ON THE POWERS OF VOICULESCU‘S CIRCULAR ELEMENT

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Abstract. The main result of the paper is that for a circular element c in a C^*-probability space, $\left(c^n, c^{n^*}\right)$ is an R-diagonal pair in the sense of Nica and Speicher for every $n = 1, 2, 3 \ldots$. The coefficients of the R-series are also determined explicitly, and are found to be identical to the generalized Catalan numbers of parameter $n-1$.

0. Introduction

Circular elements of a C^*-probability space play a central role in Voiculescu‘s free probability theory, and they provide one of the most frequently used $*$-distributions (see e.g. [9]). From our point of view the important characterization of non-commutative random variables are based on the R-series of certain pairs. The R-series of a pair is an infinite power series of two non-commuting indeterminates, and can be taken as the free analogue of the Fourier transform [5]. Now an element c is circular, if the R-series of the pair (c, c^*) is of the form

$$R_{(c,c^*)}(X_1, X_2) = X_1X_2 + X_2X_1.$$

As a natural extension of the notion of circular element, we will call a pair (a_1, a_2) of non-commutative random variables a circular pair, if its R-series is

$$R_{(a_1,a_2)}(X_1, X_2) = X_1X_2 + X_2X_1.$$

In [6] Nica and Speicher introduced the notion of R-diagonal pairs in the following way. The pair (x, y) of non-commutative random variables is R-diagonal, if the R-series of (x, y) is of the form

$$R_{(x,y)}(X_1, X_2) = \sum_{k=1}^{\infty} \alpha_k (X_1X_2)^k + \sum_{k=1}^{\infty} \alpha_k (X_2X_1)^k$$

for some sequence $\{\alpha_k\}_{k=1}^{\infty}$ of complex coefficients. The aim of this definition was to establish a class of pairs of non-commutative random variables, which contains the pair (c, c^*) obtained from a circular element c, and (u, u^*) obtained from a Haar unitary element u. Haar unitary elements form another important type of

Date. November 3, 1998.

non-commutative random variables [8], characterized by the R-series of the pair (u, u^*) having the form

$$R_{(u,u^*)}(X_1, X_2) = \sum_{k=1}^{\infty} (-1)^{k-1} c_{k-1} (X_1 X_2)^k + \sum_{k=1}^{\infty} (-1)^{k-1} c_{k-1} (X_2 X_1)^k ,$$

where $c_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number [3].

The main result of this paper (stated as Theorem 3.2) is that for any circular pair (b_1, b_2), the pair (b_1^n, b_2^n) is R-diagonal for every $n \geq 1$. The sequence of coefficients $\{\alpha_k\}_{k=1}^{\infty}$ of the R-series of (b_1^n, b_2^n) is also determined and found to be identical to $\left\{ {}_{(n-1)}c_k \right\}_{k=1}^{\infty}$, the sequence of the generalized Catalan numbers of parameter $(n-1)$ [3] given by the formula

$${}_{(n-1)}c_k = \frac{1}{(n-2)\,k+1}\binom{(n-1)\,k}{k} \quad (k = 0, 1, 2, \ldots; \quad n = 2, 3, 4, \ldots).$$

The structure of the paper is the following. In Section 1 we review the necessary elements of non-commutative probability theory. In Section 2 we state and prove some combinatorial lemmas concerning rather special types of partitions of certain sets. These lemmas are used for the proof of our main result, Theorem 3.2, which can be found in Section 3, along with some remarks and consequences.

1. Preliminaries

In this section we briefly review some basic elements of non-commutative probability theory used throughout the paper. For further details see e.g. [2, 10] and references listed there.

If $\mathcal{A}$ is a unital algebra over $\mathbf{C}$ and $\varphi : \mathcal{A} \to \mathbf{C}$ is a linear functional normalized by $\varphi(\mathbf{1}) = 1$, then we call the pair $(\mathcal{A}, \varphi)$ a *non-commutative probability space*. In this case the elements of $\mathcal{A}$ are called *non-commutative random variables* and for $a \in \mathcal{A}$, $\varphi(a^n)$ is called the n^{th} *moment* of a. If we require in addition that $\mathcal{A}$ is a C^*-algebra and φ is positive, then $(\mathcal{A}, \varphi)$ is called a *C^*-probability space*.

In the rest of this section $(\mathcal{A}, \varphi)$ stands for a non-commutative probability space.

We denote the set of all infinite power series of the non-commuting indeterminates $X_1, X_2, \ldots, X_k$ by $C_\infty \langle X_1, X_2, \ldots X_k \rangle$. For $F \in C_\infty \langle X_1, X_2, \ldots X_k \rangle$ $[coef\,(i_1, i_2, \ldots, i_m)]\,(F)$ means the coefficient of $X_{i_1} X_{i_2} \ldots X_{i_m}$ in F ($m \geq 1$ and $1 \leq i_1, i_2, \ldots, i_m \leq k$). If $V = \{h_1, h_2, \ldots, h_r\} \subset \{1, 2, \ldots, m\}$ for some $h_1 < h_2 < \cdots < h_r$, then let $(i_1, i_2, \ldots i_m | V)$ mean $(i_{h_1}, i_{h_2}, \ldots, i_{h_r})$.

