Fermi convolution

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ABSTRACT. Below a new kind of convolution is introduced for probability measures, whose combinatorics is related to non-crossing partitions without inner blocks other than singletons - the partitions corresponding to the fermionic creation and annihilation operators and Pauli's principle.

1. Introduction

It is well known that classical, free and boolean convolution is related to all partitions, non-crossing partitions and interval partitions, respectively. This means that for probability measures with finite moments of all orders (e.g. compactly supported measures) in the moment-cumulant formula

$$m_n(\mu) = \sum_{\nu} \prod_{V \in \nu} \tilde{s}_{|V|}(\mu), \tag{1.1}$$

the summation is taken over all partitions, non-crossing partitions and interval partitions, respectively, where the (classical, free or boolean) cumulants $\tilde{s}_n$ are additive with respect to the corresponding convolution $\tilde{*}$:

$$\tilde{s}_n(\mu_1 \tilde{*} \mu_2) = \tilde{s}_n(\mu_1) + \tilde{s}_n(\mu_2).$$

Below a new convolution is investigated, which we call 'Fermi convolution', as it is related to the set of those non-crossing partitions, in which all the inner blocks are singletons, i.e. the partitions corresponding to the fermionic creation and annihilation operators (and Pauli's principle). The definition of this operation is so simple that at the first sight it might suggest that it is not a new type of convolution at all. Namely, the Fermi convolution of two probability measures μ_1 and μ_2 with mean λ_1 and λ_2, respectively, is the shift of the boolean convolution of μ_1^0 and μ_2^0 by $\lambda_1 + \lambda_2$, where μ_1^0 and μ_2^0 are the zero-mean shifts of μ_1 and μ_2. As for probability measures with zero mean their Fermi and boolean convolutions coincide, many properties of the boolean convolution remain valid in the Fermi case. One important consequence of this fact is that the fermionic central limit theorem follows immediately from the boolean one.

However, for measures with non-zero mean there are some important differences between the Fermi and the boolean cases. One of these is that the boolean convolution of a measure μ with δ_a (the Dirac measure in a) is not necessarily the shift of μ by the amount of a - unlike in the classical, free and fermionic cases. The limit distributions of the corresponding Poisson limit theorems are also different, and in the Fermi case one obtains the 'fermionic Poisson-law' of Saitoh and Yoshida[4].

2. Preliminaries and definition of Fermi convolution - The combinatorial level

For a finite set S, $\pi = \{V_1, V_2, \ldots, V_p\}$ is a *partition* of S, if the (non empty) sets V_i are disjoint and the union of them is S. The sets V_i are called the *blocks* of the partition. The number of elements in a block V_i is denoted by $|V_i|$. The block V will be called *singleton*, if $|V_i| = 1$. For a linearly ordered set S we call the partition π of S *crossing*, if in π there are two blocks $V_i \neq V_j$ and elements $v_1, v_2 \in V_i$, $w_1, w_2 \in V_j$ such that $v_1 < w_1 < v_2 < w_2$. A partition of a linearly ordered set is called *non-crossing*, if it is not crossing. The block $V = \left\{v_1, \ldots, v_{|V|}\right\}$ of a non-crossing partition ν is called *inner block*, if there exists a block $U = \left\{u_1, \ldots, , u_{|U|}\right\}$ of ν such that $u_1 < v_1 < u_{|U|}$. A partition ν of a linearly ordered set S is called *interval partition*, if it is non-crossing and has no inner blocks. The set of all interval partitions of the set $\{1, 2, \ldots, k\}$ will be denoted by $IP(k)$. Finally, a partition ν of S is called *pair partition,* if every block of ν contains exactly two elements.

For further details of the theory of partitions see e.g. Refs. 1, 3 and references listed there.

