ANNALS OF MATHEMATICS
Vol. 44, No. 4, October, 1943

SOME REMARKS ON SET THEORY

BY P. ERDÖS

(Received February 1, 1943)

This paper contains a few disconnected results on the theory of sets.

I. Sierpinski[1] proved that under the assumption of the continuum hypothesis there exists a single valued function $f(x)$ whose inverse function is also single valued and which maps the sets of measure 0 into sets of first category and whose inverse function maps the sets of first category into sets of measure 0. He stated the problem[2] whether a function exists which has the above property and also the following one: It maps the sets of first category into sets of measure 0 and its inverse function maps the sets of measure 0 into sets of first category. Thus the function would interchange the sets of measure 0 and the sets of first category. We shall prove that such a function exists. Our proof will be very similar to that of Sierpinski: we will of course assume that the continuum hypothesis holds.

Construction of $f(x)$: It can be shown[3] that a transfinite sequence G_α of G_δ sets of measure 0 and a transfinite sequence F_α of F_σ sets of first category exists ($\alpha < \Omega_1$, Ω_1 is the first ordinal number of the third number class) having the following properties: 1) $G_1 \cup F_1 = R$, $G_1 \cap F_1 = \Lambda$ (R denotes the set of all real numbers), 2) every set of measure 0 is contained in some G_α and every set of first category is contained in some F_α 3) $A_\alpha = G_\alpha - G_\alpha \cap \bigcup_{\beta<\alpha} G_\beta$, $B_\alpha = F_\alpha - F_\alpha \cap \bigcup_{\beta<\alpha} F_\beta$ both have the power of the continuum, for every α. We evidently have $G_1 = \bigcup_{\alpha>1} B_\alpha$, $F_1 = \bigcup_{\alpha>1} A_\alpha$. Hence we can construct a function $f(x)$ in such a way that $f(A_\alpha) = B_\alpha$ for every $\alpha > 1$, and that $f f(x) = x$ for every x. The function $f(x)$ is clearly a single valued function whose inverse $f^{-1}(x)$ is also single valued. Since, in addition $f(x)$ coincides with its inverse, we have only to show that $f(x)$ maps both the sets of measure 0 onto sets of first category, and the sets of first category onto sets of measure 0. But both of these statements are obvious. For let G be any set of measure 0; by assumption $G \subset G_\alpha$ for some α and $f(G) \subset \underset{\beta<\alpha}{U} F_\beta$, which is a set of first category. Similarly let F be any set of first category; by assumption $F \subset F_\alpha$ for some α, and $f(F) \subset \underset{\beta<\alpha}{U} G_\beta$ which is a set of measure 0: This completes the proof.

II. Let m be a cardinal number. Two sets A and B in Euclidean space are called m-equivalent if they can be split into m summands $A = UA_\alpha$, $B = UB_\alpha$, $A_{\alpha_i} \cap A_{\alpha_j} = B_{\alpha_i} \cap B_{\alpha_j} = 0$, and $A_\alpha \cong B_\alpha$. (The sign $\cong$ denotes congruence.)

[1] W. Sierpinski, Fund. Math. Vol. 22, p. 276–278.

[2] Ibid.

[3] Ibid.

Banach and Tarski[4] proved that in three space any two sets containing open sets are finitely equivalent, and that on the line and the plane any two sets containing open sets are countably equivalent.

Professor Tarski[5] communicated to me the following result of Lindenbaum: There exist 2^c linear sets no two of which are countably equivalent. This result was never published, and Tarski does not remember the details of the proof. I have succeeded in proving that if m is any cardinal number $< c$, then there exist 2^c linear sets no two of which are m-equivalent. I do not know whether my proof differs from that of Lindenbaum, but I have thought it might be worth publishing, since the result has some interesting applications.

First we remark that it is easy to construct 2^n subsets of an infinite set A of power n such that the symmetric difference $(x - y) \cup (y - x)$ of any two subsets x and y has the power n. It is sufficient to divide A into n mutually exclusive subsets of power n, and to consider the unions of all these subsets.