Let $a_1, a_2, \ldots, a_k \in \mathcal{A}$. $M_{(a_1, a_2, \ldots, a_k)} \in C_\infty \langle X_1, X_2, \ldots X_k \rangle$ is called the *moment generating series* of the k-tuple $(a_1, a_2, \ldots, a_k)$, if

$$[coef\,(i_1, i_2, \ldots, i_m)]\left(M_{(a_1, a_2, \ldots, a_k)}\right) = \varphi\,(a_{i_1} a_{i_2} \ldots a_{i_m})$$

for $m \geq 1$ and $1 \leq i_1, i_2, \ldots, i_m \leq k$ [7].

Let $\mathcal{E}_k$ consist of the formal infinite sums

$$T = \sum_{m,n=0}^{\infty} c_{i_1, i_2, \ldots, i_m; j_1, j_2, \ldots, j_n} Z_{i_1} Z_{i_2} \ldots Z_{i_m} Y_{j_1} Y_{j_2} \ldots Y_{j_n} ,$$

where $c_{i_1, i_2, \ldots, i_m; j_1, j_2, \ldots, j_n} \in \mathbf{C}$, $i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_n \in \{1, 2, 3, \ldots, k\}$, $Z_1, Z_2, \ldots, Z_k; Y_1, Y_2, \ldots, Y_k$ are non-commuting formal symbols, and there exists an $N \in \mathbf{N}$ such that $c_{i_1, i_2, \ldots, i_m; j_1, j_2, \ldots, j_n} = 0$ whenever $n > N$. The multiplication of the sums are governed by the computational rules $Y_i Z_i = 1$ and $Y_i Z_j = 0$ when $i \neq j$, while we add the elements of $\mathcal{E}_k$ in the usual way. The condition on the coefficients makes sure that in the multiplication of two infinite sums we have to add only finitely many numbers. On $\mathcal{E}_k$ we define a normalized linear functional ω by $\omega(T) = c$, where c is the constant term of the infinite sum T.

Let $a_1, a_2, \ldots, a_k \in \mathcal{A}$. $R_{(a_1,a_2,\ldots,a_k)} \in C_\infty \langle X_1, X_2, \ldots X_k \rangle$ is called the *R-series* of the k-tuple $(a_1, a_2, \ldots, a_k)$, if for $T \in \mathcal{E}_k$,

$$T = 1 + \sum_{m=0}^{\infty} [coef\,(i_1, i_2, \ldots, i_m)] \left(R_{(a_1,a_2,\ldots,a_k)}\right) Z_{i_m} Z_{i_{m-1}} \ldots Z_{i_1},$$

$\omega(Y_{i_1} T Y_{i_2} T \ldots Y_{i_l} T) = \varphi(a_{i_1} a_{i_2} \ldots a_{i_l})$ for $l \geq 1$ and $1 \leq i_1, i_2, \ldots i_l \leq k$ [5].

It is important, that for any k-tuple $(a_1, a_2, \ldots, a_k)$ the moment generating series $M_{a_1,a_2,\ldots,a_k}$ and the R-series $R_{a_1,a_2,\ldots,a_k}$ exist and are uniquely determined, moreover, according to the so-called *moment-cumulant formula* [7]

(1.1)

$$[coef\,(i_1, \ldots, i_m)] \left(M_{(a_1,\ldots,a_k)}\right) = \sum_{\pi \in NC(k)} \prod_{V \in \pi} [coef\,(i_1, \ldots, i_m | V)] \left(R_{(a_1,\ldots,a_k)}\right),$$

where the summation is over all non-crossing partitions of $\{1, 2, \ldots, k\}$ and the product is over the blocks of the partition π. (The notions concerning non-crossing partitions are described in the beginning of Section 2.) Note that the momentum-cumulant formula is invertible, that is the coefficients of the R-series of any k-tuple can also be expressed uniquely by the coefficients of the moment generating series of the k-tuple (though that formula is somewhat more complicated (see e.g. [6])).

Let $a_1, a_2 \in \mathcal{A}$. The pair (a_1, a_2) is an *R-diagonal pair*, if the R-series of the pair is given by $R_{(a_1,a_2)}(X_1, X_2) = \sum_{k=1}^{\infty} \alpha_k (X_1 X_2)^k + \sum_{k=1}^{\infty} \alpha_k (X_2 X_1)^k$ for some sequence $\{\alpha_k\}_{k=1}^{\infty}$ of complex coefficients [6].

An element c of a C^*-probability space $(\mathcal{A}, \varphi)$ is called *circular*, if the R-series of the pair (c, c^*) is given by $R_{(c,c*)}(X_1, X_2) = X_1 X_2 + X_2 X_1$.

