ON THE GAPS BETWEEN ZEROS OF TRIGONOMETRIC POLYNOMIALS

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Abstract. We show that for every finite set $0 \notin S \subset \mathbb{Z}^d$ with the property $-S = S$, every real trigonometric polynomial f on the d dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with spectrum in S has a zero in every closed ball of diameter $D(S)$, where

$$D(S) = \sum_{\lambda \in S} \frac{1}{4||\lambda||_2} \quad ,$$

and investigate tightness in some special cases.

1. Introduction and presentation of the main results

We are interested in the following question: for real trigonometric polynomials with a given spectrum, what is the maximal possible distance between two consecutive real zeros? Of course, if we can use enough frequencies for building our trigonometric polynomial, the size of the largest gap can be as close to 2π (or to 1, if one uses the transformation used throughout this paper) as we want. The question gets more interesting when the number of frequencies is relatively small.

The first theorem is a very general result, partially answering the above question. We state it for multivariate trigonometric polynomials on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Theorem 1. *Let $0 \notin S \subset \mathbb{Z}^d$ be a finite set s.t. $-S = S$. Let*

$$f(x) = \sum_{\lambda \in S} c(\lambda) \exp 2\pi i \langle x, \lambda \rangle$$

be a real value trigonometric polynomial on $\mathbb{T}^d$. Then f has a zero in any closed ball of diameter $D(S)$ where

$$D(S) := \sum_{\lambda \in S} \frac{1}{4||\lambda||_2} \quad ,$$

$$||\lambda||_2 = \sqrt{\langle \lambda, \lambda \rangle} = \sqrt{\textstyle\sum_{j=1}^d \lambda_j^2} \quad .$$

Note that the requirement $-S = S$ is necessary because we are interested in real trigonometric polynomials.

How sharp is this theorem? To answer that we have to analyze specific cases. From now on we concentrate on the one dimensional situation. For a finite set

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$0 \notin S \subset \mathbb{Z}$ define

$$M(S) := \sup_{\operatorname{spec} f \subset S} \sup_{\substack{I \subset [0,1] \\ f|_I \neq 0}} |I| \quad ,$$

where $\operatorname{spec} f$ is the spectrum of f i.e. the support of the the Fourier transform $\hat{f}$. The second supremum is over all intervals I such that f is never zero on I. Here and also later on $|I|$ stands for the length of I. With this notation theorem 1 is shortened to the formula

$$M(S) \leq D(S) \quad .$$

The easiest case to investigate is when the spectrum is a (discrete) interval, or rather, two intervals situated symmetrically around 0. It turns out that in this case it is possible to get an explicit formula for the supremum (maximum does not exist) of the gap size:

Theorem 2. *For $S = [-N-K, -N] \cup [N, N+K] \subset \mathbb{Z}$ with N, $K \in \mathbb{N}$,*

$$M(S) = \frac{K+1}{2N+K} \quad .$$

As the proof will show, we can also demonstrate an explicit formula for the trigonometric polynomial corresponding to the supremum (being a polynomial with "touching zeros" — e.g. non-negative on an interval of length $(K+1)/(2N+K)$). We see that the general result applied for this specific case is sharp only asymptotically when $K \ll N$. This leaves many questions. As examples, we formulate some of them:

Question 1. Theorem 2 can be generalized to linear progressions with stepping b smaller than twice the starting element, i.e. when $S = \pm \{N + nb : n \in 0, 1, 2, \ldots K\}$ for some $K, b \in \mathbb{N}$, $b < 2N$ (see Theorem 3 on page 4). What happens when $b \geq 2N$? Can an explicit formula for $M(S)$ be found in this case too?

Question 2. What is the maximum when the spectrum points are perfect squares, i.e. $S = \{\pm n^2\}_{n=N}^{N+K}$?

Theorem 1 states that in this case the maximum tends to zero as $N \to \infty$ even if $K \to \infty$, a result which by itself is intriguing. However, is the asymptotic truly $O(1/N)$ or is it even lower?

