The favoured solutions of the moment problem

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Abstract. The sequence of Catalan numbers is known to be the moments of the Wigner law. In the paper a characterization of this sequence is given in terms of some determinants and the moment sequence of the q-Gaussian law is characterized in a similar fashion.

0. Introduction

According to the Hamburger moment problem, a sequence $\{a_n\}_{n=0}^{\infty}$ is the moment sequence of a positive measure μ, if and only if the determinants

$$\Delta_n^{(\mu)} = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n+1} & \dots & a_{2n} \end{vmatrix}$$

are non-negative for every $n \in \mathbf{N}$ (see e.g. [**ST**]). In this case μ is a probability measure if and only if $\Delta_0 = a_0 = 1$. Bellow we will study the following question: for which probability distributions will all the Δ_n be 1? The answer will give a new, highly non-linear characterization of the Catalan numbers. In the end of the paper we will also give the sequence of determinants $\{\Delta_n\}_{n=0}^{\infty}$ corresponding to the q-Gaussian distribution (see [**LM**]).

1. Results

1.1. Theorem. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. Then the sequence $\{a_n\}_{n=0}^{\infty}$ is identical to the sequence of the Catalan numbers if and only if*

$$\Delta_n = \begin{vmatrix} a_0 & 0 & a_1 & 0 & a_2 & \dots \\ 0 & a_1 & 0 & a_2 & 0 & \dots \\ a_1 & 0 & a_2 & 0 & a_3 & \dots \\ 0 & a_2 & 0 & a_3 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \ddots & \cdot \\ \dots & \dots & \dots & \dots & \dots & a_n \end{vmatrix} = 1 \tag{1.1}$$

for every $n \in \mathbf{N}$.

Proof. As from the relation

$$\Delta_{n+1} = a_{n+1}\Delta_n + f(a_0, a_1, \ldots, a_n)$$

(where f is a certain function of the numbers $a_0, a_1, \ldots, a_n$) a_{n+1} is determined uniquely by the numbers $a_0, a_1, \ldots, a_n$, Δ_n and Δ_{n+1}, it is enough to show that the sequence of the Catalan numbers satisfy (1.1) for every n.

It is known (see e.g. [**HP**]) that the moments of the standard Wigner law

$$w = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{[-2;2]}(x)\,dx$$

is given by

$$m_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ c_k, & \text{if } n = 2k, \end{cases}, \tag{1.2}$$

where c_k is the kth Catalan number.

It is also known (see e.g. [**ST**]) that for every symmetric measure μ (with even density function) whose all moments exist, for the corresponding determinants $\Delta_n^{(\mu)}$

$$\frac{\Delta_n^{(\mu)}}{\Delta_{n-1}^{(\mu)}} = \beta_1\beta_2\ldots\beta_n \tag{1.3}$$

holds, where the numbers β_i are the parameters in the continued fraction form of the Cauchy transform of the given measure, that is

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z-x} = \cfrac{1}{z - \cfrac{\beta_1}{z - \cfrac{\beta_2}{z - \ddots}}}. \tag{1.4}$$

In [**Ch**] it is also shown that for the weighted shift S defined by

$$S\delta_n = b_{n+1}\delta_{n+1} \qquad \left(\forall n \in \mathbf{Z}^+\right)$$

on $l^2(\mathbf{Z}^+)$, where $\{\delta_n\}_{n=0}^\infty$ is the usual base in $l^2(\mathbf{Z}^+)$ and $\{b_n\}_{n=0}^\infty$ is a bounded sequence, the distribution of the 'field' operator $S + S^*$ (where S^* is the adjoint of S) in δ_0 is given by the measure ν whose Cauchy transform is given by

$$G_\nu(z) = \int_{-\infty}^{\infty} \frac{d\nu(x)}{z-x} = \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{z - \ddots}}}. \tag{1.5}$$

(The distribution of $S+S^*$ in δ_0 is ment in the usual way via the moments given by $m_n = \langle \delta_0, (S+S^*)^n \delta_0 \rangle$.)

If we take $b_n = 1$ for every n, S is the usual shift acting as

$$S\delta_n = \delta_{n+1} \qquad \left(\forall n \in \mathbf{Z}^+\right),$$

and in this case the distribution of $S+S^*$ in δ_0 is the standard Wigner law (see e.g. [**HP**]). From this and $(1.2-5)$, as $c_0 = 1$, the statement of the theorem follows. □

The above statement also means, that among the symmetric probability measures the standard Wigner law is the only one, for which the corresponding sequence $\left\{\Delta_n^{(\mu)}\right\}_{n=0}^{\infty}$ is identically 1, so it can be called the 'favoured solution' of the Hamburger moment problem.