Let now $\{a_\alpha\}$ be a Hamel base ($\alpha < \omega_\xi$, ω_ξ is the smallest ordinal belonging to the power of the continuum.) and let A_β($\beta < \omega_\eta$, ω_η the smallest ordinal belonging to 2^c) be a family of subsets of this Hamel base such that the symmetric difference $(A_{\beta_1} - A_{\beta_2}) \cup (A_{\beta_2} - A_{\beta_1})$ has always the power c. Denote by U_β the set of real numbers of the form $\sum c_k a_k$ where the c_k are rational numbers and the a_k belong to A_β. Now we show that for $\beta_1 \neq \beta_2$ U_{β_1} and U_{β_2} are not m-equivalent. We can clearly assume that A_{β_2} contains c elements not contained in A_{β_1}. A being a set of numbers and x an arbitrary number, let us denote by $A + x$ the set of all numbers $z + x$ where z belongs to A. Also we denote by $A^{(y)}$ the reflection of A with respect to y. It suffices to show that if $\{x_\mathfrak{S}\}$ and $\{y_\mathfrak{S}\}$ are two sets of power m ($\mathfrak{S} < \omega_\delta$, ω_δ is the smallest ordinal number belonging to m) then the union of all the sets $U_{\beta_1} + x_\mathfrak{S}$, $U_{\beta_1}^{(y_\mathfrak{S})}$ does not contain U_{β_2}. And this is clear for if we denote by a_ξ the elements of the Hamel base necessary to express the $x_\mathfrak{S}$ and the $y_\mathfrak{S}$ (the power of the a_ξ is clearly $\leqq m$) our set $\bigcup U_{\beta_1} + x_\mathfrak{S}$, $U_{\beta_1}^{(y_\mathfrak{S})}$ can therefore be generated by the elements of A_{β_1} and by at most m other elements of the Hamel basis; while U_{β_2} is generated by the elements of A_{β_2}, and the latter set contains c elements which do not belong to A_{β_1}. This completes our proof.

III. A set B of real numbers is said to be of absolute measure 0 if it is finitely equivalent to a subset of an arbitrarily small interval. It is said to be of absolute measure α if for every ϵ it is finitely equivalent to a subset of an interval of length $\alpha + \epsilon$, and a subset of it is finitely equivalent with the interval of length $\alpha - \epsilon$.[6]

It is well known that the power of Lebesgue measurable sets mod null sets is of power c, but that the power of Lebesgue measurable sets is 2^c. Tarski[7]

[4] Banach and Tarski, Fund. Math. Vol. 6, p. 244–278.

[5] Oral communication.

[6] Tarski, Fund. Math. Vol. 30, p. 218–253. This paper contains the definition and all the properties used of absolute measure used in this proof.

[7] Oral communication.

posed the problem: What is the power of absolutely measurable sets mod sets of absolute measure 0? (It is of course clear the power of absolutely measurable sets is 2^c.)[8] We are going to prove that the power in question is 2^c.

First it is clear that it suffices to prove that the power of all sets in the interval $(0, 1)$ mod sets of absolute measure 0 is of power 2^c. For if we take any set A in $(0, 1)$ and translate it by 1, take its complement in $(1, 2)$ denote it by B, then $A + B$ has absolute measure 1, hence if A_1 and A_2 are not congruent mod sets of absolute measure 0, $A_1 + B_1$ and $A_2 + B_2$ are also not congruent. This is a strong indication of the truth of our theorem, since it is well known that the power of all sets mod sets of Lebesgue measure 0 is also 2^c.

To prove our theorem it clearly suffices to show that there exist in the interval $(0, 1)$ c disjoint sets whose absolute measure is not 0, for by taking all possible sums of these sets we clearly get 2^c sets no two of which are congruent mod sets of absolute measure 0.

Let now $\{a_\alpha\}$ be a Hamel base with $a_1 = 1$. Split it into c disjoint sets of power c. Denote these sets by V_β. We define the sets R_β as follows: $x \in R_\beta$ if and only if $0 \leqq x \leqq 1$ and $x = \sum_{i=1}^{k} c_i a_{\alpha_i}$, the c_i rational and different from 0 and $\alpha_1 < \alpha_2 < \cdots \alpha_k$, $a_{\alpha_k} V_\beta$. (For $i < k$ a_{α_i} does not have to belong to V_β.) We are going to prove that the disjoint sets R_β are not of absolute measure 0. In fact we shall show that R_β is not finitely equivalent with any subset of $(0, \frac{1}{2})$. For suppose that R_β is finitely equivalent with a subset of $(0, \frac{1}{2})$. This would mean that there exist sets $U_1, U_2, \cdots, U_r$ whose sum is the interval $(0, \frac{1}{2})$, and real numbers $x_1, x_2, \cdots x_j, y_1, y_2, \cdots y_l$, $k + l = r$, such that, R_β is contained in $\bigcup U_i + x_i$, $U_j^{(y_j)}$, the sets U_i are supposed to be mutually exclusive. Let a_k be the a_α of largest index which occurs in the representation of the x_i and y_j and denote by R_β' those elements of R_β' in whose representation the a_α of largest index has an index $> k$. Then if w is an element of R_β' and $w \in U_i + x_i$ there exists a $z \in U_i$ with $z = w - x_i$, hence $z \in R_\beta'$, also if $w \in U_j^{(y_j)}$ there again exists a $z \in U_j$ with $z = 2y_j - w$ hence $z \in R_\beta'$. Similarly if $z \in R_\beta'$ we have $z + x_i \in R_\beta'$ and $2y_j - z \in R_\beta'$. Thus we see that $(R_\beta' \cap U_i) + x_i = R_\beta' \cap (U_i + x_i)$ and $(R_\beta' \cap U_j)^{y_j} = R_\beta' \cap U_j^{y_j}$, hence we conclude that $R_\beta' \cap (0, \frac{1}{2})$ is finitely equivalent to $R_\beta' = R_\beta' \cap (0, \frac{1}{2}) \cup R_\beta' \cap (\frac{1}{2}, 1)$. On the other hand a translation by $\frac{a_1}{2} = \frac{1}{2}$ shows that $R_\beta' \cap (0, \frac{1}{2}) \cong (R_\beta' \cap (0, \frac{1}{2})) + \frac{1}{2} = R_\beta' \cap (\frac{1}{2}, 1)$. Thus $R_\beta' \cap (0, \frac{1}{2})$ would be finitely equivalent with $R_\beta' \cap (0, \frac{1}{2}) \cup (R_\beta' \cap (0, \frac{1}{2})) + \frac{1}{2}$. A general theorem of Lindenbaum and Tarski[9] shows that this is not possible, which completes our proof.