Question 3. What can we say when S is a random set? In this case we should handle random trigonometric polynomials where instead of the coefficients, the frequencies are chosen randomly.

Question 4. What about nets? A net of order n is a set with 2^{n+1} elements, defined by parameters $\{a_0, \ldots, a_n\}$ as follows: $S := \pm \{a_0 + \sum_{i=1}^n \{0, a_i\}\}$. In many questions nets behave like generalized arithmetic sequences. Is it true in our case as well?

We should remark that the specific case of $k = 1$ in theorem 2, together with the extremal polynomial, has been known for a while. The question was asked in [T65] and answered in [B84], which actually showed that this polynomial maximizes the **measure** of the positive part. The same polynomial reappears attributed to Arnold in [T97] and again in [GS00] (which also handles the question of positive coefficients). For our needs, though, the case $k = 1$ is not enough to make a meaningful comparison between M and D since we are even more interested in the case when they are both small, and the proofs in these papers do not seem to generalize nicely. See also [Y96] for inexact multidimensional results. Finally, [BH84, MDS99] deal with the related question of completely positive polynomials with some bounds on the zeroth coefficient.

2. Proof of Theorem 1 and Theorem 2

Proof of theorem 1. We will use induction on the cardinality of S. Due to the symmetry of S, we have $\#S = 2k$, $k = 0, 1, 2, \ldots$ The case $k = 0$ is trivial as then $f \equiv 0$.

In general, given $k > 0$, assume we have proven the statement for all sets S with cardinality smaller than $2k$. Take a set S with cardinality $2k$ and a trigonometric polynomial $f(x) = \sum_{\lambda \in S} c(\lambda) \exp 2\pi i \langle x, \lambda \rangle$. Arguing by contradiction, assume that f does not have a zero, say is strictly positive, on a d-dimensional ball of radius R centered at $y = (y_1, \ldots, y_d)$, which we denote as usual by $B(y, R)$. Pick a frequency $\nu = (\nu_1, \ldots, \nu_d) \in S$, and translate f by $\mu := \frac{\nu}{2||\nu||_2^2}$. Since f is strictly positive on the ball $B(y, R)$, the translate is strictly positive on the translated ball $B(y - \mu, R)$. Adding the translate to $f(x)$ we get a new function

$$\tilde{f}(x) := f(x) + f(x + \mu)$$

which is strictly positive on the closed ball

$$B\left(y - \tfrac{1}{2}\mu, R - \tfrac{1}{2}||\mu||_2\right) \subseteq B(y, R) \cap B(y - \mu, R) \tag{1}$$

whose diameter is $2R - \frac{1}{2||\nu||_2}$.

The Fourier expansion of the translated function is

$$\begin{aligned} f(x + \mu) &= \sum_{\lambda \in S} c(\lambda) \exp 2\pi i \langle x + \mu, \lambda \rangle \\ &= \sum_{\lambda \in S} c(\lambda) \exp 2\pi i \langle \mu, \lambda \rangle \exp 2\pi i \langle x, \lambda \rangle \end{aligned}$$

and thus

$$\tilde{f}(x) = \sum_{\lambda \in S} c(\lambda)(1 + \exp 2\pi i \langle \mu, \lambda \rangle) \exp 2\pi i \langle x, \lambda \rangle .$$

Since for $\lambda = \pm\nu$ we get $\exp 2\pi i \langle \mu, \lambda \rangle = e^{\pm\pi i} = -1$, the spectrum $S_{\tilde{f}} := \operatorname{spec}(\tilde{f})$ of $\tilde{f}$ is contained in S but definitely does not contain $\pm\nu$. Thus $D(S) - \frac{1}{2||\nu||_2} \geq D(S_{\tilde{f}})$, and so we find that we have a new set $S_{\tilde{f}}$ of cardinality smaller than $2k$ and a trigonometric polynomial $\tilde{f}$ with spectrum $S_{\tilde{f}}$ which is strictly positive on the closed ball $B(y - \mu, R(S_{\tilde{f}}))$ of radius $R(S_{\tilde{f}}) = \frac{1}{2}D(S_{\tilde{f}})$ centered at $y - \mu$. This contradicts our inductive hypothesis. □

