

P. ERDÖS and P. TURÁN: *On a problem in the theory of uniform distribution. II.*

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9. After this first reduction of our problem we transform it in the following way. Let for the polynomial $\psi(z)$ defined in (8.4)

$$M_\vartheta^* = \max_{|z|=\vartheta} |\psi(z)| = \max_{|z|=\vartheta} |\psi_1(z)| = \max_{|z|=\vartheta} \left| \prod_{v=1}^n (1 - ze^{-i\varphi_v}) \right|. \quad (9.1)$$

Then

$$\max_{|z|=\vartheta} \log |\psi_1(z)| = \log M_\vartheta^*. \quad (9.2)$$

Since $\log \psi_1(z)$ is regular for $|z| = \vartheta$ we have by the classical theorem of HADAMARD-BOREL-CARATHEODORY that for $|z| \leq \frac{\vartheta}{2}$, since $\log \psi_1(0) = 0$,

$$|\log \psi_1(z)| \leq 2 \log M_\vartheta^*$$

i.e.

$$\left| \sum_{v=1}^n \log (1 - ze^{-i\varphi_v}) \right| \leq 2 \log M_\vartheta^*$$

or denoting $\sum_{v=1}^n e^{-ki\varphi_v}$ by s_k ,

$$\left| \sum_{k=1}^{\infty} \frac{z^k}{k} s_{-k} \right| \leq 2 \log M_\vartheta^*, \quad |z| \leq \frac{\vartheta}{2}. \quad (9.3)$$

Then CAUCHY's estimation gives

$$\left. \begin{aligned} |s_k| = |s_{-k}| &\leq k \frac{2 \log M_\vartheta^*}{\left(\frac{\vartheta}{2}\right)^k} = \\ &= 2k \left(\frac{2}{\vartheta}\right)^k \log M_\vartheta^* < \left(\frac{4}{\vartheta}\right)^k \log M_\vartheta^* = \left(\frac{4}{\vartheta}\right)^k \frac{n}{g^*(n, \vartheta)}. \end{aligned} \right\} \quad (9.4)$$

Owing to the trivial inequality $|s_k| \leq n$ the inequality (9.4) is restrictive only for those k 's for which

$$k \leq \frac{\log g^*(n, \vartheta)}{\log \frac{4}{\vartheta}}. \quad (9.5)$$

Hence the proof of Theorem II is reduced to the question whether or not the inequalities (9.4) with the restriction (9.5) involve equidistribution

of the φ_v 's mod 2π . In other words we reduced the proof of Theorem II to the proof of Theorem III with

$$\psi(k) = \frac{n}{g^*(n, \vartheta)} \left(\frac{4}{\vartheta}\right)^k, \quad m = \left\lceil \frac{\log g^*(n, \vartheta)}{2 \log \frac{4}{\vartheta}} \right\rceil. \quad \dots \quad (9.6)$$

Since

$$\begin{aligned} \sum_{r=1}^m \frac{\psi(k)}{k} &< \frac{n}{g^*(n, \vartheta)} \left(\frac{4}{\vartheta}\right)^{\frac{m}{2}} \left(1 + \log \frac{m}{2}\right) + \frac{n}{g^*(n, \vartheta)} \left(\frac{4}{\vartheta}\right)^m \cdot \frac{2}{m} \cdot \frac{m}{2} < \\ &< \frac{n}{g^*(n, \vartheta)} \left(\frac{4}{\vartheta}\right)^m \left\{1 + 2 \left(\frac{\vartheta}{4}\right)^{\frac{m}{2}} \log \frac{m}{2}\right\} \equiv \\ &\equiv \frac{2n}{g^*(n, \vartheta)} \left(\frac{4}{\vartheta}\right)^m \equiv \frac{2n}{\sqrt{g^*(n, \vartheta)}} < \frac{3n}{\log g^*(n, \vartheta)} \end{aligned}$$

and

$$\frac{n}{m+1} < 2 \log \frac{4}{\vartheta} \cdot \frac{n}{\log g^*(n, \vartheta)}$$

we obtain — anticipating Theorem III — that for every $0 \leq \alpha < \beta \leq 2\pi$ we have

$$\left| \sum_{\alpha \leq \varphi_v \leq \beta \pmod{2\pi}} 1 - \frac{\beta - \alpha}{2\pi} n \right| < 5C \log \frac{4}{\vartheta} \cdot \frac{n}{\log g^*(n, \vartheta)}$$

i.e. Theorem II will be proved.

