Deformed Poisson-laws as certain transform of deformed Gaussian-laws

Ferenc Oravecz

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Abstract. We define a natural transformation of certain symmetric probability measures, and show that it brings the Gaussian-law as well as several deformed analogue of it to the Poisson-law or to the corresponding deformed Poisson-law, respectively. An important example is the q-Gaussian-law with its transform being the q-Poisson-law of Saitoh and Yoshida. By analogy we define an even more general class of deformed Poisson distributions, and show that, as in the classical and the q-case, these deformed Poisson-laws can be achieved as the distributions of certain operators.

I. Introduction

It is well known, that the bosonic field operator of the symmetric Fock Space is distributed in the vacuum state according to the Gaussian-law. In the setup of free probability theory the distribution of the field operator of the full Fock space is Wigner's semicircle-law[1]. If one takes the q-deformed Fock space[2] with the corresponding creation and annihilation operators l_q and l_q^* (which satisfy the so-called q-commutation relation $l^*l - qll^* = \mathbf{1}$), observing the distribution of the field operator $l_q + l_q^*$ in the vacuum state one arrives to the q-deformed Gaussian-law[3]. Going further on in generalization, in the setup of interacting Fock spaces Accardi and Bozejko proved that every symmetric probability measure with finite moments of all orders can be obtained as the vacuum distribution of the corresponding field operator and this way may be called a deformed Gaussian-law[4].

On the other hand, in[5] Hudson and Parthasarathy showed that the Poisson-law can be given as a vacuum distribution of the linear combination of the number operator, the field operator and the identity operator of the symmetric Fock space. Speicher proved the same result for the free Poisson-law, using operators of the full Fock space[6]. Finally, Saitoh and Yoshida obtained similar results for their q-Poisson-law[7] in the setup of q-Fock space in their recent paper[8]. (Note, that in the literature several different distributions are called the q-deformed Poisson-law (see e.g.[9,10]); when we speak about the q-Poisson distribution bellow, we always mean that of Saitoh and Yoshida.)

In this paper we define a transformation $\mathcal{F}$ of those symmetric probability measures which are unique solution of some Hamburger moment problem, and show that the $\mathcal{F}$-transform of the Gaussian-law and the q-Gaussian-law is the Poisson-law and the q-Poisson-law, respectively. Based on these facts, we define an even more general class of deformed Poisson-distributions, namely the distributions which arise as the $\mathcal{F}$-transform of some deformed Gaussian-law.

We show that, as in the classical and the q-case, these deformed Poisson-laws can be given as the distribution of the selfadjoint operators $(l_0 + \sigma \mathbf{1})(l_0^* + \sigma \mathbf{1})$, where l_0 and l_0^* are certain deformed creation and annihilation operators, σ is the square root of the parameter of the given deformed Poisson-law and $\mathbf{1}$ stands for the identity operator.

II. Preliminaries

The pair $(\mathcal{A}, \varphi)$ is called a noncommutative probability space if $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional with $\varphi(\mathbf{1}) = 1$. If $\mathcal{A}$ is a C^*-algebra and φ is a state, then $(\mathcal{A}, \varphi)$ is called a C^*-probability space. An element a of $\mathcal{A}$ is called a noncommutative random variable, $\varphi(a)$ is the expectation value of a, while $\varphi(a^n)$ is the nth moment of a. The distribution of $a \in \mathcal{A}$ is the linear functional

$$\begin{array}{rcll} \mu_a : \mathbb{C}(X) & \to & \mathbb{C} & \\ X^k & \mapsto & \varphi(a^n) & \qquad k = 0, 1, 2, \ldots, \end{array}$$

where $\mathbb{C}[X]$ stands for the algebra of polynomials in one variable.

If a is a selfadjoint element of a C^*-probability space, then its distribution can be realized as a probability measure on $\mathbb{R}$.

