External problems for finite sets and convex balls — A survey

G.O.H. Katona

Abstract

Let \( F \) be a finite set of \( n \) elements, \( F \subseteq \mathbb{N} \). Define \( \mathcal{P}(F) \) as the family of all non-empty subsets of \( F \). Consider the profile vectors \( (x_1, \ldots, x_n) \) for the \( \mathcal{F} \) and \( \mathcal{C} \) families of \( F \) and \( \mathbb{C} \) are the families of convex subsets of \( F \). The profile vectors of the \( \mathcal{F} \) and \( \mathcal{C} \) families are also described for some \( k \) connections. The largest one of these families is the one with \( k = n \). The authors are interested in the number of \( \mathcal{F} \) and \( \mathcal{C} \) families.

5. Introduction

Let \( F \) be a finite set of \( n \) elements. A family of \( F \) is said to be \( \mathcal{C} \) if \( F \subseteq \mathbb{C} \). A family of \( F \) is said to be \( \mathcal{F} \) if \( F \subseteq \mathcal{F} \). It is easy to see that \( \mathcal{F} \) contains subsets of \( F \) that are \( \mathcal{C} \). The largest one of these families is the one with \( n = 2 \). The authors are interested in the number of \( \mathcal{F} \) and \( \mathcal{C} \) families.

Theorem 1 (Katona [9]). The maximum number of members of an \( \mathcal{F} \) family is

\[
\binom{n}{\left\lceil \frac{n}{2} \right\rceil}
\]

In some applications (see [9]), however, one has

\[
\left\lceil \frac{n}{2} \right\rceil \geq \frac{n}{2}
\]

*Corresponding author.

This work was supported by the National Science Foundation.

Coding was done by the Technical Information Center, Institute for Advanced Study.
For inclusions-free families. This question has been solved in [29]. Let us consider the following objective function. Let $F(x)$ be a real function and try to find

$$
\max \sum_{i=1}^{n} F(x_i)
$$

for all inclusions-free families.

Introduce the instance $\mu(F) = \sum_{i=1}^{n} F(F(x_i), x_i)$. The base 1-dimensional vector $F(F(x_i), x_i)$ is added to the profile vector of $F$, then (1) is written in the equivalent form

$$
\max \sum_{i=1}^{n} F(x_i)
$$

when the maximum is taken for all profile vectors of inclusions-free families as $F$.

It is obvious that $F(F(x_i), x_i)$ is a function. We have to find an inequality such that this function remains a profile vector. It is obvious that it is sufficient to consider the interior or convex hull of the set of all profile vectors. One of these convex hulls will achieve the maximum in (2).

**Theorem 2.** The extreme points of the convex hull of the set of profile vectors of inclusions-free families are the convex vector and

$$
[0, \ldots, 0, a, \ldots, a]
$$

where the common component is $\frac{1}{a}$.

It is easy to see that these vectors are profile vectors. The empty family and the family consisting of all constant subsets serve as constructions. The theorem claims, on the one hand, that all profile vectors are convex linear combinations of the above vectors and, on the other hand, that the extreme points of the convex hull correspond to these points. The problem can also be described by the hypothesis. These hypothesis hypotheses are only considering 1 or 2 extreme points (since the or 2 points are opposite). We write this in the "inclusion-free" showing which side is in the convex hull.

$$
\sum_{i=1}^{n} a_i \geq 1.
$$

The above hypothesis gives an equivalent description of the convex hull of the profile vectors of inclusions-free families. However, the signs in the last one are reversed, the
The profit vector of an inclusion-fine family satisfies the inequality

$$\sum_{i \in I} x_i \geq 1.$$ 

This is Theorem 2. It is only another form of the old Yen’s inequality. This is not new, as we will see, in the case of other classes of families.

Knowing Theorem 3 is very easy to determine

$$\sum_{i \in I} x_i \leq \frac{1}{2} \sum_{i \in I} \frac{1}{x_i}.$$ 

Since inclusion-fine families have only to calculate \(\sum_{i \in I} x_i\) for the extreme points and make the largest arc \(\max_{i \in I} x_i\), each (1) gives the maximum.

Theorem 3 gives a necessary condition for a vector to be a profit vector of an inclusion-fine family. However, this is not a sufficient condition. For all cases, in the current hull, the theorem is not a sufficient condition for the profit vector. A sufficient condition for the profit vector is more than what is stated in Theorem 3. Unfortunately, this theorem is very complicated, where some obvious simplification is needed so that we can simplify the profit vector to the profit vector or its extreme points in (1-3). That is, it is hard to use the necessary and sufficient conditions. This makes it desirable to find a good sufficient condition.