Sierpinski[10] constructed a set k of real numbers of power c whose complement has also power c, and such that if $k \cong k'$ then the power of $k' \cap (R - k)$ [as before R denotes the set of all real numbers] is $<c$. It is easy to see that if we define R_β as in III but remove the restriction $0 \leqq x \leqq 1$. Our set R_β has the required property.

---

[8] This statement follows from the fact that there exist sets of absolute measure 0 having power c.

[9] Lindenbaum and Tarski.

[10] W. Sierpinski, Fund. Math. Vol. 19, p. 22–28.

We are going to prove that Sierpinski's theorem can not be improved i.e. if m is a power $<c$ there exists a number x_0 such that $k + x_0 \cap R - k$ has power $\geqq m$. Let y_α be any set of real numbers of power m, and suppose that our theorem does not hold. Then both $k + y_\alpha \cap R - k$ and $(R - k) + y_\alpha \cap k$ have power $<m$ for all α. Therefore since $m^2 = m$ it is easy to see that there exists a $z \in k$ and a $w \in R - k$ such that $z + y_\alpha \in k$ and $w + y_\alpha \in R - k$ for all α. But then clearly $x_0 = w - z$ has the required property, which completes our proof.[11]

IV. Let $f(x)$ be a continuous function in the closed interval $(0, 1)$. Denote by E the set for which

$$\overline{\lim_{h \to +0}} \frac{f(x + h) - f(x)}{h} < \infty.$$

Jarnik[12] proved that E is not countable. We are going to give a very simple proof that E is of power c. (It is easy to see that the complement of E is an F_σ, thus from the fact that E is not enumerable it immediately follows that E is of power c.)

Let x_0 be a number <1 for which $\overline{\lim} \dfrac{f(x + h) - f(x)}{h} < \infty$. We can of course assume that such a number exists. Let $N > \dfrac{f(1) - f(x_0)}{1 - x_0}$, and consider the set of numbers for which $\dfrac{f(y) - f(x_0)}{y - x_0} \geqq N$. Consider the greatest such y and denote it by y_N. Clearly $y_N < 1$. Hence evidently

$$\frac{f(y_N + h) - f(y_N)}{h} < N \quad \text{for} \quad h < 0.$$

Thus y_N belongs to E. Also we have $f(y_N) - f(x) = N(y_N - x)$, hence for $N_1 > N_2, y_{N_1} < y_{N_2}$. Thus the power of points y_N is c, which completes the proof.

Professor Anthony P. Morse communicated to me the following proof of Jarnik's theorem which he obtained some time ago: Choose k so that if we put $g(x) = f(x) - kx$ we shall have $g(0) > g(1)$. Now take any number c such that $g(0) > c > g(1)$. There clearly exists an x such that $g(x) = 0$. Let x_c be the largest such x. It is easily seen that

$$\overline{\lim_{h \to +0}} \frac{g(x_c + h) - g(x_s)}{h} \leqq 0$$

and hence

$$\overline{\lim_{h \to +0}} \frac{f(x_c + h) - f(x_c)}{a} \leqq k.$$

Thus x_c belongs to E. The power of points x_c is clearly equal to that of the continuum, which completes the proof.

University of Pennsylvania

[11] This proof is due to Mr. P. Lax. Oral communication.

[12] Jarnik, Fund. Math. Vol. 23, p. 1–8.