2. Combinatorial background

For a finite set S, $\pi = \{V_1, V_2, \ldots, V_p\}$ is a *partition* of S, if the (non empty) sets V_i are disjoint and the union of them is S. The sets V_i are called the *blocks* of the partition. The number of blocks in a partition π is denoted by $|\pi|$, the number of elements in a block V_i is denoted by $|V_i|$. For a linearly ordered set S we call the partition π of S *crossing*, if in π there are two blocks $V_i \neq V_j$ and elements $v_1, v_2 \in V_i$, $w_1, w_2 \in V_j$ such that $v_1 < w_1 < v_2 < w_2$. A partition of a linearly ordered set is called *non-crossing*, if it is not crossing. A partition π of S is called *pair-partition*, if every block of π contains exactly two elements of S. The set of all non-crossing partitions and that of all non-crossing pair-partitions of the set $\{1, 2, \ldots, k\}$ is denoted by $NC(k)$ and $NCP(k)$, respectively.

For further details of the theory of partitions see e.g. [1, 4] and references listed there.

Definition 2.1. *Let $i:\mathbf{N}\to\{1,2\}$ be an arbitrary function and let $k,n\in\mathbf{N}$. For any fixed function i let*

$$H_{i(1),i(2),\dots,i(k)} = \left\{ \begin{array}{c} \pi\in NC(k)\,|\text{for every } V=\{p_1,p_2,\dots,p_{|V|}\}\in\pi \\ |V| \text{ is even and } i(p_j)\neq i(p_{j+1}) \text{ for } j=1,2,\dots,|V|-1 \end{array} \right\},$$

$$_{(n)}\tilde{H}_{i(1),i(2),\dots,i(k)} = \left\{ \begin{array}{c} \nu\in NCP(nk)\,|\text{for every } V=\{p,q\}\in\nu \\ i\left(\left]\frac{p}{n}\right[\right)\neq i\left(\left]\frac{q}{n}\right[\right) \end{array} \right\},$$

where $]x[$ means the smallest integer not smaller than x.

*Moreover we will call a partition π of $\{1,2,\dots,k\}$an n-**partition**, if every block of π contains exactly n elements. We denote the set of all non-crossing n-partitions of $\{1,2,\dots,k\}$ by $NC_n(k)$.*

The cardinality of $NC_n(ln)$ is known to be the l^{th} generalized Catalan number of parameter n given by $_nc_l=\frac{1}{(n-1)l+1}\binom{n\cdot l}{l}$ $(n,l\in\mathbf{N})$ (see. e.g. [8]). Concerning the generalized Catalan numbers see also [3].

Lemma 2.2. Let the function $i:\mathbf{N}\to\{1,2\}$ be the following

$$i(j)=\left\{ \begin{array}{ll} 1 & \text{if} \quad j \text{ is odd and} \\ 2 & \text{if} \quad j \text{ is even,} \end{array} \right. \quad \text{or} \quad i(j)=\left\{ \begin{array}{ll} 2 & \text{if} \quad j \text{ is odd and} \\ 1 & \text{if} \quad j \text{ is even.} \end{array} \right.$$

Then the cardinality of $_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$ equals the cardinality of $NC_{n+1}(k(n+1))$.

Proof. We prove our statement by giving a bijection between the sets $_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$ and $NC_{n+1}(k(n+1))$.

First let b be the function defined by

$$\begin{array}{rcl} b & : & \{1,2,\dots,2k\cdot n\}\to\{1,2,\dots,k(n+1)\} \\ s & \mapsto & b(s)=\left\{ \begin{array}{ll} \frac{j}{2}(n+1)+l & \text{if} \quad j \quad \text{is even and} \\ \frac{j+1}{2}(n+1) & \text{if} \quad j \quad \text{is odd,} \end{array} \right. \end{array}$$

where $j=\left[\frac{s-1}{n}\right]$ and $l=s-jn$, where $[x]$ stands for the integer part of x.

For any $\nu\in{}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$ let $T(\nu)$ be the partition of $\{1,2,\dots,k(n+1)\}$ whose blocks are defined the following way.
(1) If $\{s_1,s_2\}\in\nu$, then let $b(s_1)$ and $b(s_2)$ be in the same block of $T(\nu)$.
(2) If r_1 is in the same block as r_2, and r_2 is in the same block as r_3, then let r_1 and r_3 be in the same block, too (requirement of transitivity).

Condition (1) and (2) clearly defines an equivalence relation, which gives this way a partition of $\{1,2,\dots,k\cdot(n+1)\}$.

We show that $T(\nu)\in NC_{n+1}(k\cdot(n+1))$ for every $\nu\in{}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$.