Bellow we will use the simple model when $\mathcal{A} = \mathcal{B}(l^2(\mathbb{Z}^+))$ with its standard orthonormal basis $\{\delta_n\}_{n=0}^{\infty}$, and for $a \in \mathcal{A}$ we take $\varphi(a) = \langle a\delta_0, \delta_0 \rangle$.

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying

$$b_n = 0 \qquad \Rightarrow \qquad b_m = 0 \qquad \forall m \geq n, \tag{II.1}$$

and let us take the weighted shift operator l on $l^2(\mathbb{Z}^+)$ defined by

$$l\delta_n = b_{n+1}\delta_{n+1} \qquad n = 0, 1, 2, \ldots \ . \tag{II.2}$$

For the adjoint of l we have

$$\begin{array}{rcll} l^*\delta_n & = & b_n\delta_{n-1} & \qquad n = 1, 2, 3, \ldots \ , \\ l^*\delta_0 & = & 0. & \end{array} \tag{II.3}$$

Note that this model is basically the same as the interacting Fock space model in[4].

The matrices

$$A_n = \begin{pmatrix} 0 & b_1 & 0 & \cdots & \cdots & 0 \\ b_1 & 0 & b_2 & \cdots & \cdots & 0 \\ 0 & b_2 & 0 & b_3 & \cdots & 0 \\ 0 & 0 & b_3 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & & b_{n-1} & 0 \end{pmatrix}$$

are the truncation of the field operator $l^* + l$. It is not difficult to check, that the eigenvalues of A_n are the roots of the monic polynomial $P_n(x)$ defined by the recursion

$$\begin{aligned} P_{n+1}(x) &= xP_n(x) - b_n^2 P_{n-1}(x), \\ P_{-1}(x) &= 0, \qquad P_0(x) = 1. \end{aligned}$$

According to the theory of orthogonal polynomials (see e.g.[11]), for such a sequence of monic polynomials there exist a symmetric probability measure μ, for which $\{P_n(x)\}_{n=0}^{\infty}$ form an OPS (orthogonal polynomial sequence), whose Cauchy-transform $G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z-x}$ has the continued fraction form:

$$G_\mu(z) = \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{z - \ddots}}};$$

and that the moments $\langle (l^* + l)^m \delta_0, \delta_0 \rangle$ of $l^* + l$ are given by

$$\langle (l^* + l)^m \delta_0, \delta_0 \rangle = \lim_{n\to\infty} \langle A_n^m \delta_0, \delta_0 \rangle = \int_{-\infty}^{\infty} x^m d\mu(x),$$

that is, the distribution of $l^* + l$ is just μ. (See also[12].)

Changing the direction of the above procedure one can check, that every symmetric probability measure with finite moments of all orders can be the distribution of some field operator of the above form. For further details see[4].

Examples II.1.

1. *Bosonic case.* Let $b_n = \sqrt{n}$ for ever $n = 1, 2, 3, \ldots$. In this case the polynomials $P_n(x)$ are the monic Hermite polynomials and the measure μ is the standard Gaussian-law $N(0,1)$.

2. *Fermionic case.* Let $b_1 = 1, \quad b_2 = b_3 = \ldots = 0$. Now μ is the Bernoulli-law $\frac{1}{2}\left(\delta_{-1}(x) + \delta_1(x)\right)$ of two atoms.

3. *Free case.* Let $b_n = 1$ for every $n = 1, 2, 3, , \ldots$. In this case the measure μ is the standard Wigner-law, whose density function is

$$w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\chi_{[-2;2]}(x)$$

($\chi_{[-2;2]}(x)$ stands for the characteristic function of the interval $[-2;2]$).