We will use the following notations. By $\mathcal{P}$ we denote the set of probability measures on $\mathbb{R}$, $\mathcal{P}^1$, $\mathcal{P}^2$, and $\mathcal{P}^\infty$ are the subsets of probability measures with finite mean, finite mean and variance, and with finite moments of all orders, respectively. A lower index 0 indicates vanishing mean, i.e., $\mathcal{P}_0^1$ $(\mathcal{P}_0^2, \mathcal{P}_0^\infty)$ are the probability measures with zero mean (and finite variance, and finite moments of all orders, respectively). By

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \qquad \text{and} \qquad \mathbb{C}^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$$

we denote the upper and lower complex half planes, respectively.

Definition 2.1. *Let $\mu_1, \mu_2 \in \mathcal{P}^2$. Let $\lambda_i = m_1(\mu_i)$ $(i = 1, 2)$. The **Fermi convolution** of μ_1 and μ_2 is the probability measure $\mu = \mu_1 \bullet \mu_2$, which is the shift of the boolean convolution of the zero-mean shifts of μ_1 and μ_2 by $\lambda_1 + \lambda_2$ to the right, that is*

$$d\mu(x) = d\tilde{\mu}(x - (\lambda_1 + \lambda_2)),$$

where

$$\tilde{\mu} = \mu_1^0 \uplus \mu_2^0,$$

and μ_i^0 *is the shift of* μ_i *by* λ_i *so that* μ_i^0 *has zero mean* $(i = 1, 2)$ *and* $\uplus$ *stands for the boolean convolution*[5].

Remark 2.1. The reason of defining the Fermi convolution only on the set $\mathcal{P}^2$ and not on $\mathcal{P}^1$ (as one would expect) is that we could not prove (though we believe it is true) that the convolution of arbitrary measures from $\mathcal{P}^1$ belongs to $\mathcal{P}^1$ again, which would be necessary for example for the definition of multiple Fermi convolution products. As we will see it later, for $\mathcal{P}^2$ no such problem occurs.

Note, that the Fermi convolution defined above coincides with the special case of the c-free convolution of Božejko, Leinert and Speicher[2] when $\nu_i = \delta_{m_1(\mu_i)}$, that is (using the notation of Ref. 2) $\left(\mu_1 \bullet \mu_2, \delta_{m_1(\mu_1)+m_1(\mu_2)}\right) = \left(\mu_1, \delta_{m_1(\mu_1)}\right) \boxplus \left(\mu_2, \delta_{m_1(\mu_2)}\right)$.

According to Pauli's principle, at the same time only one fermion can be in the same state. This is reflected in the formula $l^2 = 0$, where l is the creation operator of a certain state.

Non-crossing pair partitions are sometimes represented by a product of creation (l) and annihilation (l^*) operators in the following way: to the first (second) element of any block corresponds a creation (annihilation) operator, and the order in the product have to be inverted. For example, the pair partition $\nu = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$ of the set $\{1, 2, 3, 4, 5, 6\}$ can be represented by the product $l^*ll^*l^*ll$.

For non-crossing partitions a similar one-to-one representation can be given. Let us represent the singleton blocks of a non-crossing partition by the identity operator ($\mathbf{1}$), the first (last) element of every block (of at least two elements) by a creation (annihilation) operator, while the other elements of every block (of at least three elements) by the product (ll^*), again writing these representing terms from right to left. For example, the partition $\nu = \{\{1\}, \{2, 4, 7\}, \{3\}, \{5, 6\}\}$ of the set $\{1, 2, 3, 4, 5, 6, 7\}$ is represented by the product $l^*l^*l\,(ll^*)\,\mathbf{1}l\mathbf{1}$. Here, just as in the above representation of non-crossing pair partitions, every inner block of at least two elements means the presence of at least two particles (of the same state) in our hypotetic system. Therefore (or simply by the relation $l^2 = 0$), if one wants to model a one-state fermionic system by this representation, the contributing terms will correspond to non-crossing partitions that do not contain inner blocks other than singletons. Let us denote these partitions of n elements by $AIP\,(n)$ (as they are 'almost interval partitions'). It is clear that we have the relations $IP\,(n) \subset AIP\,(n) \subset NC\,(n) \subset P\,(n)$, where $NC\,(n)$ and $P\,(n)$ stand for the set of non-crossing partitions and for the set of all partitions, respectively.

