). \tag{8}$$

4. Bull. de l'Inst. Math. et Mec. a l'Univ Tomask (1956) p. 101-105.

Thus by changing the order of summation by partial summation[5] and by (8)

$$\sum_{d=1}^{\infty} \frac{1}{d\,l(d)} = \sum_{\tau=1}^{\infty} \frac{\varepsilon_\tau}{\tau} = \sum_{k=1}^{\infty} \frac{t_k}{k(k+1)} < c_6 \sum_{k=1}^{\infty} \frac{\log\log b_k}{k^2} + c_6 \sum_{k=1}^{\infty} \frac{\log k}{k^2}, \tag{9}$$

which proves Theorem 1.

The convergence of (4) follows from Theorem 1 by putting $b_k = a^k - 1$. From (5) we obtain further that, for $a > 2$.

$$\sum_{k=1}^{\infty} \frac{1}{k\,l_a(k)} < c_7 \log\log a. \tag{10}$$

(10) is a sharpening of a result of Landau[6] and was previously proved by Turan and myself[7] by a different method. It is not hard in fact to deduce from (9) a further sharpening of (10)

$$\max_{1 \leq a \leq x} \sum_{d/a} \frac{1}{d} < \max_{2 \leq a \leq x+1} \sum_{k=1}^{\infty} \frac{1}{k\,l_a(k)} < \max_{1 \leq a \leq x} \sum_{d/a} \frac{1}{d} + c_8. \tag{11}$$

The first inequality of (11) is trivial. The second follows easily from (9) and the well known inequality

$$\max_{1 \leq a \leq x^k} \sum_{d/a} \frac{1}{d} - \max_{1 \leq a \leq x} \sum_{d/a} \frac{1}{d} < c_9 \log k.$$

It is possible that the right side of (11) can be replaced by $\max\limits_{1 \leq a \leq x} \sum\limits_{d/a} \frac{1}{d} + o(1)$, but this I can not prove.

Theorem 1 is the best possible in the following sense: if $b_1 < b_2 < \cdots$

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5. The partial summation is permitted here $\sum_{k=1}^{\infty} \frac{\log\log b_k}{k^2} < \infty$ clearly implies $\liminf \frac{\log\log b_k}{k} = 0$ (in fact it implies $\lim \frac{\log\log b_k}{k} = 0$. Originally in Theorem 1. I had the extra condition $\lim \frac{\log\log b_k}{k} = 0$. The fact that this condition is unnecessary was pointed out to me by de Bruijn.
6. Acta Arithmetica Vol. 1.
7. 4. *ibid* (1935), 144-147.

fails to satisfy the relation $\sum_{k=1}^{\infty} \frac{\log\log b_k}{k} < \infty$, there exists a sequence $B_1 < B_2 < \cdots$ for which $B_k \leqq b_k$ and $\sum_{d=1}^{\infty} \frac{1}{d\, l(d)} = \infty$. To see this put

$$B_k = \prod_{p < \frac{\log b_k}{2}} p$$

From the well known result $\prod_{p \leqq y} p < 4^y$ it follows that $B_k < b_k$. ε_r and t_k have the same meaning as in (6) with B_k replacing b_k. We evidently have.

$$t_k = \prod_{p \leqq \frac{\log b_k}{2}} \left(1 + \frac{1}{p}\right) > c_{10} \operatorname{loglog} b_k. \tag{12}$$

If $\sum_{k=1}^{\infty} \frac{\log\log b_k}{k^2} = \infty$ we obtain by partial summation, and (12)

$$\sum_{d=1}^{\infty} \frac{1}{d\, l(d)} = \sum_{r=1}^{\infty} \frac{\varepsilon_r}{r} \geqq \sum_{k=1}^{\infty} \frac{t_k}{k(k+1)} > c_{10} \sum_{k=1}^{\infty} \frac{\operatorname{loglog} b_k}{k^2} = \infty \qquad q.e.d.$$

Put $b_k = a^k - 1$. Several problems can be raised about the order of magnitude of ε_r. It seems likely that limsup $r\varepsilon_r = \infty$ but that ε_r tends to o as r tends to infinity fairly fast (possibly almost as fast as $1/r$).

I can prove that $\sum_{k=1}^{\infty} \frac{1}{k\, l_a(k)}$ has a distribution function, in other words: For every $c > 0$ the density of integers a for which $\sum_{k=1}^{\infty} \frac{1}{k\, l_a(k)} > C$ exists and tends to 0 as $C \to \infty$ and tends to 1 as $C \to 0$. The proof is not easy and we do not give it here.