4. *q-case.* Let $q \in (-1;1)$, and let $b_n^{(q)} = \sqrt{[n]_q}$ for every $n = 1, 2, 3, , \ldots,$ where

$$[n]_q = 1 + q + q^2 + q^3 + \ldots + q^{n-1}$$

is the so-called n^{th} q-natural number (see e.g.[13,14]). In this case the polynomials $P_n(x)$ are the q-Hermite polynomials and the measure μ is the so-called standard q-deformed Gaussian-law whose density function is given by

$$w_q(x) = \frac{\sqrt{1-q}}{2\pi}\prod_{k=1}^{\infty}\left(1 - q^{2k}\right)\prod_{k=1}^{\infty}\left(1 + q^k\right)\sqrt{4 - (1-q)x^2}\prod_{k=1}^{\infty}\left(1 - \frac{(1-q)q^k x^2}{\left(1+q^k\right)^2}\right)$$

(see e.g.[3]). When $q = 0$ this formula gives back Wigner's semicircle-law, while for $q \to 1$ one arrives to the standard Gaussian-law.

III. The $\mathcal{F}$-transform

In the following we introduce a transformation of the "deformed Gaussian-laws". For a measure μ with moment sequence $\{m_n\}_{n=0}^{\infty}$ let

$$\Delta_n^{(\mu)} = \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{vmatrix} \qquad \text{and} \qquad \tilde{\Delta}_n^{(\mu)} = \begin{vmatrix} m_1 & m_2 & \cdots & m_{n+1} \\ m_2 & m_3 & \cdots & m_{n+2} \\ \vdots & \vdots & & \vdots \\ m_{n+1} & m_{n+2} & \cdots & m_{2n+1} \end{vmatrix}.$$

Definition III.1. *Let Φ denote the set of those symmetric probability measures of the real line, which are unique solution of a Hamburger moment problem (i.e. whose moments are all finite, and the measure itself is uniquely determined by the requirement that it has the given moment sequence). For a symmetric probability measure $\mu \in \Phi$ its $\mathcal{F}$-**transform** $\nu = \mathcal{F}(\mu)$ is the measure ν for which*

$$\Delta_n^{(\nu)} = \Delta_n^{(\mu)} \quad \text{and} \qquad \tilde{\Delta}_n^{(\nu)} = \sigma^{2n+2}\Delta_n^{(\mu)} \qquad (n = 0, 1, 2, \ldots),$$

where σ stands for the variance of μ.

Remark III.2. The determinants $\Delta_n^{(\mu)}$ and $\tilde{\Delta}_n^{(\mu)}$ are the key objects of the moment problems (see e.g.[15]):
a. A sequence of real numbers $\{m_n\}_{n=0}^{\infty}$ is the moment sequence of a measure μ on $\mathbb{R}$ if and only if $\Delta_n^{(\mu)} \geq 0$ for every $n = 0, 1, 2, \ldots$, with $\Delta_n^{(\mu)} = 0 \Rightarrow \Delta_m^{(\mu)} = 0 \quad \forall m \geq n$.
Moreover, if $\Delta_0^{(\mu)} > 0$, $\Delta_1^{(\mu)} > 0$, $\ldots, \Delta_n^{(\mu)} > 0$ but $\Delta_{n+1}^{(\mu)} = \Delta_{n+2}^{(\mu)} = \ldots = 0$ then μ consist of $n+1$ atoms.
b. A sequence of real numbers $\{m_n\}_{n=0}^{\infty}$ is the moment sequence of a measure μ on $\mathbb{R}^+$ if and only if $\Delta_n^{(\mu)} \geq 0$ and $\tilde{\Delta}_n^{(\mu)} \geq 0$ for every $n = 0, 1, 2, \ldots$, with $\Delta_n^{(\nu)} = 0 \Rightarrow \Delta_m^{(\nu)} = 0$ and $\tilde{\Delta}_n^{(\nu)} = 0 \Rightarrow \tilde{\Delta}_m^{(\nu)} = 0 \quad \forall m \geq n$.
Moreover, if $\Delta_0^{(\mu)} > 0$, $\tilde{\Delta}_0^{(\mu)} > 0$, $\Delta_1^{(\mu)} > 0$, $\tilde{\Delta}_1^{(\mu)} > 0$, $\ldots, \Delta_n^{(\mu)} > 0$ $\tilde{\Delta}_n^{(\mu)} > 0$ but $\Delta_{n+1}^{(\mu)} = \tilde{\Delta}_{n+1}^{(\mu)} = \Delta_{n+2}^{(\mu)} = \tilde{\Delta}_{n+2}^{(\mu)} = \ldots = 0$ then μ consist of $n+1$ atoms.