Remark. A similar proof can be applied to shapes different from balls. For example, for cubes we get the following version of theorem 1: If $\operatorname{spec} f = S$ then f has a zero in every closed cube of side length $L(S)$ where

$$L(S) := \sum_{\lambda \in S} \frac{||\lambda||_\infty}{4||\lambda||_2^2} \quad .$$

The only proof element that needs modification is (1), as for cubes we have (denoting by $C(y, L)$ the cube of side length L based at y),

$$C(y, L - ||\mu||_\infty) \subset C(y, L) \cap C(y - \mu, L)$$

and $||\mu||_\infty = ||\nu||_\infty / ||\nu||_2^2$.

Proof of Theorem 2. Theorem 2 is a specific case of the following

Theorem 3. *For* $S = \{-N-Kb, \dots, -N-b, -N\} \cup \{N, N+b, \dots, N+Kb\}$ *with* $N,\ K,\ b \in \mathbb{N}$ *and* $b < 2N$ *we have*

$$M(S) = \frac{K+1}{bK+2N} \quad .$$

Proof. First we show

$$M\left(S\right) \le \frac{K+1}{bK+2N} \tag{2}$$

by proving that for every trigonometric polynomial $f\left(t\right) = \sum_{\lambda \in S} c\left(\lambda\right) e^{2\pi i \lambda t}$, if $f > 0$ on an interval I then $|I| \le \frac{K+1}{bK+2N}$. We may assume w.l.o.g. that $I = [0, a]$. One can write $f\left(x\right) = \operatorname{Re} F\left(t\right)$ with $F\left(t\right) := e^{2\pi i N t} Q\left(e^{2\pi i t}\right)$, where Q is a polynomial of degree Kb, that is

$$Q\left(z\right) = c \prod_{j=1}^{Kb} \left(z - \xi_j\right) \quad . \tag{3}$$

The condition $f > 0$ on I is equivalent to $\arg F \in (-\pi/2, \pi/2)$ on I. Since $e^{2\pi i N t}$ does a rotation of $2\pi N a$ in the positive direction as t goes from 0 to a, $f > 0$ on I requires that $Q\left(e^{2\pi i t}\right)$ does a rotation of at least $2\pi N a - \pi$ in the negative direction on I. In other words, if $\widetilde{\arg} Q$ is any continuous version of arg on the arc $\left\{e^{2\pi i t} : t \in I\right\}$ ($\widetilde{\arg}$ exists because $f|_I > 0$ gives $Q \ne 0$ on the arc, so for example take $\widetilde{\arg}\, Q\left(e^{2\pi i t}\right) := \operatorname{Im} \int_0^t \frac{Q'}{Q}$) then a necessary condition is

$$\operatorname{ind} Q := \widetilde{\arg}\, Q\left(e^{2\pi i a}\right) - \widetilde{\arg}\, Q\left(1\right) \le \pi - 2\pi N a \quad . \tag{4}$$

Clearly the index $\operatorname{ind} Q$ of Q does not depend on the choice of $\widetilde{\arg}$ (all versions of $\widetilde{\arg}$ differ by a constant $2\pi k$). As we want an upper bound for $a = |I|$, we want to maximize $-\operatorname{ind} Q$. Clearly

$$\operatorname{ind} Q = \sum_{j=1}^{Kb} \operatorname{ind}\left(z - \xi_j\right) \quad .$$