Open Problem 1: Find a good sufficient condition for a vector to be a profit vector of an inclusion-fine family by either

1. giving a (simpler) version of the simple described in Theorem 2 such that the profit vector belongs to the profit vector,

2. determining the extreme points of the set of non-profit vectors.

In the rest of the paper, we consider other families.

2. Extreme points for some classes

Let \(A\) be a class of families of subsets of \(E\). Denote \(K(E, A)\), denoting the set of profit vertices of the family belonging to \(A\).

$$\sigma(A) = \{\sigma(A) \mid \sigma(A) \in K(E, A), \sigma(A) < \sigma(A)\}.$$ 

The set of the extreme points of the convex hull of \(\sigma(A)\) is denoted by \(\lambda(A)\).
The $A$ considered in the present paper are boundary, that is, $A \subset F$. A $4$-tuples $P \times A$ for boundary $A$ is then a single way of inclusion of the set of extreme points of $P$. Multiple counting is not here to complicate some more nonsense. $P \times A$ is the set of all convex extreme points of $P$ if $A$ is the set of all convex extreme points of $P$. $P \times A$ is the set of all convex extreme points of $P$.

Proposition 5. Suppose that $A$ is boundary. Then an element of $A(a)$ can be obtained by taking some components of $A(a)$, the $a$.

The signature of the proposition is that if it is sufficient to determine the set $A(a)$. Any element of $A(a)$ is an element of some set of subset $A(a)$. Thus should be independently of any other elements.

We will now prove some important propositions.

We now prove a family of propositions. If $A(a)$ does not contain an $A(a)$ that is the statement of proposition $A(a)$. Furthermore, $P$ is said to be...
Theorem 4 (Hilton et al. [11]). The extremal eigenvalues of the convex hull of the set of projectors on the minimizing elements of the families are

\[ v = v_0 \cdot n \sum_{i=1}^{k} a_i \cdot (x_i, x_i) \quad \text{and} \quad u = 0 \cdot n \sum_{i=1}^{k} a_i \cdot (x_i, x_i) \quad \text{for} \quad k = 1, 2, \ldots, n, \]

where the left and right components are non-zero.

It is easy to see that if the cone spans and \( s \), the vector obtained from \( v_0 \) by replacing \( v_0 \) by zero, we also manage to see. Any construction done that all these vectors are only positive seems to have interesting, if not desirable. It is harder to prove that all other projectors are convex linear combinations of them. This is easy to see only if we work with the fact that using such matrices.

Theorem 5 (Hilton et al. [11]). Suppose that \( x \) is a convex linear combination of the given extreme points. There is, however, another method which is shown to be efficient in most of the cases. It is the so-called spectral method [17]. For a reliable overview of \( x \) and consider only those subsets of \( x \) which have an extreme point. The methods introduced for finding optimal projections are based on the intuition that it is not easy to find such a projection that is better than the claimed one. Then a weighted double counting leads to the desired result.

Let us see the consequence of Theorem 4.

Theorem 6 (Hilton et al. [11]). Suppose that \( x \) is a convex linear combination of the given extreme points. The methods introduced for finding optimal projections are based on the intuition that it is not easy to find such a projection that is better than the claimed one. Then a weighted double counting leads to the desired result.
The above expression is equal to:
\[
\max \left( \sum_{i=1}^{n} \left( \sum_{j=1}^{m} \left( r_{ij} - r_{ij}^{*} \right) \right) \right)
\]

It is an easy task to determine the maximum and the following theorem is obtained.

**Theorem 1** (Kohnen, 1963). The maximum number of numbers in an arbitrary subsequence family is 
\[
\max_{i,j} \left( \sum_{k=1}^{n} \left( t_{ik} - t_{ij} \right) \right)
\]

One can prove an inequality like the following one.

**Theorem 2** (Kohnen, 1963). If \( p \) is the profile vector of an intercrossing, inclusion-free family, then
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \left( p_{ij} - p_{ij}^{*} \right)
\]

The right-hand side of (1) is a linear function of \( p \); therefore, it achieves its maximum at an essential extreme point. However, the essential extreme points give either zero or 1 for the right-hand side, due to the triangle inequality.

The following inequality can be proved in the same way.