Remark III.3. The uniqueness requirement in Definition III.1. is automatically satisfied for compactly supported measures. Further uniqueness conditions are given in[15] for example.

Proposition III.4.
a. The measure ν of Definition III.1. exists and is supported on $\mathbb{R}^+$.
b. The moments of ν are uniquely determined by the requirements for $\Delta_n^{(\nu)}$ and $\tilde{\Delta}_n^{(\nu)}$.
c. If $\mu_1, \mu_2 \in \Phi$, $\mu_1 \neq \mu_2$ then $\nu_1 = \mathcal{F}(\mu_1) \neq \mathcal{F}(\mu_2) = \nu_2$.
d. ν is probability measure with mean σ^2.

Proof. a. As in the above definition $\Delta_n^{(\mu)}$ belongs to a measure μ, according to Remark III.2.a. $\Delta_n^{(\mu)} \geq 0$ for every $n = 0, 1, 2, \ldots$, so $\Delta_n^{(\nu)} \geq 0$ and $\tilde{\Delta}_n^{(\nu)} \geq 0$ for every $n = 0, 1, 2, \ldots$ according to their definition, and from Remark III.2.b. follows, that the measure ν of Definition III.1. exist and is supported on the non-negative half line.
b. Obvious from the structure of $\Delta_n^{(\nu)}$ and $\tilde{\Delta}_n^{(\nu)}$.
c. Starting from different $\mu_1, \mu_2 \in \Phi$ (in this case their moment sequences are different because of the uniqueness requirement in Definition III.1.), the determinant sequences $\left\{\Delta_n^{(\nu_1)}\right\}$ and $\left\{\Delta_n^{(\nu_2)}\right\}$, and $\left\{\tilde{\Delta}_n^{(\nu_1)}\right\}$and $\left\{\tilde{\Delta}_n^{(\nu_2)}\right\}$ will be different, so because of b., the moment sequences of ν_1 and ν_2 are different, making the proof complete.
d. Obvious from the definition as μ is a symmetric probability measure with variance σ. $\square$

Examples III.5.

1. $\mu = \delta_0(x)$ (one atom in the origin). This is the simplest possible case with $\left\{m_n^{(\mu)}\right\}_{n=0}^{\infty} = \{1,0,0,\ldots\}$, $\left\{\Delta_n^{(\mu)}\right\}_{n=0}^{\infty} = \{1,0,0,\ldots\} = \left\{\Delta_n^{(\mathcal{F}(\mu))}\right\}_{n=0}^{\infty}$, $\left\{\tilde{\Delta}_n^{(\mathcal{F}(\mu))}\right\}_{n=0}^{\infty} = \{0,0,0,\ldots\}$, which gives $\nu = \mathcal{F}(\mu) = \mu$. (The only case, when $\mu = \mathcal{F}(\mu)$).

2. *The Bernoulli-law of variance r.* $\mu = \frac{1}{2}(\delta_{-r}(x) + \delta_r(x))$. Computing the first few moments of $\nu = \mathcal{F}(\mu)$ from the determinants $\Delta_n^{(\mu)}$, using b. of Remark III.2. one gets the "fermionic" Poisson-law (of parameter r^2) of Saitoh and Yoshida[8]

$$\nu = \left(\frac{1}{2} + \frac{1}{2\sqrt{4r^2+1}}\right)\delta_{x_1}(x) + \left(\frac{1}{2} - \frac{1}{2\sqrt{4r^2+1}}\right)\delta_{x_2}(x),$$

where

$$x_1 = r^2 + \frac{1}{2} - \sqrt{r^2 + \frac{1}{4}}, \qquad x_2 = r^2 + \frac{1}{2} + \sqrt{r^2 + \frac{1}{4}}.$$

3. *Purely atomic measures (finite number of atoms).* Using Remark III.2. and Proposition III.3. one easily checks that if μ consists of n atoms, then $\nu = \mathcal{F}(\mu)$ also consists of n atoms.

