All Maximum 2-Part Sperner Families

Petr L. Elias
Institute of Mathematics and Computer Science, Academy of Sciences of the Czechoslovakia
Prague 1 (Czechoslovakia)

and

G. K. Kema
Mathematical Institute of the Hungarian Academy of Sciences,
Budapest H-1117 (Hungary)

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Let \( F = \{ A_1, A_2, \ldots, A_n \} \) be a partition of an interval of length \( n \). Suppose that the lengths of the intervals of the partition \( F \) satisfy the following condition: if there is no interval \( A_i \neq A_j \) in \( F \) such that \( A_i \cap A_j \neq \emptyset \), then the intervals of \( F \) cannot be a subpartition of \( L \). Let \( F \) be a permutation of \( L \). The data are distributed to the sets \( A_i \) in such a way that the partition \( F \) of \( L \) satisfies the condition.

No discussion of the maximum growth of this theorem.

1. Introduction

Let us start with a classic theorem of Sperner [3]:

Let \( F \) is a family of disjoint subsets of an element set \( X \) such that \( F = F \), holds for all \( F \), then

\[ |F| \leq n \]

Ehrenfeucht [3] and Kema [1] independently discovered that the converse of the above can be formulated while its statement is equivalent:

Let \( F = \{ A_1, A_2, \ldots, A_n \} \) be partition of \( X \) with \( |F| \geq n \). Suppose that the family \( F \) satisfies the following condition:

\[ |F_i \cap F_j| \leq 1 \text{ for all } i \neq j \]

Then

\[ |F| \leq n \]
The family satisfying (1) is called 2-part Speiser families. The main
use of the present paper is to determine all maximal 2-part Speiser
families, that is, the ones with equality in (1). It is worth mentioning that
all of them have the following monotonic property. If \( P \subseteq F \) then
\( |P| \leq |F| \). Thus, \( (P, \leq) = (F, \subseteq) \) imply the \( P \). This is one
but not the only way to prove that (1) is true for more than 2 parts. See [4]
for additional questions.

The proof is based on a theorem of [2]. We state it to make the paper
readable:

Let \( P \) be a 2-part Speiser family, and let \( p_0 \) denote the number of mem-
bers, \( P \subseteq F \), such that \( |P| = 1, |P| = 2 \) (or \( |P| = 2 \) in this case). The
monotonicity \( P \subseteq F \) is defined by the curve \( p_0 \). It
can be considered as a point of the sphere \( |P| \) in the 1-dimensional space.

We state a theorem which is easily verifiable and which states what
we want to prove. The theorem is a consequence of the following
assumption:

**Theorem A** A particular case of Theorem 2.1 of [2]. The extreme points
of the set of monotonicity of all 2-part Speiser families are the \( (p_0, \leq) =
(p_0, \subseteq) \) monotonic bases with \( p_0 \) or \( 1, 2 \) in (1) entry but having in
all cases the same value of \( p_0 \).

For interested readers we also suggest the recent survey paper [3] on
linear partial Speiser families.

2. **Theorem**

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i x_j = 0
\]

is a linear function of the variables \( x_i \). It follows that \( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i x_j = 0 \) is a proper linear function for some points described in
Theorem 4, and so it is for some correct linear combinations of these
points.

If, for \( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i x_j = 0 \) the points are in \( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i x_j = 0 \), then the points are in \( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i x_j = 0 \), and so on.

The partial transversals are defined accordingly. If \( (x, p_0) = (x, \leq) \) is a partial transversal of \( (x, \leq) \), then \( (x, p_0) = (x, \leq) \), and so on.