**Theorem 3** (Kohnen, 1963). If \( p \) is the profile vector of an intercrossing, inclusion-free family, then
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \left( p_{ij} - p_{ij}^{*} \right)
\]

This property of theorem illustrates the main significance of the 'essential point.' However, many other extremal theorems and inequalities follow from Theorem 4 and Theorem 6.

Let us observe the facts. A family \( A \) in a transversal \( \mathcal{P} \) of \( \mathcal{A} \). A profile \( A \) in a transversal \( \mathcal{P} \) of \( \mathcal{A} \) for the vector \( A \) of the profiles given in the profile space such that the vector \( A \) is a minimal point of the profile space. A profile \( A \) in a transversal \( \mathcal{P} \) of \( \mathcal{A} \) for the vector \( A \) of the profiles given in the profile space such that the vector \( A \) is a minimal point of the profile space.

Finally, let us examine the other branches of the tree.

Suppose that \( A \) is a profiled \( A = (A_{1}, A_{2}) \). A family \( A \) is given as \( A \) and \( B \) is the profiled \( A = (A_{1}, A_{2}) \) and \( B = (B_{1}, B_{2}) \).

The minimum of the profile space and the extreme points of the convex hull of all profile vectors of branches including a profile vector are obtained. That [13] proves theorems for these extreme points or extreme vectors.
Consider a subset \( P \) of \( \mathbb{R}^d \) with \( \mathcal{H}^d \)-finite perimeter. Let \( \mathcal{H}^{d-1} \) be the \( \mathcal{H}^{d-1} \)-measure.

Theorem 1. \( \mathcal{H}^{d-1} \)-measurable \( \mathcal{H}^{d-1} \)-finite perimeter sets are \( \mathcal{H}^d \)-measurable. For a \( \mathcal{H}^{d-1} \)-measurable set \( A \) of \( \mathcal{H}^{d-1} \)-finite perimeter, \( \mathcal{H}^{d-1} \)-measure of \( A \) is \( \mathcal{H}^d \)-measurable.

Proof. Let \( f \) be \( \mathcal{H}^{d-1} \)-measurable. Then \( f \) is \( \mathcal{H}^d \)-measurable. Since \( f \) is \( \mathcal{H}^d \)-measurable, \( f \) is \( \mathcal{H}^{d-1} \)-measurable. Therefore, \( f \) is \( \mathcal{H}^{d-1} \)-measurable.

3. Essential Facts

Suppose that the set of extreme points of \( \mathbb{A} \) is known for a certain class \( \mathcal{A} \) of families. The only theorem of (A) (i) is that the characteristic function of \( \mathbb{A} \) is determined by the characteristic function of \( \mathcal{A} \). Therefore, the characteristic function of \( \mathbb{A} \) is determined by the characteristic function of the characteristic function of \( \mathcal{A} \).

Theorem 2. \( \mathbb{A} \) is a family of \( \mathcal{A} \)-finite perimeter sets. \( \mathcal{A} \) is a family of \( \mathcal{A} \)-finite perimeter sets. Therefore, \( \mathbb{A} \) is a family of \( \mathcal{A} \)-finite perimeter sets. Therefore, \( \mathbb{A} \) is a family of \( \mathcal{A} \)-finite perimeter sets.
Let \( P \) be the convex hull of the points given in Example 4. Then \( \alpha(P) \) can be easily determined by Theorem 1. First, we have to determine only the essential concave points of \( P \). This can be done in a step-by-step procedure (see [25]). Finally, it is shown in Example 4 that the essential concave points of \( P \) are exactly those points of \( P \) which lie on the boundary of \( P \).

\begin{equation}
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} (x_{ij} - x_{i+1,j}) \right) = 0,
\end{equation}

where the indices are circular, i.e., \( n + 1 \equiv 1 \).

Observation 1: \( c_{ij} = 0 \) if and only if \( \alpha_{ij} \) is a convex point.

Observation 2: \( a_{ij} = 0 \) if and only if \( \alpha_{ij} \) is an essential concave point.

Theorem 9 (Hilbert and Hohloch [26]). The essential concave points of the convex hull of the given set of points are the following ones: Let \( x_{ij} = \alpha_{ij} \) and \( l_{ij} = \beta_{ij} \) be any integers.

\begin{equation}
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} (x_{ij} - x_{i+1,j}) \right) = 0,
\end{equation}

where the indices are circular, i.e., \( n + 1 \equiv 1 \).

