SUPPLEMENTARY NOTE

BY

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Theorem 2 of the above paper runs as follows: Let

$$a_k \geqslant 0, \sum_{k=1}^{n} a_k(s_{n-k}+k) = n^2+O(n) \; (s_m = \sum_{k=1}^{m} a_k). \qquad (1)$$

Then
$$s_n = n+O(1). \qquad (2)$$

I dealt with this result in a lecture at the University of Illinois this summer and several remarks were made by the audience which I propose to discuss here.

Reiner asked whether anything more can be deduced if in (1) we assume that the error term is $o(n)$. If we put $a_1 = 3/2$, $a_{2k+1} = 2$ for $k > 1$, $a_{2k} = 0$, then $\sum_{k=1}^{n} a_k(s_{n-k}+k) = n^2+o(1)$, but $s_n \neq n+o(1)$. On the other hand if we assume that there exists an $\epsilon > 0$ so that for $k > k_0$, $a_k < 2-\epsilon$, then indeed $\sum_{k=1}^{n} a_k(s_{n-k}+k) = n^2+o(n)$ implies $s_n = n+o(1)$. We do not give the proof since it follows that of the original theorem closely.

Hua raised the following questions: What can be deduced if we assume that $a_k \geqslant 0$ and $\sum_{k=1}^{n} ka_k = \frac{1}{2}n^2+O(n)$, also $a_k \geqslant 0$, and $\sum_{k=1}^{n} a_k(s_{n-k}+k) = \frac{1}{2}n^2+O(n)$?

Here I prove

THEOREM 1. *Let* $a_k \geqslant 0$ *and* $\sum_{k=1}^{n} k.a_k = \frac{1}{2}n^2+O(n)$, *then*

$$s_n = n+O(\log n). \qquad (3)$$

and (3) *is best possible.*

To prove (3) put $s_n = n + A_n$. Denote $\max_{m \leqslant n} |A_n| = \bar{A}_n$. We can assume that $\bar{A}_n \to \infty$ (for otherwise (3) holds and there is nothing to prove). Since $\bar{A}_n \to \infty$ we can choose arbitrarily large values of n so that $\bar{A}_n = |A_n|$, and in fact it will be clear from the proof that without loss of generality we can assume $\bar{A}_n = A$. We have

$$\sum_{k=1}^{n} k\,a_k = n\,s_n - \sum_{k=1}^{n-1} s_k = n\,(n+\bar{A}_n)$$
$$- \sum_{k=1}^{n-1} (k+A_k) \geqslant \tfrac{1}{2} n^2 + O(n) + \frac{n}{2}(\bar{A}_n - \bar{A}_{n/2}) \quad (4)$$

(if $n/2 < k \leqslant n$ we replace A_k by $\bar{A}_n$, if $k \leqslant n/2$ we replace A_k by $\bar{A}_{n/2}$). If (3) does not hold then clearly $\overline{\lim}\ \bar{A}_{n/\log n} = \infty$, or for every C there exist infinitely many n so that $\bar{A}_n - \bar{A}_{n/2} > C$. But then from (4)

$$\sum_{k=1}^{n} k.a_k > \tfrac{1}{2} n^2 + \frac{C}{2} n + O(n),$$

which contradicts the assumptions of Theorem 1 (since C can be chosen arbitrarily large), which proves (3).

The fact that (3) cannot be improved is immediately clear by putting $a_k = 1 + 1/k$.

THEOREM 2. *Let* $a_k \geqslant 0$, $\sum_{k=1}^{n} a_k s_{n-k} = \frac{1}{2} n^2 + O(n)$. *Then*

$$s_n = n + o(n). \quad (5)$$

The error term cannot be $o(n^{1/2})$.

To prove this it suffices to assume that $a_k \geqslant 0$ and $\sum_{k=1}^{n} a_k\, s_{n-k} = \frac{1}{2} n^2 + o(n^2)$. Put $F(x) = \sum_{k=1}^{\infty} a_k x^k$, $F(x)^2 = \sum_{k=1}^{\infty} b_k x^k$. Clearly

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} a_k\, s_{n-k} = \tfrac{1}{2}n^2 + o(n^2).$$

Thus

$$\lim_{x=1}\,(1-x)^2 F(x)^2 = 1 \text{ or } \lim_{x=1}\,(1-x)\,F(x) = 1.$$

Hence by the well-known Tauberian theorem of Hardy and Littlewood $s_n = n + o(n)$.

By putting $a\,(n\,!)^2 = n\,!$, $a_m = 0$ if $(n\,!)^2 < m \leqslant (n\,!)^2 + n\,!$, $a_m = 1$ otherwise, we immediately obtain that the error term in (5) cannot be $o(n^{1/2})$.

Let $f(x)$ be an increasing function satisfying $f(x) \leqslant x$, $f'(x) \leqslant 1$. $f^{-1}(x)$ is defined by $f[f^{-1}(x)] = x$. Then we have

THEOREM 3. *Let* $a_k \geqslant 0$ *and*

$$S_n = \sum_{k=1}^{n} a_k\,[s_{f^{-1}[f(n)-f(k)]} + f(k)] = f(n)^2 + O(f(n)). \quad (6)$$

Then

$$s_n = f(n) + O(1). \quad (7)$$

REMARK: If $f(x) = x$ we obtain our original theorem that (1) implies (2), also $f(x) = x^\alpha$, $0 < \alpha \leqslant 1$, $f(x) = \log x$ satisfy the conditions of Theorem 3.

PROOF OF THEOREM 3. Denote $[f(n)] = N_s$, (i.e. $f^{-1}(N) = n + \delta$, $|\delta| < 1$) $\sum_{r<f(n)<r+1} a_k = A_r$. We have from (6)

$$S_{f^{-1}(N+1)} - S_{f^{-1}(N)} = O(N) \geqslant N A_N \text{ or } A_N < c.$$

Thus from (6) by a simple computation, we have

$$\sum_{r=1}^{N} A_r(A_1 + \ldots + A_{N-k_r} + k_r) = N^2 + O(N)$$

which by our theorem clearly implies (7).