10. Before turning to the proof of Theorem III we sketch the corresponding reasoning for Theorem I. In this case — as we remarked in ¹⁾ — the general case can be reduced to the case when all the roots lie on the unit circle. Then in (9.2) M_ϑ^* is replaced by $\max_{|z|=1} |\psi(z)| = M$. Applying

the theorem of HADAMARD-BOREL-CARATHEODORY to the interior circle $|z| = \varrho$, where we determine ϱ suitable later, we obtain

$$|\log \psi_1(z)| \equiv \frac{2 \log M}{1 - \varrho}, \quad |z| \equiv \varrho$$

resp.

$$\left| \sum_{k=1}^{\infty} \frac{z^k}{k} s_{-k} \right| < \frac{2 \log M}{1 - \varrho} \quad \dots \quad (10.1)$$

CAUCHY's estimation gives

$$|s_k| = |s_{-k}| \equiv k \frac{2 \log M}{1 - \varrho} \cdot \frac{1}{\varrho^k}.$$

Choosing $\varrho = 1 - \frac{1}{k+1}$

$$\left. \begin{aligned} |s_k| &\leq 20 k^2 \log M \\ k &= 1, 2, \dots \end{aligned} \right\} \dots \dots \dots (10.2)$$

This gives according to the corollary of Theorem III an error term $O(n^{1/2})$ only. Hence Theorem I seems to be much deeper. This seems to justify the use of more difficult analytical tools in the proof of 1).

11. For the proof of theorem III we need some simple auxiliary considerations.

Let

$$R = \int_0^{2\pi} \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} t \right)^4 \frac{1}{\sin \frac{t}{2}} dt \dots \dots \dots (11.1)$$

Obviously

$$\left. \begin{aligned} R &> 4 \int_0^{2\pi} \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} t \right)^4 dt = 4 \left(\left[\frac{m}{2} \right] + 1 \right)^3 \int_0^{\pi \left(\left[\frac{m}{2} \right] + 1 \right)} \left(\frac{\sin y}{y} \right)^4 dy > \\ &> \frac{1}{2} \left(\frac{m}{2} \right)^3 \int_0^{\frac{\pi}{2}} \left(\frac{\sin y}{y} \right)^4 dy > c_1 m^3 \end{aligned} \right\} (11.2)$$

where c_1 and later c_2, \dots denote numerical constants. Further

$$\left. \begin{aligned} R &= 2 \int_0^{\pi} \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} t \right)^4 \frac{1}{\sin \frac{t}{2}} dt \leq 2\pi^4 \int_0^{\pi} \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} t \right)^4 dt < \\ &< \frac{\pi^4}{4} \left(\left[\frac{m}{2} \right] + 1 \right)^3 \int_0^{\infty} \left(\frac{\sin y}{y} \right)^4 dy < c_2 m^3. \end{aligned} \right\} (11.3)$$

12. Let a be a parameter subjected only to the restriction

$$\pi \leq a \leq \frac{10}{m+1} \dots \dots \dots (12.1)$$

and let

$$\pi_m(x, a) = \frac{1}{R} \int_{-x}^{a-x} \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{\sin \frac{t}{2}} t \right)^4 dt. \quad \dots \quad (12.2)$$

We have also

$$\pi_m(x, a) = \frac{1}{R} \int_0^a \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{\sin \frac{t-x}{2}} (t-x) \right)^4 dt \quad \dots \quad (12.3)$$

and since the integrand of (12.2) is an even function of t

$$\pi_m(x, a) = \frac{1}{R} \int_{x-a}^x \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{\sin \frac{t}{2}} t \right)^4 dt. \quad \dots \quad (12.4)$$

Since generally

$$\left(\frac{\sin \frac{k+1}{2} y}{\sin \frac{y}{2}} \right)^2 = (k+1) + k \cdot 2 \cos y + (k-1) 2 \cos 2y + \dots + 1 \cdot 2 \cos ky,$$

we obtain at once that the integrand of (12.2), i.e. $\pi_m(x, a)$ itself, is a trigonometric polynomial of order $\leq m$

$$\pi_m(x, a) = a_0(a) + \sum_{1 \leq \nu \leq m} (a_\nu(a) \cos \nu x + b_\nu(a) \sin \nu x). \quad \dots \quad (12.5)$$

13. We need some information about the coefficients in (12.5). Evidently; using (12.3),

$$\left. \begin{aligned} a_0(a) &= \frac{1}{2\pi} \int_0^{2\pi} \pi_m(x, a) dx = \frac{1}{2\pi R} \int_0^a dt \int_0^{2\pi} \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{\sin \frac{t-x}{2}} (t-x) \right)^4 dx = \\ &= \frac{1}{2\pi R} \int_0^a dt \cdot R = \frac{a}{2\pi}. \end{aligned} \right\} \quad (13.1)$$