THEOREM 2. *Let $b_1 < b_2 < \cdots$ be a sequence of integers satisfying*

$$\operatorname{loglog} b_k < c_{11} \log k, \qquad k = 1, 2, \cdots \tag{13}$$

Let $f(d)$ be any increasing function for which $\sum_{d=1}^{\infty} \frac{1}{d\, f(d)}$ converges. Then $\sum_{d=1}^{\infty} \frac{1}{d\, f(l(d))}$ also converges. In fact

(14) $$\sum_{d=1}^{\infty} \frac{1}{d f(l(d))} < c_{12} \sum_{d=1}^{\infty} \frac{1}{d f(d)}.$$

Theorem 2 clearly applies for $b_k = a^k - 1$. Thus Theorem 2 implies that $\sum_{k=1}^{\infty} \frac{1}{k(\log [l_a(k)])^{1+\varepsilon}}$ converges.

(13) and (7) implies that

(15) $$t_\tau \leqq \sum_{d/b_1 \cdots b_\tau} \frac{1}{d} < c_{13} \log \tau.$$

Thus by changing the order of summation, by partial summation and by (15) we have

(16) $$\sum_{d=1}^{\infty} \frac{1}{d f(l(d))} = \sum_{\tau=1}^{\infty} \frac{\varepsilon_\tau}{f(\tau)} = \sum_{\tau=1} \frac{t_\tau}{f(\tau)} \frac{f(\tau+1) - f(\tau)}{f(\tau+1)} <$$
$$c_{13} \sum_{\tau=1}^{\infty} \frac{\log \tau}{f(\tau)} \frac{f(\tau+1) - f(\tau)}{f(\tau+1)}.$$

the partial summation can be used only if $\lim t_\tau / f(\tau) = 0$. But this is satisfied, since by (15) $t_\tau < c_{13} \log \tau$ and the convergence of $\sum_{d=1}^{\infty} \frac{1}{d f(d)}$, $f(d)$ increasing implies $\log \tau / f(\tau) \to 0$. This last statement is well known and can be seen as follows: The convergence of $\sum_{d=1}^{\infty} \frac{1}{d f(d)}$ implies that $\sum_{\tau^{\frac{1}{2}}}^{\tau} \frac{1}{d f(d)}$ tends to 0 as τ tends to ∞. But

$$\sum_{\tau^{\frac{1}{2}}}^{\tau} \frac{1}{d f(d)} \geqq \frac{1}{f(\tau)} \sum_{\tau^{\frac{1}{2}}}^{\tau} \frac{1}{d} > \frac{1}{3} \frac{\log \tau}{f(\tau)} \longrightarrow 0 \qquad q. e. d.$$

Now we prove the following.

LEMMA 1. *Assume that $f(d)$ is increasing and that $\sum_{d=1}^{\infty} \frac{1}{d f(d)}$ converges. Then*

$$\sum = \sum_{\tau=1}^{\infty} \frac{\log \tau}{f(\tau)} \frac{f(\tau+1) - f(\tau)}{f(\tau+1)} < 16 \sum_{\tau=1}^{\infty} \frac{1}{\tau f(\tau)}.$$

(16) and Lemma 1 clearly implies Theorem 2. To prove the lemma put

$$\sum_k = \sum_{e^{2^k}}^{e^{2^{k+1}}-1} \frac{\log \tau}{f(\tau)} \frac{f(\tau+1)-f(\tau)}{f(\tau+1)}.$$

First we estimate Σ_k. Put $\tau_1 = e^{2^k}$ and denote by τ_i the smallest τ for which $f(\tau_i) > 2f(\tau_{i-1})$. Let $j-1$ be the greatest index for which

$$\tau_{j-1} < e^{2^{k+1}}.$$

Clearly $j > 1$ but j can be 2. Put $\tau_j = e^{2^{k+1}}$. Then clearly

$$\sum_k = \sum_{i=1}^{j-1} \sum_{\tau_i}^{\tau_{i+1}-1} \frac{\log \tau}{f(\tau)} \frac{f(\tau+1)-f(\tau)}{f(\tau+1)}. \tag{17}$$