While $IP(n)$, $NC(n)$ and $P(n)$ form a lattice for every n with the usual partial order of partitions (i.e.: $\nu \leq \pi$ if and only if each block of ν is contained in a block of π), this is not true for $AIP(n)$. (The simplest counterexample is obtained from the elements $\nu_1 = \{\{1\}, \{2,3\}, \{4\}\}$ and $\nu_2 = \{\{1,4\}, \{2\}, \{3\}\}$ of $AIP(4)$, whose supremum is $\nu = \{\{1,4\}, \{2,3\}\} \notin AIP(4)$.)

Based on the sets $AIP(n)$ one can define the 'Fermi cumulants' of a measure with finite moments of all orders.

Definition 2.2. *Let μ be a probability measure with finite moments of all orders. Then its **Fermi cumulants** γ_k are defined by the formula*

$$m_n(\mu) = \sum_{\nu \in AIP(n)} \prod_{V \in \nu} \gamma_{|V|}(\mu) \qquad \text{for } n = 1,2,3,\ldots \tag{2.1}$$

Note that (3.1) is invertible, that is, there exist a formula of the form $\gamma_k(\mu) = f(m_1(\mu), m_2(\mu), \ldots, m_k(\mu))$ for every $k = 1,2,3,\ldots$, so the above definition has sense. Therefore the sequences $\{m_n(\mu)\}_{n=1}^{\infty}$ and $\{\gamma_n(\mu)\}_{n=1}^{\infty}$ determine each other uniquely.

The corollary of the next proposition shows that for measures with finite moments of all orders the Fermi cumulants behave additively under the operation of Fermi convolution.

Proposition 2.1. *Let $\mu \in \mathcal{P}^{\infty}$. Let μ^0 be the zero mean shift of μ. Then $\gamma_n(\mu) = \gamma_n(\mu^0)$ for every $n \geq 2$.*

Proof. The moments of μ can be given by the moments of μ^0 as

$$\begin{aligned} m_n(\mu) &= \int_{-\infty}^{\infty} x^n d\mu(x) = \int_{-\infty}^{\infty} x^n d\mu^0(x-\lambda) = \int_{-\infty}^{\infty} (y+\lambda)^n d\mu^0(y) \\ &= \int_{-\infty}^{\infty} \sum_{k=0}^{n} \binom{n}{k} y^k \lambda^{n-k} d\mu^0(y) = \sum_{k=0}^{n} \binom{n}{k} m_k(\mu^0) \lambda^{n-k}. \end{aligned}$$

On the other hand

$$m_n(\mu^0) = \sum_{\nu \in AIP(n)} \prod_{V \in \nu} \gamma_{|V|}(\mu^0) = \sum_{\nu \in IP(n)} \prod_{V \in \nu} \gamma_{|V|}(\mu^0) = \sum_{\nu \in IP^*(n)} \prod_{V \in \nu} \gamma_{|V|}(\mu^0),$$

where $IP^*(n)$ is the set of those interval partitions in which every block contains at least two elements. The last two equations hold because $\gamma_1(\mu^0) = m_1(\mu^0) = 0$.

With these (2.1) gives

$$\sum_{\nu \in AIP(n)} \prod_{V \in \nu} \gamma_{|V|}(\mu) = \lambda^n + \sum_{k=2}^{n} \binom{n}{k} \lambda^{n-k} \cdot \left(\sum_{\nu \in IP^*(k)} \prod_{V \in \nu} \gamma_{|V|}(\mu^0) \right). \tag{2.2}$$