From the definition of $_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$ for our special function $i:\mathbf{N}\to\{1,2\}$ it follows that if $\{s_1,s_2\}\in\nu$, then one of $j_1=\left[\frac{s_1-1}{n}\right]$, $j_2=\left[\frac{s_2-1}{n}\right]$ is odd, while the other is even. Let j_1 be odd, for example. Then for every $s_l\in\{1,2,\dots,2kn\}$ of the form $s_l=j_1n+l$ $(l=1,2,\dots,n)$ $b(s_l)=\frac{j_1+1}{2}(n+1)$. Let us take the block V of $T(\nu)$ that contains $\frac{j_1+1}{2}(n+1)\in\{1,2,\dots,k(n+1)\}$. Let the pair of s_l be r_l in ν (that is let $r_l\in\{1,2,\dots,2kn\}$ such that $\{r_l,s_l\}\in\nu$ - such an r_l exists uniquely for all $l=1,2,\dots,n$ as ν is a pair-partition). As $\left[\frac{s_l-1}{n}\right]=j_1$ is odd, $j_l=\left[\frac{r_l-1}{n}\right]$ must be even. *Let* the "decomposition" of r_l be $r_l=j_ln+h_l$. Then because of (1) $b(r_l)=\frac{j_l}{2}(n+1)+h_l$ is also in V for every $l=1,2,\dots,n$. This means that V

has at least $n+1$ elements, namely $b(s_l)$ (which is the same for every l) and $b(r_l)$ (which is different for different l). Taking any element x of $\{1,2,\ldots,2kn\}$ for which $b(x)=\frac{j_l+1}{2}(n+1)$ or $b(x)=\frac{j_l+1}{2}(n+1)+h_l$ $(l=1,2,\ldots,n)$, the element $b(y)$ defined by the pair of x is already in V because of (1), so (2) cannot put any other element in V. This means that V has exactly $n+1$ elements. As the same argument works for every block of $T(\nu)$, we have that in $T(\nu)$ every block consist of exactly $n+1$ elements. It is also clear from the above argument, that in every block V in $T(\nu)$ there is exactly one element of the form $\frac{j}{2}(n+1)$ (let us call such an element b-odd); while the rest is of the form of $\frac{j}{2}(n+1)+l$ for some $l=1,2,\ldots,n$ (we will call this type of elements b-even).

Let us suppose, that for some $\nu\in{}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$ $T(\nu)$ is crossing. Then there are $V,U\in T(\nu)$ $(V\neq U)$, $v_1,v_2\in V$, $u_1,u_2\in U$ such that $v_1<u_1<v_2<u_2$. If one of v_1 and v_2 and one of u_1 and u_2 is b-odd, then there are blocks $\{s_V,r_V\}$ and $\{s_U,r_U\}$ of ν such that $b(s_V)=v_1; b(r_V)=v_2; b(s_U)=u_1$ and $b(r_U)=u_2$, where $s_V<s_U<r_V<r_U$. This means that ν is also crossing, which contradicts to the fact that $\nu\in_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$. If among v_1,v_2,u_1,u_2 there is only one b-odd (let us say one of v_1 and v_2), then, if the (uniquely existing) b-odd element u_0 of the block U is smaller than v_1 or bigger than v_2, then instead of u_2 we can take u_0 to get the "crossing elements" v_1,v_2,u_1,u_0 (if $u_0<v_1$ then $u_0<v_1<u_1<v_2$ hold, if $u_0>v_2$ then $v_1<u_1<v_2<u_0$ hold); while if u_0 is between v_1 and v_2, then v_1,v_2,u_0,u_2 will be good crossing elements containing two b-odd elements, leading to the same contradiction. If there is no b-odd element among v_1,v_2,u_1,u_2, then, if $u_0<v_1$ or $u_0>v_2$ then v_1,v_2,u_1,u_0, while if $v_1<u_0<v_2$ then v_1,v_2,u_0,u_2 are good crossing elements with one b-odd among them, leading again to the same contradiction. This finally gives that for every $\nu\in{}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$ $T(\nu)$ is non-crossing, which proves that $T(\nu)\in NC_{n+1}(k(n+1))$ for every $\nu\in{}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$.

Next we show that T is injective, that is for $\nu_1\neq\nu_2$ $T(\nu_1)\neq T(\nu_2)$.

Let $\nu_1,\nu_2\in{}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$. We will call $s\in\{1,2,\ldots,2kn\}$ j-odd or j-even, if in its decomposition $s=jn+l$ j is odd, or even, respectively. The definition of ${}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$ shows that in every $\nu\in{}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$ every block of ν contains exactly one j-odd and one j-even element. So if $\nu_1\neq\nu_2$, then there is at least one $s\in\{1,2,\ldots,2kn\}$ j-even element for which its pair in ν_1 and that in ν_2 are different. *Let* us suppose, that $\{s,r_1\}\in\nu_1$; $\{s,r_2\}\in\nu_2$ and that $r_1=j_1n+l_1$; $r_2=j_2n+l_2$, where $j_1\neq j_2$. In this case in $T(\nu_1)$ $b(s)$ is in the same block as $b(r_1)=\frac{j_1+1}{2}(n+1)$, while in $T(\nu_2)$ $b(s)$ is in the same block as $b(r_2)=\frac{j_2+1}{2}(n+1)\neq b(r_1)$. As $b(r_1)$ and $b(r_2)$ are both b-odd, they cannot be in the same block, so $T(\nu_1)$ and $T(\nu_2)$ are different.