Remark 1.2. According to the basic properties of determinants, Theorem 1.1 also shows that the determinants

$$\tilde{\Delta}_n = \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ . & . & . & . \\ c_n & c_{n+1} & \dots & c_{2n} \end{vmatrix}$$

equal 1 for every n, too (here c_n stands for the nth Catalan number again). Of course it is not true any more, that for determinants of such structure the sequence of the Catalan numbers is the only sequence with this property. But with the additional requirements that the determinants

$$\hat{\Delta}_n = \begin{vmatrix} c_1 & c_2 & \dots & c_n \\ c_2 & c_3 & \dots & c_{n+1} \\ . & . & . & . \\ c_n & c_{n+1} & \dots & c_{2n} \end{vmatrix}$$

should also equal 1 for every $n = 1, 2, \dots$, the sequence of the Catalan numbers is unique solution again (due to basic properties of determinants). As the sequence of the Catalan numbers equals the moment sequence of the free Poisson law of parameter 1 (see e.g. [**OP**])

$$\mu = \frac{\sqrt{4-(x-2)^2}}{2\pi x}\chi_{[1;4]}(x)\,dx,$$

we may say, that the free Poisson law is the 'favoured solution' of the Stieltjes moment problem (see e.g. [**ST**]). From the proof of Theorem 1.1 it also follows, that in the continued fraction form of the Cauchy transform

$$G_\mu(z) = \cfrac{1}{z-\alpha_1-\cfrac{\beta_1}{z-\alpha_2-\cfrac{\beta_2}{z-\alpha_3-\ddots}}}$$

of the free Poisson law of parameter 1 all the coefficients β_n equal 1.

In the following theorem the q-Gaussian (or q-normal) distribution is involved. For further details on this family of distributions see e.g. [**LM**].

Theorem 1.3. *Let $q \in (-1;1)$. Let $\left\{m_n^{(q)}\right\}_{n=0}^\infty$ be the moment sequence of the q-Gaussian law whose density function is given by*

$$C_q\sqrt{1-\frac{(1-q)\,x^2}{4}}\prod_{k=1}^{\infty}\left(1-\frac{(1-q)\,q^2x^2}{(1+q^k)^2}\right),$$

where

$$C_k = \frac{\sqrt{1-q}}{\pi}\prod_{k=1}^{\infty}\left(1-q^{2k}\right)\prod_{k=1}^{\infty}\left(1+q^k\right).$$

Then the for the determinants

$$\Delta_n^{(q)} = \begin{vmatrix} m_0^{(q)} & m_1^{(q)} & \dots & m_n^{(q)} \\ m_1^{(q)} & m_2^{(q)} & \dots & m_{n+1}^{(q)} \\ \cdot & \cdot & \cdot & \cdot \\ m_n^{(q)} & m_{n+1}^{(q)} & \dots & m_{2n}^{(q)} \end{vmatrix} \tag{1.6}$$

$$\Delta_n^{(q)} = \prod_{k=1}^{n}[k]_q! \tag{1.7}$$

*holds for every n, where $[k]_q!$ is the product of the first k q-natural number (see e.g. [**AI**]). Conversely, if* (1.7) *holds for the determinants* (1.6) *(for every n), than the sequence of numbers appearing in the determinants* (1.6) *is identical to the moment sequence of the corresponding q-Gaussian distribution.*

Proof. The proof is the same as in the case of Theorem 1.1; the only difference is that the corresponding weighted shift operator now acts according to

$$S\delta_n = \sqrt{[n+1]_q}\,\delta_{n+1} \qquad \left(\forall n \in \mathbf{Z}^+\right)$$

(see e.g. [**JSW**]). □

Remark 1.4. In the case of $q = 0$ we get back the statement of Theorem 1.1. When $q \to -1$ the above formula gives $\Delta_0 = \Delta_1 = 1$, $\Delta_n = 0$ (for $n > 1$), as one would expect for the distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ of the 'fermionic' field operator. Finally, the case $q \to 1$ corresponds to the standard Gaussian law (the distribution of the 'bosonic' field operator), when for the determinants $\Delta_n^{(\mu)}$ one gets

$$\Delta_n^{(\mu)} = \prod_{k=1}^{n} k!.$$

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