In the left hand side of (2.2) let us to take the partition $\nu_0 = \{\{1\}, \{2\}, \ldots, \{n\}\} \in AIP(n)$ (i.e. the partition where all the blocks are singletons) separately. This partition gives λ^n. Now every $\nu \in AIP(n) \setminus \{\nu_0\}$ can be obtained from a partition $\tilde{\nu} \in IP^*(k)$ $(k \leq n)$, where $\tilde{\nu}$ is the partition which remains after taking all the singleton blocks of ν away. (From $\tilde{\nu}$ we get back ν by inserting all the eliminated singleton blocks to their original site.) It is clear that every $\tilde{\nu} \in IP^*(k)$ $(k \geq 2)$ can be the 'reduction' of a partition $\nu \in AIP(n) \setminus \{\nu_0\}$ $(n \geq k)$ (usually several such partition will do), while by inserting singleton blocks into different partitions of $IP^*(k)$ we always get different partitions of $AIP(n)$. It is easy to check that in order to get all the partitions of the set $AIP(n) \setminus \{\nu_0\}$ by the above method (by inserting the appropriate number of singletons), we have to take in account every $\tilde{\nu} \in IP^*(k)$ $(k \geq 2)$ and every possible insertions of singletons (of appropriate number). For the latter we have $\binom{n}{n-k} = \binom{n}{k}$ possibilities. With these in mind the left hand side of (2.2) can be written as

$$\lambda^n + \sum_{k=2}^{n} \binom{n}{k} \lambda^{n-k} \cdot \left(\sum_{\nu \in IP^*(k)} \prod_{V \in \nu} \gamma_{|V|}(\mu) \right)$$

(here λ^{n-k} comes from the $n-k$ singletons). Comparing this with the right hand side, we conclude $\gamma_n(\mu) = \gamma_n(\mu^0)$ for $n \geq 2$. $\square$

Corollary 2.1. *Let $\mu_1, \mu_2 \in \mathcal{P}^\infty$. Let $\hat{\mu} = \mu_1 \bullet \mu_2$ be the Fermi convolution of μ_1 and μ_2. Then $\hat{\mu} \in \mathcal{P}^\infty$ and*

$$\gamma_n(\hat{\mu}) = \gamma_n(\mu_1) + \gamma_n(\mu_2) \qquad \textit{for } n = 1, 2, 3, \ldots$$

Proof. As for probability measures with zero mean and with finite moments of all orders their Fermi cumulants are the same as their boolean cumulants, the statement of the corollary is direct consequence of Proposition 2.1. and the additivity of the boolean cumulants with respect to boolean convolutin. $\square$

3. The analytic theory of the Fermi convolution

For a probability measure μ its Cauchy transform G_μ is given by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z-x} \qquad \left(z \in \mathbb{C}^+\right),$$

while its self-energy[5] K_μ is

$$K_\mu(z) = z - \frac{1}{G_\mu(z)} \qquad \left(z \in \mathbb{C}^+\right).$$

Let us define the $\tilde{B}$-transform of $\mu \in \mathcal{P}^2$ by

$$\tilde{B}_\mu(z) = \lambda z + z K_{\mu^0}\left(\frac{1}{z}\right),$$

where λ is the mean of μ and μ^0 is the zero mean shift of μ. Since a measure $\mu \in \mathcal{P}^2$ is uniquely determined by its Cauchy transform G_μ, the same is true for $\tilde{B}_\mu$. Furthermore, $\tilde{B}_\mu$ can be written in the form

$$\tilde{B}_\mu(z) = \sum_{n=1}^{\infty} \gamma_n(\mu) z^n.$$

In the next theorem we show that for measures in $\mathcal{P}^2$ the $\tilde{B}$-transform behave additively with respect to the operation of Fermi convolution.

Theorem 3.1. *Let $\mu_1, \mu_2 \in \mathcal{P}^2$. Let $\mu = \mu_1 \bullet \mu_2$ be the Fermi convolution of μ_1 and μ_2. Then*

$$\tilde{B}_\mu(z) = \tilde{B}_{\mu_1}(z) + \tilde{B}_{\mu_2}(z).$$

Furthermore, $\mu \in \mathcal{P}^2$, and the mean of μ is the sum of the means of μ_1 and μ_2.