Next we show that if there is no $s\in\{1,2,\ldots,2kn\}$ j-even such that in the decomposition of its pair in ν_1 and in ν_2 $j_1=\left[\frac{r_1-1}{n}\right]$ and $j_2=\left[\frac{r_2-1}{n}\right]$ are different, then ν_1 must be identical to ν_2. In this case the set $\{s_1,s_2,\ldots,s_n\}$ of the pairs of the n j-odd elements of the form $r_l=jn+l$ $(l=1,2,\ldots,n)$ in ν_1 is the same as that in ν_2, for every $j=1,2,\ldots,2k-1$. But, as it can easily be checked, there is only one pair-partition of the subset $\{s_1,s_2,\ldots,s_n,r_1,r_2\ldots,r_n\}\subset\{1,2,\ldots,2kn\}$ which is non-crossing and in which every pair contains one s_l and one $r_{\bar{l}}$, i.e. which can be a "subpartition" of a $\nu\in{}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(2k)}$. This means that all the blocks of ν_1 are the same as the blocks of ν_2, proving the injectivity of T.

Finally we show that T is surjective, that is for every $\pi\in NC_{n+1}(k(n+1))$ there exists a $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$ such that $T(\nu) = \pi$. This we prove by induction in k.

For $k = 1$ the only element of $NC_{(n+1)}(n+1)$ is $\pi = \{1, 2, \dots, n+1\}$. The partition $\nu = \{\{1, 2n\}, \{2, 2n-1\}, \dots, \{n, n+1\}\} \in {}_{(n)}\tilde{H}_{i(1),i(2)}$ clearly satisfies $T(\nu) = \pi$. Let us suppose that T is surjective for k. We show that this implies the same for $k+1$. As $\pi \in NC_{(n+1)}((k+1)(n+1))$ is non-crossing and every block of π contains exactly $n+1$ elements, there is at least one block of π that contains the elements $\{p, p+1, p+2, \dots, p+n\}$ for some $p \in \{1, 2, \dots, k(n+1)\}$. Among these elements there is exactly one which is of the form of $j(n+1)$ for some $j = 1, 2, \dots, k+1$. We denote this element by q. Let us eliminate the block $\{p, p+1, p+2, \dots, p+n\}$ of π, this way we get a shorter partition $\tilde{\pi} \in NC_{(n+1)}(k(n+1))$. According to our assumption there exists a $\tilde{\nu} \in {}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$ such that $\tilde{\pi} = T(\tilde{\nu})$. Let us take the set $\{1, 2, \dots, 2kn\}$ with its partition $\tilde{\nu}$. Let $s = \left[\frac{p}{n+1}\right] 2n + \left(p - \left[\frac{p}{n+1}\right](n+1)\right)$ if $q \neq p$ and $s = \frac{p}{n+1} 2n - (n-1)$ if $q = p$. This assures that $b(s) = p$. Now let us put $2n$ extra elements between $s-1$ and s, and renumber our set so that we get the set $\{1, 2, \dots, 2(k+1)n\}$. Now let ν be the following pair-partition of this set. For the elements that came from our original set $\{1, 2, \dots, 2kn\}$, their pairs in ν are defined by $\tilde{\nu}$, while for the $2n$ new elements we define their pairs by $\{s_{h_0,j}, s_{h_0-j}\}$ for $j = 1, 2, \dots, h_0$ and $\left\{s_{h_0,n-\tilde{j}}, s_{h_0+1+\tilde{j}}\right\}$ for $\tilde{j} = 0, 1, \dots, n-h_0-1$, where $h_0 = q-p$; $s_h = \left[\frac{p+h}{n+1}\right] 2n + \left(p + h - \left[\frac{p+h}{n+1}\right](n+1)\right)$ for $h = 0, 1, \dots, n$; $h \neq h_0$, and $s_{h_0,l} = 2\left(\frac{p+h_0}{n+1} - 1\right) n + l$ $(l = 1, 2, \dots, n)$.

As the given pair-partition of the $2n$ new elements is non-crossing and $\tilde{\nu} \in {}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)}$ is also non-crossing, ν is non-crossing. Moreover (as it can be verified easily) the pair of every j-odd element of $\{1, 2, \dots, 2(k+1)n\}$ is j-even, which assures that $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2(k+1))}$. In $T(\nu)$ the block that contains q is clearly $\{p, p+1, p+2, \dots, p+n\}$, while all the other blocks of $T(\nu)$ are defined by $\tilde{\nu}$ (taken as a subpartition of ν), so $T(\nu) = \pi$, which means that T is surjective for $k+1$, and so for all $k = 1, 2, 3, \dots$.