Theorem 10 (Hilbert and Hohloch [26]). The essential concave points of the convex hull of the given set of points are the following ones: Let \( x_{ij} = \alpha_{ij} \) and \( l_{ij} = \beta_{ij} \) be any integers.

\begin{equation}
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} (x_{ij} - x_{i+1,j}) \right) = 0,
\end{equation}

where the indices are circular, i.e., \( n + 1 \equiv 1 \).

4. Applications

Theorem 11 (Hilbert and Hohloch [26]). The essential concave points of the convex hull of the given set of points are the following ones: Let \( x_{ij} = \alpha_{ij} \) and \( l_{ij} = \beta_{ij} \) be any integers.

\begin{equation}
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} (x_{ij} - x_{i+1,j}) \right) = 0,
\end{equation}

where the indices are circular, i.e., \( n + 1 \equiv 1 \).

Theorem 12 (Hilbert and Hohloch [26]). The essential concave points of the convex hull of the given set of points are the following ones: Let \( x_{ij} = \alpha_{ij} \) and \( l_{ij} = \beta_{ij} \) be any integers.

\begin{equation}
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} (x_{ij} - x_{i+1,j}) \right) = 0,
\end{equation}

where the indices are circular, i.e., \( n + 1 \equiv 1 \).

Theorem 13 (Hilbert and Hohloch [26]). The essential concave points of the convex hull of the given set of points are the following ones: Let \( x_{ij} = \alpha_{ij} \) and \( l_{ij} = \beta_{ij} \) be any integers.

\begin{equation}
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} (x_{ij} - x_{i+1,j}) \right) = 0,
\end{equation}

where the indices are circular, i.e., \( n + 1 \equiv 1 \).

Theorem 14 (Hilbert and Hohloch [26]). The essential concave points of the convex hull of the given set of points are the following ones: Let \( x_{ij} = \alpha_{ij} \) and \( l_{ij} = \beta_{ij} \) be any integers.
Theorem 9 (Edmonds and Mahi [12]). Let \( \mathcal{F} \) be an inclusion-free family and \( F \subseteq C \) an integer. Suppose that
\[
(\lambda) \quad \sum_{\lambda \in \mathcal{F}} e_\lambda = 0.
\]

Then the average size of the members of \( \mathcal{F} \) is at least \( k \).

Let \( p \) be the profile of \( \mathcal{F} \). The condition of the theorem can be written in the form
\[
(\beta) \quad \sum_{\lambda \in \mathcal{F}} e_\lambda = 0.
\]

That is, \( p \) is "above" the hypograph
\[
(\gamma) \quad \sum_{\lambda \in \mathcal{F}} e_\lambda = 0.
\]

In the convex hull of inclusion-free families, this means that the convex hull of \( \mathcal{F} \) should be cut by this plane. Since the new projection by \( \mathcal{F} \) the extreme points of \( I \) can be obtained by investigating the edges of \( F \), \( F \) has some type of edge. They either contain one or two extreme edges, or they contain two symmetric vectors. Their form is described by the following equation:
\[
(\delta) \quad e_\lambda = e_\mu = -e_\nu = -e_\rho = 0, \quad e_\sigma = e_\tau = -e_\upsilon = -e_\xi = 0.
\]

Thus we have \( e_\mu = -e_\mu = -e_\nu = -e_\rho = 0 \) for all \( \mu \).

The intersection of (1) and (3) satisfies \( e_\mu = \sum_{\lambda \in \mathcal{F}} e_\lambda \). It is equal to \( 0 \) if \( \mu < k \) or \( \mu > k \), and it is the extreme point only for \( \mu = k \) where the first non-zero component is the \( k \)th one, \( x(1) = \ldots = x(k-1) = 0 \).

The intersection of (2) and (3) has
\[
(\sigma) \quad e_\lambda = e_\mu = -e_\nu = -e_\rho = 0, \quad e_\sigma = e_\tau = -e_\upsilon = -e_\xi = 0.
\]

(Since \( \lambda \leq \mu \), the \( x \) and \( y \) components are non-zero.) As we see, the intersection plane, that is \( F \), is the intersection of the edge plane of \( F \) and \( F \), \( k \) dimensions. The extreme points of \( F \) are \( e_\mu = \ldots = e_k = -e_\nu = \ldots = -e_\rho = 0 \) and all extreme points are located in the boxes \( Z(1) = \ldots = Z(k) = -1 \).

We have shown that the average size of the members of \( \mathcal{F} \) is at least \( k / |\mathcal{F}| \) when
\[
(\theta) \quad s \sum_{\lambda \in \mathcal{F}} e_\lambda = 0.
\]