Proof. For probability measures of $\mathcal{P}_0^2$ the linearity of the $\tilde{B}$-transform follows immediately from the linearity of the self-energy with respect to the boolean convolution.

In the general case let λ_i be $m_1(\mu_i)$, and μ_i^0 be the zero mean shift of μ_i $(i = 1, 2)$. Since $\mu = \mu_1 \bullet \mu_2$ is the shift of $\mu_1^0 \bullet \mu_2^0$ by $\lambda = \lambda_1 + \lambda_2$, μ has mean λ, and

$$\begin{aligned}
\tilde{B}_\mu(z) &= (\lambda_1 + \lambda_2) z + z K_{\mu_1^0 \bullet \mu_2^0}\left(\frac{1}{z}\right) \\
&= \lambda_1 z + \lambda_2 z + z K_{\mu_1^0}\left(\frac{1}{z}\right) + z K_{\mu_2^0}\left(\frac{1}{z}\right) \\
&= \tilde{B}_{\mu_1}(z) + \tilde{B}_{\mu_2}(z).
\end{aligned}$$

$\mu \in \mathcal{P}^2$ is a consequence of Proposition 3.2. in Ref. 5. $\square$

Remark 3.1. The Fermi convolution is a commutative and associative operation on $\mathcal{P}^2$, with δ_0 as neutral element: $\mu \bullet \delta_0 = \mu$ for every $\mu \in \mathcal{P}^2$.

Examples. 1. $\delta_a \bullet \delta_b = \delta_{a+b}$ for all $a, b \in \mathbb{R}$.

2. As in the classical and the free case (but not the boolean case) $\mu \bullet \delta_a$ is the shift of μ by the amount a for all $\mu \in \mathcal{P}^2$, $a \in \mathbb{R}$. Therefore the zero-mean shift of a measure $\mu \in \mathcal{P}^2$ with mean λ is $\mu^0 = \mu \bullet \delta_{-\lambda}$.

Remark 3.2. As a consequence of the analogous property of the boolean case and commutativity and associativity of the Fermi convolution, all probability measures $\mu \in \mathcal{P}^2$ are infinitely divisible in the Fermi sense.

The classical, free and boolean Poisson distributions of parameter λ ($\lambda \geq 0$) are characterized by the requirement that their classical, free and boolean cumulants, respectively, all equal λ. By analogy we define the Fermi-Poisson distribution μ_λ of parameter λ by the requirement $\gamma_n(\mu_\lambda) = \lambda$ for all n.

This requirement leads to the $\tilde{B}$-transform

$$\tilde{B}_{\mu_\lambda}(z) = \sum_{n=1}^{\infty} \lambda z^n = \frac{\lambda z}{1-z} \qquad (\text{for } |z| < 1), \tag{3.1}$$

so the $\tilde{B}$-transform of the μ_λ^0 zero-mean shift of μ_λ is given by

$$\tilde{B}_{\mu_\lambda^0}(z) = \tilde{B}_{\mu_\lambda}(z) - \lambda z = \frac{\lambda z^2}{1-z}.$$

For the Cauchy transform of μ_λ^0 this gives

$$G_{\mu_\lambda^0}(z) = \frac{1}{z - \dfrac{\lambda}{z-1}},$$

hence

$$G_{\mu_\lambda}(z) = \int_{-\infty}^{\infty} \frac{d\mu_\lambda(x)}{z-x} = \int_{-\infty}^{\infty} \frac{d\mu_\lambda^0(x-\lambda)}{z-x} = G_{\mu_\lambda^0}(z-\lambda) = \frac{1}{z - \lambda - \dfrac{\lambda}{z-\lambda-1}},$$

which corresponds to the Fermi-Poisson distribution of Saitoh and Yoshida[4]:

$$\mu_\lambda = \left(\frac{1}{2} + \frac{1}{2\sqrt{4\lambda+1}}\right)\delta_{x_1}(x) + \left(\frac{1}{2} - \frac{1}{2\sqrt{4\lambda+1}}\right)\delta_{x_2}(x), \tag{3.2}$$

where

$$x_1 = \lambda + \frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}, \qquad x_2 = \lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}.$$