We conclude that T is a ${}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(2k)} \to NC_{n+1}(k \cdot (n+1))$ bijection, which proves our statement. $\boxdot$

Lemma 2.3. *For any function $i : N \to \{1, 2\}$ and for all $k \in \mathbf{N}$, $n = 2, 3, 4, \dots$*

$$\#_{(n)}\tilde{H}_{i(1),i(2),\dots,i(k)} = \sum_{\pi \in H_{i(1),i(2),\dots,i(k)}} \prod_{V \in \pi} {}_{(n-1)}C_{\frac{|V|}{2}}$$

Proof. For any $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(k)}$ let $L(\nu)$ be the partition of $\{1, 2, \dots, k\}$ whose blocks are defined the following way.
(1) If $\{s_1, s_2\} \in \nu$, then let $p_1 = \left]\frac{s_1-1}{n}\right[$ and $p_2 = \left]\frac{s_2-1}{n}\right[$ be in the same block of $L(\nu)$, where $]x[$ means the smallest integer not smaller than x.
(2) If r_1 is in the same block as r_2, and r_2 is in the same block as r_3, then let r_1 and r_3 be in the same block, too (requirement of transitivity).

Condition (1) and (2) clearly defines an equivalence relation, which gives this way a partition of $\{1, 2, \dots, k\}$.

Let $s_l(p) = (p-1)n + l$ $(l = 1, 2, \dots, n, p = 1, 2, \dots k)$. We show that for every $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\dots,i(k)}$ if $V = \{p_1, p_2, \dots, p_{|V|}\}$ $(p_1 < p_2 < \dots < p_{|V|})$ is a block of

$L(\nu)$, then $\{s_n(p_j), s_1(p_{j+1})\} \in \nu$ for $j = 1, 2, \ldots |V-1|$, and $\{s_n(p_{|V|}), s_1(p_1)\} \in \nu$. As ν is a non-crossing pair-partition and as for $\{s_1, s_2\} \in \nu$ $i\left(\left]\frac{s_1}{n}\right[\right) \neq i\left(\left]\frac{s_2}{n}\right[\right)$, the pair of $s_1(p)$ must be an element $s_n(q)$ for some $q = 1, 2, \ldots, k$, and the pair of $s_n(p)$ must be an element $s_1(\tilde{q})$ for some $\tilde{q} = 1, 2, \ldots, k$, for any $p = 1, 2, \ldots, k$. (Otherwise the number of elements s between $s_1(p)$ and its pair, for which $i\left(\left]\frac{s}{n}\right[\right) = 1$ would differ from the number of those for which $i\left(\left]\frac{\tilde{s}}{n}\right[\right) = 2$, so ν could not belong to ${}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$.) Now, if the pair of $s_n(p_j)$ is not $s_1(p_{j+1})$, then p_j and p_{j+1} could not be in the same block of $L(\nu)$ (again, because ν is non-crossing).

The pair of the element $s_n(p_{|V|})$ can not be $s_1(p)$ for $p > p_{|V|}$ or $p < p_{|V|}$, because in this case $p_{|V|}$ would not be the largest, or p_1 would not be the smallest element in V. But if the pair of $s_n(p_{|V|})$ is $s_1(p)$ for some $p_1 < p < p_{|V|}$, then - as ν is non-crossing - p_1 and $p_{|V|}$ could not be in the same block of $L(\nu)$. From this follows that the pair of $s_n(p_{|V|})$ is necessarily $s_1(p_1)$.

After this we show that for every $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$ $L(\nu)$ is non-crossing. Let us suppose, that for some $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$ $L(\nu)$ is crossing. Then there are $V, U \in L(\nu)$ $(V \neq U)$, $v_1, v_2 \in V$, $u_1, u_2 \in U$ such that $v_1 < u_1 < v_2 < u_2$. Let $\tilde{v}_1$ be the largest element of V which is still smaller than u_1; let $\tilde{v}_2$ be the smallest element of V which is still bigger than u_1; let $\tilde{u}_1$ be the largest element of U which is still smaller than $\tilde{v}_2$, and let $\tilde{u}_2$ be the smallest element of U which is still bigger than $\tilde{v}_2$ (such elements clearly exist). This implies that there is no element of V between $\tilde{v}_1$ and $\tilde{v}_2$ nor element of U between $\tilde{u}_1$ and $\tilde{u}_2$, and $\tilde{v}_1 < \tilde{u}_1 < \tilde{v}_2 < \tilde{u}_2$ still hold. But, as we saw it above, in this case $\{s_n(\tilde{v}_1), s_1(\tilde{v}_2)\} \in \nu$ and $\{s_n(\tilde{u}_1), s_1(\tilde{u}_2)\} \in \nu$. As $s_n(\tilde{v}_1) < s_n(\tilde{u}_1) < s_1(\tilde{v}_2) < s_1(\tilde{u}_2)$, ν would also be crossing, which contradicts to $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$.

Next we show that for every $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$ and for every block $V = \{p_1, p_2, \ldots, p_{|V|}\}$ $(p_1 < p_2 < \cdots < p_{|V|})$ of $L(\nu)$ $|V|$ is even and $i(p_j) \neq i(p_{j+1})$ $(j = 1, 2, \ldots, |V|-1)$. We saw above that $\{s_n(p_j), s_1(p_{j+1})\} \in \nu$, from which $i(p_j) = i\left(\left]\frac{s_n(p_j)-1}{n}\right[\right) \neq i\left(\left]\frac{s_1(p_{j+1})-1}{n}\right[\right) = i(p_{j+1})$ follows, while $\{s_n(p_{|V|}), s_1(p_1)\}$ $\in \nu$ gives $i(p_1) = i\left(\left]\frac{s_1(p_1)-1}{n}\right[\right) \neq i\left(\left]\frac{s_n(p_{|V|})-1}{n}\right[\right) = i(p_{j+1})$, what proves our statement. This means that $L(\nu) \in H_{i(1),i(2),\ldots,i(k)}$ for every $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$.