Further using (12. 2)

$$a_\nu(a) = \frac{1}{\pi} \int_0^{2\pi} \pi_m(x, a) \cos \nu x \, dx = -\frac{1}{\pi \nu} \int_0^{2\pi} \frac{\partial \pi_m}{\partial x} \sin \nu x \, dx =$$

$$= -\frac{1}{\pi \nu R} \int_0^{2\pi} \sin \nu x \left\{ \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} (a-x) \right)^4 + \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} x \right)^4 \right\} dx$$

i.e.

$$|a_\nu(a)| \cong \frac{1}{\pi \nu R} \int_0^{2\pi} \left\{ \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} (a-x) \right)^4 + \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} x \right)^4 \right\} dx = \frac{2}{\pi \nu}, \quad 1 \cong \nu \cong m; \quad (13. 2)$$

similarly

$$|b_\nu(a)| \cong \frac{2}{\pi \nu} \left. \begin{matrix} \\ 1 \cong \nu \cong m. \end{matrix} \right\} \dots \dots \dots (13. 3)$$

14. We need also some information about the shape of the graph of $\pi_m(x, a)$. The definition of R and representation (12. 2) give immediately for every real x

$$0 \cong \pi_m(x, a) \cong 1. \dots \dots \dots (14. 1)$$

We consider $\pi_m(x, a)$ in the interval

$$\frac{2}{m+1} \cong x \cong a - \frac{2}{m+1} \dots \dots \dots (14. 2)$$

(this has a meaning owing to (12. 1)). Using the representation (12. 2) and the estimation (11. 3) we have in the range (14. 2)

$$\pi_m(x, a) > \frac{1}{c_2 m^3} \int_0^{\frac{1}{1 + \left[\frac{m}{2} \right]}} \left(\frac{\sin \left[\frac{m}{2} \right] + 1}{2} t \right)^4 dt =$$

$$= \frac{1}{c_2 m^3} \left(\left[\frac{m}{2} \right] + 1 \right)^3 \int_0^{\frac{1}{2}} \left(\frac{\sin y}{y} \right)^4 dy > c_3. \quad (14. 3)$$

Further, for $a < x \leq \frac{3}{2}\pi$ we have, using the estimation (11.2) and the representation (12.4)

$$\left. \begin{aligned} \pi_m(x, a) &< \frac{1}{c_1 m^3} \int_{x-a}^{\frac{3}{2}\pi} \left(\frac{\sin \frac{[\frac{m}{2}] + 1}{2} t}{\sin \frac{t}{2}} \right)^4 dt < \frac{\pi^4}{c_1 m^3} \int_{x-a}^{\frac{3}{2}\pi} \left(\frac{\sin \frac{[\frac{m}{2}] + 1}{2} t}{t} \right)^4 dt < \\ &< \frac{\pi^4}{c_1 m^3} \left(\frac{[\frac{m}{2}] + 1}{2} \right)^3 \int_{\frac{[\frac{m}{2}] + 1}{2}(x-a)}^{\infty} \left(\frac{\sin y}{y} \right)^4 dy < c_4 \int_{\frac{m}{4}(x-a)}^{\infty} \frac{dy}{y^4} < \frac{c_5}{(m(x-a))^3}. \end{aligned} \right\} (14.4)$$

Finally, for $\frac{3}{2}\pi \leq x \leq 2\pi$ we have from the estimation (11.2) and the representation (12.4), since $x - a \geq \frac{3}{2}\pi - \pi = \frac{\pi}{2}$

$$\left. \begin{aligned} \pi_m(x, a) &< \int_{\frac{\pi}{2}}^x \left(\frac{\sin \frac{[\frac{m}{2}] + 1}{2} t}{\sin \frac{t}{2}} \right)^4 dt < c_6 \int_{\frac{\pi}{2}}^x \left(\frac{\sin \frac{[\frac{m}{2}] + 1}{2} t}{2\pi - t} \right)^4 dt = \\ &= c_6 \int_{2\pi - x}^{\frac{3\pi}{2}} \left(\frac{\sin \frac{[\frac{m}{2}] + 1}{2} y}{y} \right)^4 dy < \frac{c_7}{(m(2\pi - x))^3} \end{aligned} \right\} (14.5)$$

15. Now we are going to prove the following

Lemma. Assuming (4.1) and (4.2) we have for the number N of the φ_v 's lying in an arbitrary interval of length $\frac{10}{m+1}$ with $m > 20$ the inequality

$$N < c_8 \left(\frac{n}{m+1} + \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu} \right) \dots \dots \dots (15.1)$$