For measures which not consist of a finite number of atoms, the method used above for determining $\mathcal{F}(\mu)$ is hopeless. In the following we give a more powerful tool.

It is well known (see e.g.[16]) that for every probability measure μ on $\mathbb{R}$ with finite moments of all orders there exist two sequences of real numbers (the so-called Jacobi parameters) $c_n^{(\mu)} \in \mathbb{R}$ and $\lambda_n^{(\mu)} \geq 0$, such that the monic OPS $\{P_n(x)\}_{n=0}^{\infty}$ associated to μ satisfies the relations

$$\begin{aligned}
P_n(x) &= \left(x - c_n^{(\mu)}\right)P_{n-1}(x) - \lambda_{n-1}^{(\mu)}P_{n-2}(x) \qquad \text{for } n = 1,2,3,\ldots; \\
P_{-1}(x) &= 0, \qquad P_0(x) = 1 \qquad \text{and} \\
\int_{-\infty}^{\infty} P_n(x)\,P_m(x)\,d\mu(x) &= \delta_{n,m}\lambda_1^{(\mu)}\lambda_2^{(\mu)}\ldots\lambda_n^{(\mu)}.
\end{aligned}$$

Moreover (see e.g.[15]) the Cauchy transform $G_\mu(z) = \int_{-\infty}^{\infty}\frac{d\mu(x)}{z-x}$ of μ has the continued fraction form

$$G_\mu(z) = \cfrac{1}{z - c_1^{(\mu)} - \cfrac{\lambda_1^{(\mu)}}{z - c_2^{(\mu)} - \cfrac{\lambda_2^{(\mu)}}{z - c_3^{(\mu)} - \ddots}}},$$

and

$$\lambda_n^{(\mu)} = \frac{\Delta_{n-2}^{(\mu)} \Delta_n^{(\mu)}}{\left(\Delta_{n-1}^{(\mu)}\right)^2} \qquad \text{for } n = 1, 2, 3, \ldots, \tag{III.1}$$

with $\Delta_{-1}^{(\mu)} = 1$ by definition.

The above measure μ is symmetric if and only if $c_n^{(\mu)} = 0$ for each n, and μ is supported in a finite number of points if and only if $\lambda_n^{(\mu)} = 0$ for some n.

Theorem III.6. *If $\nu = \mathcal{F}(\mu)$ for some $\mu \in \Phi$, then*
a. $\lambda_n^{(\nu)} = \lambda_n^{(\mu)}$ for every $n = 1, 2, 3, \ldots$,
b. $c_n^{(\nu)} = \sigma^2 + \dfrac{\lambda_{n-1}^{(\mu)}}{\sigma^2}$ for every $n = 2, 3, 4, \ldots$, with $c_1 = \sigma^2$, where σ is the variance of μ.

Proof. From the definition of $\mathcal{F}(\mu)$ and $(III.1)$ a. follows immediately.
b. To the moment sequence $m_0, m_1, m_2, \ldots$ of ν corresponds the monic OPS $\{P_n(x)\}_{n=0}^{\infty}$ with the recursion

$$\begin{aligned} P_n(x) &= \left(x - c_n^{(\nu)}\right) P_{n-1}(x) - \lambda_{n-1}^{(\nu)} P_{n-2}(x) \qquad \text{for } n = 1, 2, 3, \ldots; \\ & P_{-1}(x) = 0, \qquad P_0(x) = 1 \ . \end{aligned} \tag{III.2}$$