We show that L is surjective, that is for any $\pi \in H_{i(1),i(2),\ldots,i(k)}$ there exists (several) $\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$ such that $L(\nu) = \pi$. Let $\pi = \{V_1, V_2, \ldots, V_{|\pi|}\}$, and let $V_j = \left\{p_1^{(j)}, p_2^{(j)}, \ldots, p_{|V_j|}^{(j)}\right\}$ $\left(p_1^{(j)} < p_2^{(j)} < \cdots < p_{|V_j|}^{(j)}, j = 1, 2, \ldots |\pi|\right)$. Let ν be the pair-partition of $\{1, 2, \ldots, kn\}$ which consists of the blocks $\left\{s_1\left(p_1^{(j)}\right), s_n\left(p_{|V_j|}^{(j)}\right)\right\}$, $\left\{s_n\left(p_m^{(j)}\right), s_1\left(p_{m+1}^{(j)}\right)\right\}$ and $\left\{s_l\left(p_{2t-1}^{(j)}\right), s_{n+1-l}\left(p_{2t}^{(j)}\right)\right\}$, where $j = 1, 2, \ldots |\pi|$, $m = 1, 2, \ldots, |V_j| - 1$, $t = 1, 2, \ldots, \frac{|V_j|}{2}$ and $l = 2, 3, \ldots, n-1$. This pair-partition ν is clearly an element of ${}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(k)}$, and as one can easily see $L(\nu) = \pi$.

Note, that the blocks $\left\{s_1\left(p_1^{(j)}\right), s_n\left(p_{|V_j|}^{(j)}\right)\right\}$ and $\left\{s_n\left(p_m^{(j)}\right), s_1\left(p_{m+1}^{(j)}\right)\right\}$ $(m = 1, 2, \ldots, |V_j| - 1,\ j = 1, 2, \ldots, |\pi|)$ of ν already defines all the blocks of $L(\nu)$; and the fact that ν is non-crossing assures that the rest of the blocks of ν do not define any other elements into any of the blocks of $L(\nu)$. This means that on the elements $\left\{s_l\left(p_t^{(j)}\right)\right\}$ (where $l = 2, 3, \ldots, n-1$; $t = 1, 2, \ldots, |V_j|$ and $j =$

$1, 2, \ldots, |\pi|$) any other non-crossing pair-partition $\tilde{\nu}$ - given in such a way that if $\{s_1, s_2\} \in \tilde{\nu}$, then $i\left(\left]\frac{s_1-1}{n}\right[\right) \neq i\left(\left]\frac{s_2-1}{n}\right[\right)$ - satisfies $L(\tilde{\nu}) = \pi$. Taking in account that for any $\pi \in H_{i(1),i(2),\ldots,i(k)}$, for every block $V_j = \left\{p_1^{(j)}, p_2^{(j)}, \ldots, p_{|V_j|}^{(j)}\right\}$ of π $i\left(p_m^{(j)}\right) \neq i\left(p_{m+1}^{(j)}\right)$, and that the subpartitions of $\tilde{\nu}$ that define different blocks of $\pi = L(\tilde{\nu})$ are independent from each other, Lemma 2.2. completes the proof. $\boxdot$

3. Results

Definition 3.1. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $b_1, b_2 \in \mathcal{A}$. We will call the pair (b_1, b_2) a **circular pair**, if its R-series is $R_{(b_1,b_2)}(X_1, X_2) = X_1X_2 + X_2X_1$.*

Theorem 3.2. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, $b_1, b_2 \in \mathcal{A}$, and let (b_1, b_2) be a circular pair. Then the pair (b_1^n, b_2^n) is R-diagonal for every $n = 2, 3, 4, \ldots$. Moreover, the R-series of (b_1^n, b_2^n) is given by*

$$R_{\left(b_1^n, b_2^n\right)}(X_1, X_2) = \sum_{k=1}^{\infty} {}_{(n-1)}c_k \, (X_1X_2)^k + \sum_{k=1}^{\infty} {}_{(n-1)}c_k \, (X_2X_1)^k,$$

where ${}_{(n-1)}c_k = \frac{1}{(n-2)k+1}\binom{(n-1)k}{k}$ is the k^{th} generalized Catalan number of parameter $(n-1)$.

Proof. In the following we will use the notations of Definition 2.1.