Now

$$\sum_{\tau_i}^{\tau_{i+1}-1} \frac{\log \tau}{f(\tau)} \frac{f(\tau+1)-f(\tau)}{f(\tau+1)} \leqq \frac{\log \tau_{i+1}}{f(\tau_i)} \sum_{\tau_i}^{\tau_{i+1}-1} \frac{f(\tau+1)-f(\tau)}{f(\tau+1)} \tag{18}$$

$$< 2\frac{\log \tau_{i+1}}{f(\tau_i)} \leqq \frac{2^{k+2}}{f(\tau_i)}.$$

Thus from (17) and (18)

$$\sum_k < 2^{k+2} \sum_{i=1}^{j-1} \frac{1}{f(\tau_i)} < \frac{2^{k+3}}{f(e^{2^k})}. \tag{19}$$

Now

$$\sum_{\tau=1}^{\infty} \frac{1}{\tau f(\tau)} \geqq \sum_{\tau=0}^{\infty} \sum_{e^{2^\tau}}^{e^{2^{\tau+1}}-1} \frac{1}{\nu f(\nu)} > \sum_{\tau=0}^{\infty} \frac{1}{f(e^{2^\tau})} \sum_{e^{2^\tau}}^{e^{2^{\tau+1}}-1} \frac{1}{\nu} > \frac{1}{2} \sum_{\tau=0}^{\infty} \frac{2^\tau}{f(e^{2^\tau})}. \tag{20}$$

Hence from (19) and (20)

$$\sum = \sum_{k=0}^{\infty} \left(\sum_k\right) < \sum_{\tau=0}^{\infty} \frac{2^{\tau+3}}{f(e^{2^\tau})} < 16 \sum_{\tau=1}^{\infty} \frac{1}{\tau f(\tau)},$$

which proves our lemma. Thus the proof of Theorem 2 is complete.

THEOREM 3. *Let $f(x)$ be a polynomial with integer coefficients. We have*

$$\sum_{k=1}^{x} \sigma_{-1}(f(a^k)) = Ax + o(x).$$

Without loss of generality we can assume that the coefficients of $f(x)$ have no common factor. For simplicity we further assume that the constant term of $f(x)$ is relatively prime to a. It will be clear from the proof that it would be easy to omit these assumptions.

Denote by $g_x(d)$ the number of solutions of the congruence

$$f(a^k) \equiv 0 \pmod{d},\ 0 < k \leqq x.$$

By interchanging the order of summation we have

$$\sum_{k=1}^{x} \sigma_{-1}(f(a^k)) = \sum_{d=1}^{\infty} \frac{g_x(d)}{d} = \sum\nolimits_1 + \sum\nolimits_2 \tag{21}$$

where in Σ_1, $l_a(d) \leqq x$, and in Σ_2, $l_a(d) > x$, ($l_a(d)$ is the exponent of $a \pmod{d}$).

Denote by $v(d)$ the number of distinct prime factors of d. A well known theorem of Nagell[8] states that the number of solutions of

$$f(r) \equiv 0 \pmod{d},\ 0 < k \leqq d$$

is less than $s^{v(d)}$ where s is a constant depending only on the polynomial $f(x)$. Therefore the number of solutions of

$$f(a^k) \equiv 0 \pmod{d},\ 0 < k \leqq l_a(d) \tag{22}$$

is at most $s^{v(d)}$ (the numbers a^i, $0 < i \leqq l_a(d)$ are all incongruent (mod (d)). Therefore for the d in Σ_2 $g_x(d) < s^{v(d)}$. Thus

8. Journal de Math. Second series, Vol. 4 (1921). See also L.K.Hua, Journal of the London Math. Soc. (1938).

$$(23)\qquad \sum_2 \leqq \sum_2 \frac{s^{\nu(d)}}{d} < \sum_{d \mid \prod_{k=1}^{x} f(a^k)} \frac{s^{\nu(d)}}{d} < \prod_{p \mid \prod_{k=1}^{x} f(a^k)} \left(1 + \frac{s}{p} + \frac{s}{p^2} + \cdots\right)$$

$$= \prod_{d \mid \prod_{k=1}^{x} f(a^k)} \left(1 + \frac{s}{p-1}\right) < \exp\left(\sum_{p \mid \prod_{k=1}^{x} f(a^k)} \frac{2s}{p}\right) < (\log x)^{c_{14}} = o(x).$$

The last step of (23) is based on the well known inequality

$$(24)\qquad \sum_{p \mid y} \frac{1}{p} < c_{15} \log\log\log y.$$

Thus it is enough to consider $\sum_1$. The sequence a^k is periodic modd (its period is $l_a(d)$). Thus for fixed d $\lim g_x(d)/x$ exists.