To the moment sequence $m_1, m_2, m_3, \ldots$ corresponds the monic OPS $\{P_n^*(x)\}_{n=0}^{\infty}$ with the recursion

$$\begin{aligned} P_n^*(x) &= (x - c_n^*) P_{n-1}^*(x) - \lambda_{n-1}^* P_{n-2}^*(x) \qquad \text{for } n = 1, 2, 3, \ldots; \\ & P_{-1}^*(x) = 0, \qquad P_0^*(x) = 1 \ . \end{aligned} \tag{III.3}$$

It is also known (see e.g.[11]), that in this case

$$P_n^* = \frac{1}{x}\left[P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)} P_n(x)\right] \qquad \text{for } n = 0, 1, 2, \ldots. \tag{III.4}$$

(Note, that the measure ν^* corresponding to the moment sequence $m_1, m_2, m_3, \ldots$ is not necessarily a probability measure, its total weight is m_2.) It is clear, that the determinants $\Delta_n^{(\nu^*)}$ corresponding to the moment sequence $m_1, m_2, m_3, \ldots$ are the same as the determinants $\tilde{\Delta}_n^{(\nu)}$ corresponding to the moment sequence $m_0, m_1, m_2, \ldots$. From this we get

$$\lambda_n^* = \frac{\Delta_{n-2}^{(\nu^*)} \Delta_n^{(\nu^*)}}{\left(\Delta_{n-1}^{(\nu^*)}\right)^2} = \frac{\tilde{\Delta}_{n-2}^{(\nu)} \tilde{\Delta}_n^{(\nu)}}{\left(\tilde{\Delta}_{n-1}^{(\nu)}\right)^2} = \frac{\Delta_{n-2}^{(\mu)} m_2^{n-1} \Delta_n^{(\mu)} m_2^{n+1}}{\left(\Delta_{n-1}^{(\mu)}\right)^2 m_2^{2n}} = \frac{\Delta_{n-2}^{(\mu)} \Delta_n^{(\mu)}}{\left(\Delta_{n-1}^{(\mu)}\right)^2} = \lambda_n^{(\nu)}$$

for $n = 1, 2, 3, \ldots$.

Now writing $(III.4.)$ into $(III.3.)$, comparing the result with $(III.2.)$ (using the fact, that $\{P_n(x)\}_{n=0}^{N}$ is a basis for the vector subspace of polynomials of degree at most N), after some easy calculation, in which we use that $\lambda_n^* = \lambda_n^{(\nu)}$, we get

$$c_n^{(\nu)} = c_1^{(\nu)} + \frac{\lambda_{n-1}^{(\mu)}}{c_1^{(\nu)}} \qquad \text{for every} \qquad n = 2, 3, 4, \ldots .$$

To get the value of $c_1^{(\nu)}$ let us take the polynomial sequence $\{S_n(x)\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} S_{2m}(x) &= P_m(x) \qquad \text{and} \\ S_{2m+1}(x) &= xP_m^*(x) \qquad \text{for every } m = 0, 1, 2, \ldots . \end{aligned}$$

This means that $\{S_n(x)\}_{n=0}^{\infty}$ is the monic OPS corresponding to the moments $m_0, 0, m_1, 0, m_2, 0, \ldots$, with the recursion

$$\begin{aligned} S_n(x) &= xS_{n-1}(x) - \gamma_{n-1}S_{n-2}(x) \qquad \text{for } n = 1, 2, 3, \ldots ; \\ S_{-1}(x) &= 0, \qquad S_0(x) = 1 . \end{aligned}$$

With these at hand we have (see e.g.[11])

$$c_1^{(\nu)} = \gamma_1 = \int_{-\infty}^{\infty} d\nu^*(x) = m_2 = \sigma^2 ,$$

which completes the proof. $\square$

Examples III.7.

