NOTE ON AN ELEMENTARY PROBLEM OF INTERPOLATION

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The unique polynomial of degree $n-1$ assuming the values $y_1, y_2, \cdots, y_n$ at the abscissas $x_1, x_2, \cdots, x_n$, respectively, is given by the Lagrange interpolation formula

$$(1) \qquad L_n(x) = y_1 l_1(x) + y_2 l_2(x) + \cdots + y_n l_n(x),$$

where

$$(2) \qquad l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n,$$

(fundamental polynomials of the Lagrange interpolation) and the polynomial $\omega(x)$ is defined by

$$(3) \qquad \omega(x) = c(x - x_1)(x - x_2) \cdots (x - x_n),$$

where c denotes an arbitrary constant not equal to zero.

In this note we prove the following theorem:

THEOREM. *In the Lagrange interpolation formula let* $x_k = x_k^{(n)} = \cos (2k-1)\pi/2n = \cos \theta_k^{(n)}$, $(k = 1, 2, \cdots, n)$, *which implies* $\omega(x) = T_n(x) = \cos (n \arc \cos x) = \cos n\theta$ *(Tschebycheff polynomial). Then*

$$(4) \qquad | l_k^{(n)}(x) | = \left| \frac{\omega(x)}{\omega'(x_k)(x - x_k)} \right| < \frac{4}{\pi}, \qquad -1 \leqq x \leqq +1,$$

for all n and k, and furthermore

$$(5) \qquad \lim_{n\to\infty} | l_1^{(n)}(+1) | = \lim_{n\to\infty} | l_n^{(n)}(-1) | = \frac{4}{\pi}\cdot$$

In this connection Fejér* proved for all n, k, and x, $(-1 \leqq x \leqq +1)$,

$$(6) \qquad | l_k^{(n)}(x) | < 2^{1/2}.$$

Of course (5) implies that inequality (4) is the best possible in the following sense: For any $\epsilon > 0$ there exist values of n, k, and x,

* L. Fejér, *Lagrangesche Interpolation und die zugehörigen konjugierten Punkte*, Mathematische Annalen, vol. 106 (1932), pp. 1–55; see pp. 10, 11. This paper will hereafter be referred to as L.

$(-1 \leqq x \leqq +1)$, such that

$$
(7) \qquad \left| l_k^{(n)}(x) \right| > \frac{4}{\pi} - \epsilon.
$$

Our proof depends upon the Hermite interpolation formula which gives the unique polynomial $H(x)$ of degree $2n-1$ satisfying the conditions

$$
(8) \qquad H(x_k) = y_k, \qquad H'(x_k) = y_k', \qquad k = 1, 2, \cdots, n,
$$

where the y_k and y_k' are given numbers. It is easy to show that*

$$
(9) \qquad H(x) = \sum_{k=1}^{n} y_k v_k(x) \{l_k(x)\}^2 + \sum_{k=1}^{n} y_k' (x - x_k) \{l_k(x)\}^2,
$$

where

$$
(10) \qquad v_k(x) = 1 - (x - x_k) \frac{\omega''(x_k)}{\omega'(x_k)},
$$

$$
(11) \qquad \sum_{k=1}^{n} v_k(x) \{l_k(x)\}^2 \equiv 1.
$$

For the Tschebycheff abscissas we have

$$
(12) \quad v_k(x) = v_k^{(n)}(x) = \frac{1 - x x_k^{(n)}}{1 - (x_k^{(n)})^2}, \qquad x_k^{(n)} = \cos (2k-1)\pi/2n.
$$

Fejér proved (6) by aid of the simple inequality $v_k^{(n)}(x) \geqq 1/2$.†

We also need the following result due to M. Riesz:‡

LEMMA. *A trigonometric polynomial of degree $n-1$ assumes the maximum of its absolute value at a point whose distance from any of the roots of this trigonometric polynomial is not less than $\pi/[2(n-1)]$.*

We are now in position to prove the theorem. For $n=1$ and $n=2$

$$
\left| l_1^{(1)}(x) \right| = 1, \quad \left| l_1^{(2)}(x) \right| = \left| \frac{(\sin \pi/4) \cos 2\theta}{2(\cos\theta - \cos \pi/4)} \right|
$$

$$
= \frac{1}{2^{1/2}} \left| \cos\theta + \frac{1}{2^{1/2}} \right| < \frac{1 + 2^{1/2}}{2} < \frac{4}{\pi},
$$

* L. Fejér, *Weierstrasssche Approximation, besonders durch Hermitische Interpolation*, Mathematische Annalen, vol. 102 (1930), pp. 707–725.

† See L, p. 5.