Further by (22) for the d in $\sum_1$ (i.e. $l_a(d) \leqq x$).

$$(25)\qquad g_x(d) < x \frac{s^{\nu(d)}}{l_a(d)} + s^{\nu(d)} \leqq 2x \frac{s^{\nu(d)}}{l_a(d)}$$

Thus in view of (24) and the existence of $\lim \frac{g_x(d)}{x}$ we obtain

$$\sum_1 = Ax + o(x),\ A = \sum_{d=1}^{\infty} \frac{n_d}{d},\ \text{ where }\ n_d = \lim_{x\to\infty} \frac{g_x(d)}{x},$$

if we can prove that

$$(26)\qquad \sum_{d=1}^{\infty} \frac{s^{\nu(d)}}{d\, l_a(d)} < \infty.$$

Instead of (26) we prove the following more general

LEMMA 2. *Let $b_1 < b_2 < \cdots$ satisfy for every $\varepsilon > 0$ $\log\log b_k = o(k^\varepsilon)$. Then for every s*

$$\sum_{d=1}^{\infty} \frac{s^{\nu(d)}}{d\, l_a(d)} < \infty.$$

The proof of Lemma 2 is almost identical with that of Theorem 1. We only outline the proof. Put

$$\varepsilon_r^{(s)} = \sum_{\substack{d \mid b_r \\ d \nmid b_l,\ 1 \leqq l < r}} \frac{s^{\nu(d)}}{d}, \quad t_k^{(s)} = \sum_{r=1}^{k} \varepsilon_r^{(s)}.$$

By (24) we easily obtain $t_k^{(s)} < c_{17} (\log \log b_k)^{c_{18}} = o(k^{\varepsilon})$. Thus

$$\sum_{d=1}^{\infty} \frac{s^{\nu(d)}}{d\, l(d)} = \sum_{r=1}^{\infty} \frac{\varepsilon_r^{(s)}}{r} = \sum_{k=1}^{\infty} \frac{t_k^{(s)}}{k(k+1)} < \infty.$$

Thus the proof of Theorem 3 is complete.

THEOREM 4. *The density of the integers of the form $p + f(a^k)$ is positive*[9].

We are only going to indicate the proof, since it follows very closely the ideas of Romanoff, except that a result like Theorem 3 is needed.

We want to estimate the number of distinct integers $H(x)$ not exceeding X of the form $p + f(a^k)$. Let $k < c_{17} \log X$ where c_{17} is a sufficiently small positive constant. Then clearly $f(a^k) < x/2$. Denote now by $h(x, K)$ the number of integers of the form

$$p + f(a^k), \quad p < \frac{x}{2}$$

which are not of the form $p + f(a^l), l < k, p < x/2$. It follows from the results of Schnirelmann[10] that the number of solutions of

$$p + f(a^k) = p + f(a^l), \quad p < \frac{x}{2}$$

is less than

$$c_{18} \frac{x}{(\log x)^2} \prod_{p \mid (f(a^k) - f(a^l))} \left(1 + \frac{1}{p}\right) \tag{27}$$

9. This theorem was suggested to me in a letter of Shapiro.

10. See e. g. Landau, Neuere Ergebnisse der Additiven Zahlentheorie.

Thus from (27) we have

$$(28)\qquad h(x,k) > \pi\left(\frac{x}{2}\right) - \frac{x}{(\log x)^2}\sum_{l=1}^{k-1}\prod_{p\mid f(a^k)-f(a^l)}\left(1+\frac{1}{p}\right) >$$

$$\frac{x}{4\log x} - \frac{x}{(\log x)^2}\sum_{l=1}^{k-1}\prod_{p\mid(f(a^k)-f(a^l))}\left(1+\frac{1}{p}\right),$$

since $\pi(x/2) > x/4\log x$. Now we prove the following

LEMMA 5.

$$\sum_{l=1}^{k-1}\prod_{p\mid(f(a^k)-f(a^l))} < c_{19}\,k.$$

Assume that the lemma is already proved. Then we have from $k < c_{17}\log x$ our lemma and (28)

$$(29)\qquad h(x,k) > \frac{x}{4\log x} - c_{19}\frac{kx}{(\log)^2} > \frac{x}{8\log x}$$

if c_{17} is sufficiently small.

From (29) we have

$$H(x) \geqq \sum_{k<c_{17}\log x} h(x,k) > c_{17}\frac{x}{10},$$

which proves Theorem 4.

Thus we only have to prove the lemma. But we can suppress the proof of the lemma since it is identical with that of Theorem 5.