1. *The Gaussian-law.* The

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Hermite polynomials form an OPS for the function e^{-x^2}. The recursion for $H_n(x)$ is

$$\begin{aligned} H_{n+1}(x) &= 2xH_n(x) - 2nH_{n-1}(x) \qquad n = 0, 1, 2, \ldots ; \\ H_{-1}(x) &= 0, \qquad H_0(x) = 1. \end{aligned}$$

From these one can easily calculate the recursion for the monic OPS $\left\{\hat{H}_n(x)\right\}_{n=0}^{\infty}$ corresponding to the function $N(0, \sigma)(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}$ (see e.g.[11]), obtaining

$$\begin{aligned} \hat{H}_n(x) &= x\hat{H}_{n-1}(x) - (n-1)\sigma^2 \hat{H}_{n-2}(x) \qquad n = 1, 2, 3, \ldots ; \\ \hat{H}_{-1}(x) &= 0, \qquad \hat{H}_0(x) = 1. \end{aligned}$$

Calculating the Jacobi parameters of $\nu = \mathcal{F}(N(0,\sigma))$ according to Theorem III.6., one realizes that these are the same as those in the recursion

$$\begin{aligned} C_n^{(a)}(x) &= (x-(n-1)-a)\,C_{n-1}^{(a)}(x) - a(n-1)\,C_{n-2}^{(a)}(x) \qquad \text{for } n = 1,2,3,\ldots; \\ C_{-1}^{(a)}(x) &= 0, \qquad C_0^{(a)}(x) = 1 \end{aligned}$$

for the Charlier polynomials

$$C_n^{(a)} = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} k!\,(-a)^{n-k}$$

of parameter $a = \sigma^2$, which are known to form a monic OPS with respect to the Poisson-law of parameter a.

2. *The q-deformed Gaussian-law ($q \in [0;1)$).* The q-deformed Gaussian-law μ of variance σ can be associated to the q-Hermite polynomials with Jacobi parameters $c_n^{(\mu)} = 0$, $\lambda_n^{(\mu)} = \sigma^2 [n]_q$ $(n = 1,2,3,\ldots)$ (see e.g.[7]). For the Jacobi parameters of its $\mathcal{F}$-transform this gives $c_n^{(\nu)} = \sigma^2 + [n-1]_q$ for $n = 2,3,4,\ldots$, with $c_1 = \sigma^2$, $\lambda_n^{(\nu)} = \sigma^2 [n]_q$ $(n = 1,2,3,\ldots)$, which corresponds to the q-Poisson distribution of Saitoh and Yoshida[7]. The case $q = 0$ has some importance in its own, the corresponding measures being the Wigner-law and its $\mathcal{F}$-transform, the free Poisson-law.

In Section II. we mentioned that in the setup of interacting Fock spaces every $\mu \in \Phi$ may be taken as a deformation of the Gaussian-law. Now in the same sense we may call every measure of the form $\nu = \mathcal{F}(\mu)$ a deformation of the Poisson-law. The only criteria for a probability measure ν supported on the non-negative half line to be a deformed Poisson-law is that for the corresponding determinants $\tilde{\Delta}_n^{(\nu)} = a^n \Delta_n^{(n)}$ should hold for every $n = 0,1,2,\ldots$, where a is the mean of ν (of course this criteria requires that all the moments of ν be finite). An equivalent criteria can be deduced from Theorem III.6., namely, that for the Jacobi parameters corresponding to ν $c_n^{(\nu)} = a + \frac{\lambda_{n-1}^{(\nu)}}{a}$ should hold for every $n = 2,3,4\ldots$, with $c_1 = a$ (where a is again the mean of ν). One can easily check, that for example the exponential law $\nu(x) = \lambda e^{-\lambda x} dx$ does not satisfy these conditions, so it can not be the $\mathcal{F}$-transform of any $\mu \in \Phi$ (and therefor it is not a deformed Poisson-law).