Now it is obvious that if ξ is inside the unit disc, then $\operatorname{ind}\left(z - \xi\right)$ is positive (draw a picture!) — this is not what we want. If ξ is outside the disc, then $\operatorname{ind}\left(z-\xi\right)$ is harmonic (as a function of ξ) and therefore has no maxima in any open subset outside the disc, thus we need to investigate only the limiting behavior as ξ tends to infinity and to the unit circle.[1] Both can be calculated explicitly. When $\xi \to \infty$ clearly $\operatorname{ind}\left(z - \xi\right) \to 0$. On the other hand, a simple calculation shows that

$$\lim_{\xi \to e^{2\pi i s}} \operatorname{ind}\left(z - \xi\right) = \begin{cases} \pi a - \pi & \text{if } s \in (0, a) \\ \pi a & \text{if } s \notin [0, a] \end{cases} \tag{5}$$

where the limit is taken from the outside of the unit circle. A geometric note may be due here: the limit $\lim_{\xi \to e^{2\pi i s}} \left(\widetilde{\arg}\left(e^{2\pi i a} - \xi\right) - \widetilde{\arg}\left(1 - \xi\right)\right)$, viewed as a function of a, for $s \in I$ is not continuous — it has a $-\pi$ jump discontinuity at s, this gives the $-\pi$ factor in (5).

[1] The fact that the supremum occurs when ξ converges to the unit circle from outside can also be deduced by elementary geometric arguments.

A similar calculation shows that if $\xi \to 1$ or $\xi \to e^{2\pi i a}$ then the limit of $\operatorname{ind}(z-\xi)$ depends on the angle of approach, but in any case is bounded from below by $\pi a - \pi$. These three cases (ξ inside the disc, outside, and approaching the boundary) totally give

$$\operatorname{ind}(z-\xi) \geq \pi(a-1) \quad \forall \xi \in \mathbb{C} \setminus \left\{e^{2\pi i s} : s \in I\right\} \quad . \tag{6}$$

The case $b=1$ is now immediate from (4) and (6). The case $b>1$ is not much harder. Let $\tau = e^{2\pi i/b}$. If ξ is a zero of Q then so are $\tau\xi, \tau^2\xi, \ldots, \tau^{b-1}\xi$ so we can rearrange ξ_j such that $\xi_{j+K} = \tau\xi_j$. In particular this gives the following representation using the first K zeros:

$$Q(z) = c\prod_{j=1}^{K}\left(z^b - \xi_j^b\right) \quad .$$

Now the analog of (6) is

$$\operatorname{ind}\left(z^b - \xi^b\right) \geq \pi(ba-1) \tag{7}$$

which holds for $a < 1/b$ — we will justify this assumption later. The reasoning is similar: if ξ is inside the unit disc so are all the conjugates and the index is positive. For ξ outside the disc the index is harmonic in ξ so we need to investigate only ξ converging to the circle, but as for $a < 1/b$ only one of the conjugates of ξ may converge to the arc $\left\{e^{2\pi i s} : s \in (0,a)\right\}$. Summing b copies of (5) gives (7). Summing the K copies of (7) gives (with (4))

$$K\pi(ba-1) \leq \operatorname{ind} Q \leq \pi - 2\pi N a \tag{8}$$

which, again, holds for $a < 1/b$. As according to our conditions $2N/b > 1$, for $a = \frac{1}{b} - \epsilon$ with $\epsilon > 0$ sufficiently small (8) cannot hold, thus no such f can have an interval of length $\frac{1}{b} - \epsilon$ with no zeros. This justifies our a priori assumption $a < 1/b$, and we can use (8) to get the largest a allowed, proving (2).