‡ M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; pp. 363–364.

$$\left| l_2^{(2)}(x) \right| < \frac{1}{2^{1/2}} \left| \cos\theta - \frac{1}{2^{1/2}} \right| < \frac{1 + 2^{1/2}}{2} < \frac{4}{\pi} \cdot$$

Thus we have to consider only the case $n \geqq 3$. For the Tschebycheff abscissas we have*

$$(13) \quad l_k^{(n)}(x) = (-1)^{k+1} n^{-1} \sin\theta_k^{(n)} \frac{\cos n\theta}{\cos\theta - \cos\theta_k^{(n)}}, \qquad x = \cos\theta.$$

From (13) it follows that $l_k^{(n)}(\cos\theta)$ is a trigonometric polynomial of degree $n-1$. For $2 \leqq k \leqq n-1$, the roots of $l_k^{(n)}(\cos\theta)$ in $(0, \pi)$ are

$$(14) \qquad \theta_\nu^{(n)} = (2\nu - 1)\frac{\pi}{2n}, \qquad 1 \leqq \nu \leqq n, \nu \neq k,$$

and since $\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} = \pi/n$, $\theta_1^{(n)} - 0 = \pi - \theta_n^{(n)} = \pi/2n$, $\left| l_k^{(n)}(\cos\theta) \right|$ assumes its maximum between $\theta_{k-1}^{(n)}$ and $\theta_{k+1}^{(n)}$. Further it is clear that $\left| l_1^{(n)}(\cos\theta) \right|$ and $\left| l_n^{(n)}(\cos\theta) \right|$ assume their maxima at $\theta = 0$ and $\theta = \pi$, respectively. Let us consider first $l_1^{(n)}(x)$ and $l_n^{(n)}(x)$. According to the last remark it will be sufficient to find bounds for $\left| l_1^{(n)}(+1) \right|$ and $\left| l_n^{(n)}(-1) \right|$. From (13) we have

$$\left| l_1^{(n)}(+1) \right| = \left| l_n^{(n)}(-1) \right| = \frac{\sin\theta_1^{(n)}}{n(1 - \cos\theta_1^{(n)})} = \frac{1}{n}\cot\frac{\pi}{4n},$$

whence

$$(15) \qquad \lim_{n\to\infty} \left| l_1^{(n)}(+1) \right| = \lim_{n\to\infty} \left| l_n^{(n)}(-1) \right| = \frac{4}{\pi} \cdot$$

By differentiation we easily see that $x \cot x$ decreases if x increases so that

$$(16) \quad \left| l_1^{(n)}(+1) \right| \leqq \left| l_1^{(n+1)}(+1) \right|, \quad \left| l_n^{(n)}(-1) \right| \leqq \left| l_{n+1}^{(n+1)}(-1) \right|.$$

From (15) and (16) we obtain (7), that is, the second part of the statement.

We now prove that

$$(17) \qquad \max_{-1 \leqq x \leqq +1} \left| l_k^{(n)}(x) \right| < \left| l_1^{(n)}(+1) \right|, \qquad 2 \leqq k \leqq n-1.$$

By (16) it suffices to show that

$$(18) \quad \max_{-1 \leqq x \leqq +1} \left| l_k^{(n)}(x) \right| = \left| l_k^{(n)}(\mu_k) \right| < \left| l_1^{(2)}(+1) \right| = \tfrac{1}{2}(1 + 2^{1/2}).$$

* See L, p. 5.

In order to prove (18) we show that

$$v_k^{(n)}(\mu_k) > \frac{13}{18}, \qquad 2 \leqq k \leqq n-1. \tag{19}$$

Then (11) furnishes, since $v_k(x) \geqq 0$,

$$1 = \sum_{k=1}^{n} v_k^{(n)}(\mu_k)\left\{l_k^{(n)}(\mu_k)\right\}^2 > \frac{13}{18}\left\{l_k^{(n)}(\mu_k)\right\}^2,$$

that is,

$$\left| l_k^{(n)}(\mu_k) \right| < \left[\frac{18}{13}\right]^{1/2} < (1.4)^{1/2} < 1.2 = \tfrac{1}{2}(1 + 1.4) < \tfrac{1}{2}(1 + 2^{1/2}).$$

Let $\mu_k = \cos\phi$, $(0 < \phi < \pi)$. According to the lemma we have $|\phi - \theta_k^{(n)}| < \pi/2n$. On account of (12) it is sufficient to prove

$$\frac{1 - \cos\theta_k^{(n)}\cos(\theta_k^{(n)} + \delta)}{1 - (\cos\theta_k^{(n)})^2} \geqq \frac{13}{18}, \qquad \delta = \pm\frac{\pi}{2n}. \tag{20}$$

We can assume that $\theta_k^{(n)} < \pi/2$ and $\delta = -\pi/2n$. If we write $\theta_k^{(n)} = 3\mu$, we have $\theta_k^{(n)} + \delta \geqq 2\mu$ and $\cos\mu = t$, $(\frac{1}{2} \leqq t \leqq +1)$, hence

$$\frac{\cos(\theta_k^{(n)} + \delta) - \cos\theta_k^{(n)}}{1 - \cos\theta_k^{(n)}} \leqq \frac{\cos 2\mu - \cos 3\mu}{1 - \cos 3\mu} = \frac{4t^2 + 2t - 1}{4t^2 + 4t + 1}.$$

The last fraction is $\leqq 5/9$ so that

$$\frac{\cos\theta_k^{(n)}}{1 + \cos\theta_k^{(n)}} \frac{\cos(\theta_k^{(n)} + \delta) - \cos\theta_k^{(n)}}{1 - \cos\theta_k^{(n)}} \leqq \frac{5}{18},$$

which is equivalent to (20). This completes the proof.

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