IV. The deformed Poisson-law as the distribution of certain operators

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a monic OPS with respect to a probability measure ν, with the recursion $(III.2)$, where $c_n = c_1 + \frac{\lambda_{n-1}}{c_1}$ for $n = 2,3,4,\ldots$; $c_1 > 0$ and $\lambda_n > 0$

for every $n = 1, 2, 3, \ldots$. It follows, that ν is supported on $\mathbb{R}^+$, and that there is a symmetric probability measure $\mu \in \Phi$ such that $\nu = \mathcal{F}(\mu)$, that is, ν is a deformed Poisson-law. Similarly as in the beginning of Section 1., the eigenvalues of the matrix

$$A_n = \begin{pmatrix} c_1 & b_1 & 0 & \cdots & \cdots & 0 \\ b_1 & c_2 & b_2 & \cdots & \cdots & 0 \\ 0 & b_2 & c_3 & b_3 & \cdots & 0 \\ 0 & 0 & b_3 & c_4 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & & b_{n-1} & c_n \end{pmatrix}$$

(where $b_n = \sqrt{\lambda_n}$) are the roots of the polynomial $P_n(x)$. If the operators l, l^* are given by $(II.2)$ and $(II.3)$, and T of $l^2(\mathbb{Z}^+)$ act as

$$T\delta_n = c_{n+1}\delta_n \qquad n = 0, 1, 2, \ldots,$$

then the moments of the operator $l^* + l + T$ are

$$\langle (l^* + l + T)^m \delta_0, \delta_0 \rangle = \lim_{n\to\infty} \langle A_n^m \delta_0, \delta_0 \rangle = \int_{-\infty}^{\infty} x^m d\nu(x),$$

that is the distribution of $l^* + l + T$ is given by the measure ν.

Let us denote the mean of ν by σ^2. Then, as $c_1 = \sigma^2$, $c_n = \sigma^2 + \frac{b_{n-1}^2}{\sigma^2}$ $(n \geq 2)$, one can verify directly that

$$T = \sigma^2 \mathbf{1} + \frac{1}{\sigma^2} l l^*.$$

So finally we get that the deformed Poisson-law ν is the distribution of the self-adjoint operator $l^* + l + \sigma^2 \mathbf{1} + \frac{1}{\sigma^2} l l^* = \left(\frac{1}{\sigma} l + \sigma \mathbf{1}\right)\left(\frac{1}{\sigma} l^* + \sigma \mathbf{1}\right)$, just as in the bosonic case[5], free case[6] and q-case[8] for $q \in [0; 1)$.

Thus we have proven the following

Theorem IV.1. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers satisfying $(II.1)$. Let μ be the distribution of the operator $l^* + l$ (where l^* and l are given by $(II.2)$ and $(II.3)$ for the given sequence $\{b_n\}_{n=1}^{\infty}$). Let $\nu = \mathcal{F}(\mu)$. Then ν is the distribution of the operator $\left(\frac{1}{\sigma} l + \sigma \mathbf{1}\right)\left(\frac{1}{\sigma} l^* + \sigma \mathbf{1}\right)$, where σ is the variance of μ.* □

Remark IV.2. The result is even nicer if we take the "normalized" creation and annihilation operators $l_0 = \frac{1}{\sigma} l$, $l_0^* = \frac{1}{\sigma} l^*$ (these gives a field operator whose distribution is of variance 1); then the corresponding deformed Poisson-law ν is the distribution of the operator $(l_0 + \sigma \mathbf{1})(l_0^* + \sigma \mathbf{1})$.

We note that while Theorem IV.1. is a generalization of Theorem II.8. in[8], it is still a special case of Theorem (3) of[4].

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Ferenc Oravecz
Alfred Renyi Institute of Mathematics, The Hungarian Academy of Sciences
Reáltanoda u. 13-15, H-1053, Budapest, Hungary
e-mail: oravecz@renyi.hu

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