For an arbitrary R-diagonal pair (a_1, a_2) its R-series is of the form $R_{(a_1,a_2)}(X_1, X_2) = \sum_{k=1}^{\infty} \alpha_k (X_1X_2)^k + \sum_{k=1}^{\infty} \alpha_k (X_2X_1)^k$ for some sequence $\{\alpha_k\}_{k=1}^{\infty}$ of complex coefficients. According to the moment-cumulant formula (1.1) this implies

(3.1)
$$[coef\,(i(1), i(2), \ldots, i(m))]\left(M_{(a_1,a_2)}\right) = \sum_{\pi \in H_{i(1),i(2),\ldots,i(m)}} \prod_{V \in \pi} \alpha_{\frac{|V|}{2}}$$

for $m \geq 1$ and for arbitrary function $i : \mathbf{N} \to \{1, 2\}$.

From now on let $a_1 = b_1^n$, $a_2 = b_2^n$ for any fixed $n \in \{2, 3, 4, \ldots\}$. In this case

$$\begin{aligned}
& [coef\,(i(1), i(2), \ldots, i(m))]\left(M_{(a_1,a_2)}\right) = \varphi\left(a_{i(1)}a_{i(2)}\ldots a_{i(m)}\right) \\
&= \varphi\left(b_{i(1)}^n b_{i(2)}^n \ldots b_{i(m)}^n\right) \\
&= \varphi\left(b_{i(1)}b_{i(1)}\ldots b_{i(1)}b_{i(2)}b_{i(2)}\ldots b_{i(2)}\ldots b_{i(m)}b_{i(m)}\ldots b_{i(m)}\right) \\
(1) \quad &= \sum_{\pi \in H_{i(1),i(1),\ldots,i(1),i(2),i(2),\ldots,i(2),\ldots,i(m),i(m),\ldots,i(m)}} \prod_{V \in \pi} \alpha_{\frac{|V|}{2}} \\
(2) \quad &= \sum_{\nu \in {}_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(m)}} \prod_{V \in \nu} 1 = \#_{(n)}\tilde{H}_{i(1),i(2),\ldots,i(m)} \\
(3) \quad &= \sum_{\pi \in H_{i(1),i(2),\ldots,i(m)}} \prod_{V \in \pi} {}_{(n-1)}c_{\frac{|V|}{2}}.
\end{aligned}$$

Here the first equation comes from the definition of the moment generating series, (1) holds because (b_1, b_2) is circular pair which means that it is also R-diagonal for some sequence $\{\alpha_k\}_{k=1}^{\infty}$, (2) can be obtained taking in account that in the R-series of circular pairs the only coefficients that do not vanish are the coefficient of X_1X_2 and that of X_2X_1, which are equal to 1, and (3) follows from Lemma 2.3. This means that the moment generating series of (b_1^n, b_2^n) is given by formula (3.1) for every $n \geq 2$ ($m \geq 1$, $i : \mathbf{N} \to \{1,2\}$) for the sequence $\left\{{}_{(n-1)}c_k\right\}_{k=1}^{\infty}$. Since the moment-cumulant formula is invertible, this completes the proof. $\boxdot$

From Theorem 3.2 immediately follows

Corollary 3.3. *For a circular element c of a C^*-probability space $(\mathcal{A}, \varphi)$ the pair $(c^n, (c^*)^n)$ is R-diagonal for every $n \geq 1$.*

Remark 3.4. If $n = 2$, all the coefficients α_k are 1, that is

$$R_{\left(b_1^2, b_2^2\right)}(X_1, X_2) = \sum_{k=1}^{\infty} (X_1X_2)^k + \sum_{k=1}^{\infty} (X_2X_1)^k .$$

If $n = 3$, then the coefficients $\{\alpha_k\}_{k=1}^{\infty}$ coincides the sequence of Catalan numbers $\{c_k\}_{k=1}^{\infty}$, so

$$R_{\left(b_1^3, b_2^3\right)}(X_1, X_2) = \sum_{k=1}^{\infty} c_k (X_1X_2)^k + \sum_{k=1}^{\infty} c_k (X_1X_2)^k ,$$

where $c_k = {}_2c_k = \frac{1}{k+1}\binom{2k}{k}$.

Remark 3.5. From the proof of Theorem 3.2 the following facts are obvious for any R-diagonal pair (a_1, a_2)
(*a*) $\varphi\left(a_{i(1)}a_{i(2)} \ldots a_{i(m)}\right) = 0$ whenever m is odd,
(*b*) $\varphi\left(a_{i(1)}a_{i(2)} \ldots a_{i(m)}\right) = 0$ whenever

$$\#\{j \in \{1,2,\ldots,m\} : i(j) = 1\} \neq \#\{j \in \{1,2,\ldots,m\} : i(j) = 2\}, \quad \text{and}$$

(*c*) $\varphi\left(\bar{a}_{i(1)}\bar{a}_{i(2)} \ldots \bar{a}_{i(m)}\right) = \varphi\left(a_{i(1)}a_{i(2)} \ldots a_{i(m)}\right)$ for $m \geq 1$ and for arbitrary function $i : \mathbf{N} \to \{1,2\}$, where $\bar{a}_1 = a_2$ and $\bar{a}_2 = a_1$.

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Supported by OTKA T023447

AMS subject Classification (1991): 46L50, 06A07