In a recent paper[11] I proved the following theorem:

Let $a_1 < a_2 < \cdots$, $a_k \mid a_{k+1}$ be an infinite sequence of integers. The necessary and sufficient condition that $p+a_i$ should have positive density is that the following two conditions should hold

$$(30)\qquad a_k < c_{20}^k, \quad \sum_{d\mid a_k}\frac{1}{d} < c_{20}.$$

11. Summa Brasiliensis Math. (1951).

By similar methods as used in Theorem 5 and in the above paper I can prove that if (50) holds then $p+f(a_k)$ has positive density.

THEOREM 5. *Let $s > 0$ be sufficiently small. Then*

$$\lim_{x=\infty} \frac{1}{x} \sum_{k=1}^{x} \sigma_{-s}(a^k-1) = \infty.$$

We have by interchanging the order of summation

$$\sum_{k=1}^{x} \sigma_{-s}(a^k-1) \geqq \sum_{d=1}^{x} \left[\frac{x}{l_a(d)}\right] \frac{1}{d^s}.$$

Thus to prove Theorem 4 it will suffice to show that for s small enough

$$\sum_{d=1}^{\infty} \frac{1}{d^s\, l_a(d)} = \infty \tag{51}$$

(51) will be an immediate consequence of the following

LEMMA 4. *There exists a constant c_{21} so that for every ε and sufficiently large X the number of integers $d \leqq x$ satisfying $l_a(d) < d^\varepsilon$ is greater than $X^{c_{21}}$!*

Assume that the lemma is already proved. Then a simple argument shows that (51) diverges for every $s < c_{21}$. Thus we only have to prove the lemma. We need two further lemmas. Let X be sufficiently large.

LEMMA 5. *The number of squarefree integers not exceeding x composed of $c_{22}(\log x)^{1+c_{23}}/\log\log x$ arbitrarily given primes not exceeding $(\log x)^{1+c_{23}}$ is greater than $x^{c_{21}}$ where c_{21} depends only on c_{23}.*

This is lemma 3 of my paper "On the normal number of prime factors etc" Quarterly Journal of Math. Vol 6. (1935) p. 212.

LEMMA 6. *Let c_{23} be sufficiently small. Then the number of primes $p < (\log x)^{1+c_{23}}$ for which all prime factors of $p-1$ are less than $(\log x)^{1-c_{23}}$ is greater than $c_{22}(\log x)^{1+c_{23}}$.*

This is lemma 4 of the above paper (p. 212). (In the lemma replace $\log x$ by $(\log x)^{1-c_{23}}$ and $1+\varrho$ by $1+c_{23}/(1-c_{23})$.

Denote now by $p_1, p_2, \cdots$ the primes of Lemma 6 and by $d_1 < d_2 < \cdots < d_r \leqq x$ the squarefree integers not exceeding x composed of the p' s. By lemma 5 $r > x^{c_{23}}$ Further if k is squarefree $l_a(k)$ is clearly not greater than the least common multiple of all $p_i - 1, p_i / k$. Thus finally since all the $p_i - 1$ with p_i / d have all their prime factors not exceeding $(\log x)^{1-c_{23}}$ and each $p_i \mid d$ is less than $(\log x)^{1+c_{23}}$, we have

$$l_a(d) < [(\log x)^{1+c_{23}}]^{\pi((\log x)^{1-c_{23}})} < (\log x)^{(1+\epsilon)2\pi(\log x)^{1-c_{23}}} = o(x^\epsilon).$$

This together with $r > x^{c_{23}}$ proves Lemma 4, and thus the proof of Theorem 4 is complete.

It seems doubtful whether $\sum_{n=1}^{x} d(a^n \pm 1)$ has a satisfactory asymptotic expression. A theorem of Bang states that except when $a=2, n=6$ there always exists a prime $p \mid a^n - 1, p \nmid a^m - 1, 1 \leqq m < n$. Thus $V(a^n - 1) \geqq 2^{v(n)} - 2$ ($v(y)$ denotes the number of distinct prime factors of y). Thus

$$d(a^n - 1) \geqq \frac{1}{4} 2^{2^{v(n)}}. \tag{32}$$

Now it easily follows from the prime number theorem that

$$\max_{1 \leqq n \leqq x} v(n) > (1-\varepsilon)^{\log x / \log\log x} \tag{33}$$

Thus from (32) and (33)

$$\sum_{n=1}^{x} d(a^n - 1) > 2^{2^{(1-\varepsilon)\log x/\log\log x}} > x^A \tag{34}$$

for every A if x is sufficiently large. (34) can be shown in the same way for $\sum_{n=1}^{x} d(a^n + 1)$.

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