We turn to the proof of

$$M(S) \geq \frac{K+1}{bK+2N} \quad . \tag{9}$$

The first step will be to construct an extremal function f which will only satisfy $f \geq 0$ (i.e. will have zeros in the interval $I = [0,a]$ where $a = \frac{K+1}{bK+2N}$). The analysis above suggests that we try $f = \operatorname{Re} F$, $F(t) = e^{2\pi i N t} Q\left(e^{2\pi i t}\right)$, for Q satisfying (3), with K zeros on the arc $\left\{e^{2\pi i s} : s \in I\right\}$ (the other zeros are the conjugates, which are of course also an the unit circle but not on the arc). It turns out that it is enough to space the zeros evenly, namely define

$$Q(z) := c\prod_{j=1}^{K}\left(z^b - \tau^{jb}\right)$$

where $\tau := e^{2\pi i \eta}$ with $\eta := \frac{1}{bK+2N}$, and c is some constant such that $F(0) = Q(0) = -1$. Now $\arg F$ can be calculated explicitly, where this time arg is in the usual sense, i.e. a discontinuous function between $-\pi$ and π. Elementary geometry shows that

$$\arg\left(e^{i\varphi} - e^{i\nu}\right) \equiv \begin{cases} \frac{\nu+\varphi-\pi}{2} & \text{if } \nu > \varphi \\ \frac{\nu+\varphi+\pi}{2} & \text{if } \nu < \varphi \end{cases} \mod 2\pi$$

Summing this for all the roots we get that on each segment $(k\eta, (k+1)\eta)$ $\arg F$ is a linear function with derivative $\pi(bK+2N)$. Thus, on $(0,\eta)$ it is increasing from

$-\pi/2$ to $\pi/2$, at η it has a jump of $-\pi$, then on $(\eta, 2\eta)$ it is again linear with the same slope, so it again rises from $-\pi/2$ to $\pi/2$, then has another jump of $-\pi$ and so on. This argument works till $(K+1)\,\eta = a$. Thus we get $\arg F \in [-\pi/2, \pi/2]$ on I and therefore $f \geq 0$ on I having its only zeros at the points $\{0, \eta, 2\eta, \ldots, K\eta, a\}$.

Taking $f(t) + f(t+\epsilon)$ (whose Fourier transform is also supported on S) for ϵ small enough one gets a strictly positive function on any interval interior to I. This proves (9) and the theorem. □

3. Corollaries and remarks

Let's compare the results of theorems 1 and 2. A simple calculation shows that

$$D(S) = \sum_{n=N}^{N+K} \frac{1}{2n} = \frac{K+1}{2N+K} + O\left(\frac{K^2}{N^3}\right) \quad .$$

It is interesting to note that the true value, i.e. $M(S) = \frac{K+1}{2N+K}$ is the unique approximation of $D(S)$ by a rational function of order 1 with this error term.

The following fact is a corollary of Theorem 2:

Corollary. *For every interval $I \subset [0,1]$ with $|I| < 1$ there exists some constant α such that for all $N \in \mathbb{N}$ there exists a real trigonometric polynomial f with*

$$\operatorname{spec} f \subset [-\alpha N, -N] \cup [N, \alpha N]$$

such that $f > 0$ on I.

Proof. Let $\delta := 1 - |I|$. Let $\alpha := 1 + 4/\delta$. Apply Theorem 2 for N, $K := (\alpha - 1)N$ and $\epsilon := \delta/2$. Then as in this case

$$\left(\frac{K+1}{2N+K} - \epsilon\right) - |I| = \frac{1 + N\delta}{2N+K} > 0,$$

there exists a real trigonometric polynomial f_1 with $\operatorname{spec}(f_1) \subset [-\alpha N, -N] \cup [N, \alpha N]$ s.t. $f_1 > 0$ on an interval of length bigger than $|I|$ (namely, on the interval $\left[0, \frac{K+1}{2N+K} - \epsilon\right]$). If $I \subset \left[0, \frac{K+1}{2N+K} - \epsilon\right]$ we are done. Otherwise, we need another shift and $f(t) := f_1(t-a)$ will do, where $I = (a, b)$. □

Here is a simple but slightly disappointing corollary to theorem 3:

Corollary. *M and D may be significantly different even when they are both small.*

Proof. Take $b = K$ and $N = K^2$. Then easily

$$M(S) = \frac{1}{3}K^{-1} + O(K^{-2}) \quad D(S) = (\log 2)K^{-1} + O(K^{-3/2}).$$ □

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