This is remarkebly different from the boolean Poisson distribution[5]. For comparision we give the first few moments of the Fermi and the boolean Poisson distributions of parameter λ :

$$\begin{aligned} m_0\left(\mu_\lambda^{Fermi}\right) &= 1 \\ m_1\left(\mu_\lambda^{Fermi}\right) &= \lambda \\ m_2\left(\mu_\lambda^{Fermi}\right) &= \lambda^2 + \lambda \\ m_3\left(\mu_\lambda^{Fermi}\right) &= \lambda^3 + 3\lambda^2 + \lambda \\ m_4\left(\mu_\lambda^{Fermi}\right) &= \lambda^4 + 6\lambda^3 + 5\lambda^2 + \lambda \end{aligned} \qquad \begin{aligned} m_0\left(\mu_\lambda^{boolean}\right) &= 1 \\ m_1\left(\mu_\lambda^{boolean}\right) &= \lambda \\ m_2\left(\mu_\lambda^{boolean}\right) &= \lambda^2 + \lambda \\ m_3\left(\mu_\lambda^{boolean}\right) &= \lambda^3 + 2\lambda^2 + \lambda \\ m_4\left(\mu_\lambda^{boolean}\right) &= \lambda^4 + 3\lambda^3 + 3\lambda^2 + \lambda \end{aligned}$$

As one would expect, for $n \geq 3$ the moments m_n of the Fermi-Poisson distribution are bigger than those of the boolean one (of the same parameter).

The following limit theorem gives another reason to call the above measure μ_λ Fermi-Poisson distribution.

Theorem 3.2. (The fermionic Poisson limit theorem) *For $\lambda \geq 0$ let*

$$\mu_n = \left(1 - \frac{\lambda}{n}\right)\delta_0 + \frac{\lambda}{n}\delta_1 \qquad (n \in \mathbb{N}).$$

Then the weak limit of the following n-fold Fermi convolution is the Fermi-Poisson distribution of parameter λ:

$$w - \lim_{n \longrightarrow \infty} \mu_n \bullet \ldots \bullet \mu_n = \mu_\lambda,$$

where μ_λ is given by (3.2).

Proof. Calculating the $\tilde{B}$-transform of μ_n from the Cauchy transform and $\tilde{B}$-transform of its zero mean shift μ_n^0 one gets

$$\tilde{B}_{\mu_n}(z) = \frac{\lambda}{n}z + z\frac{n\lambda - \lambda^2}{n^2\frac{1}{z} - n^2 + 2n\lambda},$$

while the $\tilde{B}$-transform of the n-fold convolution of μ_n is

$$\tilde{B}(z) = n\tilde{B}_{\mu_n}(z) = \lambda z + z\frac{n\lambda - \lambda^2}{n\frac{1}{z} - n + 2\lambda} \longrightarrow \frac{\lambda z}{1-z} \qquad (\text{as } n \longrightarrow \infty).$$

Comparing this with (3.1) completes the proof. □

Remark 3.3. The classical, free and boolean centered Gaussian distributions (i.e. with zero mean and variance σ^2) are characterized by the requirement that the corresponding cumulants are all zero expect the second ones, which equal σ^2. As for measures with zero mean the Fermi cumulants are the same as the boolean cumulants, the 'Fermi Gaussian distribution' is the same as the boolean one (the Bernoulli law $\frac{1}{2}(\delta_{-\sigma} + \delta_{\sigma})$). For the same reason the 'fermionic central limit theorem' is an immediate consequence of the boolean one. The characterisation theorems of Cramer and Marcinkiewicz - as they also concern measures with zero mean - also follow from their boolean analogues[5].

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