Proof. Without loss of generality we may suppose that our interval is

$$\frac{2}{m+1} \equiv x \equiv \frac{12}{m+1} \dots \dots \dots (15.2)$$

We consider the polynomial $\pi_m(x, \gamma)$ where

$$\gamma = \frac{14}{m+1} \dots \dots \dots (15.3)$$

Replacing x in

$$\pi_m(x, \gamma) = a_0(\gamma) + \sum_{\nu=1}^m (a_\nu(\gamma) \cos \nu x + b_\nu(\gamma) \sin \nu x)$$

by $\varphi_1, \varphi_2, \dots, \varphi_n$ and summing we obtain owing to (13.1)

$$\sum_{l=1}^n \pi_m(\varphi_l, \gamma) = \frac{\gamma}{2\pi} n + \sum_{\nu=1}^m a_\nu(\gamma) \sum_{l=1}^n \cos \nu \varphi_l + \sum_{\nu=1}^m b_\nu(\gamma) \sum_{l=1}^n \sin \nu \varphi_l.$$

In the interval (15.2) condition (14.2) is satisfied i.e. from (14.3) and the non-negativity of $\pi_m(x, \gamma)$ we have

$$c_3 N \cong \sum_{l=1}^n \pi_m(\varphi_l, \gamma) \cong \frac{7}{\pi} \cdot \frac{n}{m+1} + \sum_{\nu=1}^m (|a_\nu(\gamma)| + |b_\nu(\gamma)|) \left| \sum_{l=1}^n e^{i\nu\varphi_l} \right|.$$

Applying (4.1), (13.2) and (13.3) we obtain further

$$c_3 N < \frac{7}{\pi} \cdot \frac{n}{m+1} + \frac{4}{\pi} \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu}. \quad \text{Q. e. d.}$$

16. Now we turn to the proof of theorem III. Let d be given satisfying

$$\frac{10}{m+1} \cong d \cong \pi \quad \dots \dots \dots (16.1)$$

and consider the polynomial $\pi_m(x, d)$. Replacing x in

$$\pi_m(x, d) = \frac{d}{2\pi} + \sum_{\nu=1}^m (a_\nu(d) \cos \nu x + b_\nu(d) \sin \nu x)$$

by $\varphi_1, \varphi_2, \dots, \varphi_n$ and summing we obtain

$$\sum_{j=1}^n \pi_m(\varphi_j, d) = \frac{d}{2\pi} n + \sum_{\nu=1}^m a_\nu(d) \sum_{j=1}^n \cos \nu \varphi_j + \sum_{\nu=1}^m b_\nu(d) \sum_{j=1}^n \sin \nu \varphi_j.$$

Arguing as before we obtain

$$\sum_{j=1}^n \pi_m(\varphi_j, d) > \frac{d}{2\pi} n - \frac{4}{\pi} \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu} \dots \dots \dots (16.2)$$

Now denoting the number of φ_ν 's in $0 \leq x \leq d$ by $N(d)$, the contribution of these φ_ν 's is, owing to $\pi_m(x, d) \leq 1$,

$$\cong N(d) \dots \dots \dots (16.3)$$

To obtain an upper estimation for the contribution of the other φ_ν 's we construct successive contiguous intervals of length $\frac{10}{m+1}$ each starting from $x = d$ and covering the interval $d \leq x \leq \frac{3}{2}\pi$. The contributions of the φ_ν in the interval

$$D_k : d + k \frac{10}{m+1} \cong x \cong d + (k+1) \frac{10}{m+1}$$

is owing to the lemma

$$< c_8 \left(\frac{n}{m+1} + \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu} \right) \max_{x \in D_k} \pi_m(x, d)$$

which is, by (14.4)

$$\begin{aligned} &< c_8 \left(\frac{n}{m+1} + \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu} \right) \max_{x \in D_k} \frac{c_5}{(m(x-d))^3} < \\ &< c_9 \left(\frac{n}{m+1} + \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu} \right) \cdot \frac{1}{k^3}. \end{aligned}$$

Hence summing over k

$$\sum_{d \leq \varphi_j \leq d+\pi} \pi_m(\varphi_j, d) < c_{10} \left(\frac{n}{m+1} + \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu} \right). \quad \dots \quad (16.4)$$

The contribution of the φ_ν lying in the remaining interval $\frac{3\pi}{2} < x \leq 2\pi$ we can estimate similarly. Combining (16.2), (16.3) and (16.4) we obtain

$$N(d) > \frac{d}{2\pi} \cdot n - c_{11} \left(\frac{n}{m+1} + \sum_{\nu=1}^m \frac{\psi(\nu)}{\nu} \right). \quad \dots \quad (16.5)$$

Obviously the same estimation holds for the number $N(c, c+d)$ of the φ_ν 's for which $c \leq \varphi_\nu \leq c+d \pmod{2\pi}$. The restriction $d \leq \pi$ is obviously unnecessary, if we replace c_{11} by $2c_{11} = c_{12}$.

To obtain the upper estimation of $N(c, c+d) - \frac{d}{2\pi} n$ we have, due to

$$N(c_1 c + d) = n - N(0, c) - N(c + d, 2\pi) \quad \text{Q. e. d.}$$

merely to apply (16.5) twice.

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