We have to minimize

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i x_j = 0
\]

for partial transversals \( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i x_j = 0 \).
It is trivially clear that (1) is maximum if the great number (7) are
poured with great (7) and the little ones with little ones. Some easy lemm
a leading to the following are:

Lemma 1. Let \( a_1, a_2, \ldots, a_n \) be integers and 1 a partial transpo
ted. Suppose that \( (a_1, a_2, \ldots, a_n) \) and \( (b_1, b_2, \ldots, b_n) \) hold and define
\( \Gamma = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n) \). Then
\[
\sum_{i=1}^{n} a_i b_i < \sum_{i=1}^{n} a_i b_i \tag{1}
\]

Proof. We have
\[
\sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n -
\sum_{i=1}^{n} a_i b_i - a_1 b_1 - a_2 b_2 - \ldots - a_n b_n
\]
which is negative, proving (1).

Lemma 2. Let \( a_1, b_1, a_2, b_2, \ldots, a_n, b_n \) and \( k, b_2, b_3, b_4, \ldots, b_n \)
be integers. If \( \Gamma \) is a partial transposed matching
\[
\sum_{i=1}^{n} a_i b_i 
\]
then (1), (2) hold.

Proof. Suppose, on the contrary, that (1), (2) hold. We will find contradic
tions distinguishing several cases. If there is no other way with compact
(1), then we have (1). This is the same as (1) and (2) hold.

If (1) holds for \( j \) but (2) fails for any \( j \) then (1) can be replaced by
(1). This is a contradiction to \( a_i b_i < a_i b_i \). The case when \( 1, j_1 \) (j
is not in the same way. Finally, suppose that (1) holds for all \( i \) and (2) hold for one. This holds for one. Then (1), (2) hold.

Similarly, suppose that (2) holds for \( (k, l) \), \( (k, l) \), and (1), (2) hold.
Lemma 1 gives the contradiction.

Lemma 3. Let \( a_1, b_1, a_2, b_2, \ldots, a_n, b_n \) and \( k, b_2, b_3, b_4, \ldots, b_n \)
be integers. If \( \Gamma \) is a partial transposed matching (1) does refer
(1), (2) hold.

Proof. Suppose, on the contrary, that none of them holds. The proof of
Lemma 2 can be repeated, since it does not have here, a contradiction only if one is involved. However, it is not yet by the impossible assumption.

Lemma 4. Let \( a_1, b_1, a_2, b_2, \ldots, a_n, b_n \) and \( k, b_2, b_3, b_4, \ldots, b_n \)
be integers. If \( \Gamma \) is a partial transposed matching (1) does refer
(1), (2), (3) hold.
Proof. Suppose that one of (1), (2), (3), (4), (5) is in $S$. Then the proof of Lemma 1 fails to contradiction, since (2), (3), (4), and (5) imply (1). Therefore, the proof of Lemma 1 fails to contradict (1), (2), (3), and (4) if and only if $a = b$. This shows that the necessary conditions satisfy the conditions of Lemma 1, so the remaining statements are true.

Now we are able to determine all partial intervals $I$ maximizing (1). However, we have to distinguish cases according to the parity of $n$ and $n'$.

Lemma 5. If $n$ and $n'$ are both even then $I$ is a partial interval $[0, \frac{p}{q}]$.

and each one of the following two cases holds for each $k = 1, 2, \ldots$ in (9), (10):

\[
\left(\frac{p}{q} + \frac{k\cdot\frac{p}{q}}{q}\right) \cup \left(\frac{p}{q} - \frac{k\cdot\frac{p}{q}}{q}\right) = \left(\frac{p}{q} + \frac{k\cdot\frac{p}{q}}{q}\right) \cup \left(\frac{p}{q} - \frac{k\cdot\frac{p}{q}}{q}\right)
\]

Proof. We use first Lemma 2 with the numbers (2), (3), (4), and (5) defined correspondingly. This proves (4), (6), (7), and (8) from the theorem. The remaining numbers satisfy the conditions of Lemma 1; therefore, either (9) holds with $a = b$. The proof of the necessity of (10) can be completed by induction.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\fill[pattern=north west lines, pattern color=gray!50] (-1,-1) rectangle (1,1);
\fill[pattern=north east lines, pattern color=gray!50] (-1,1) rectangle (1,-1);
\fill[pattern=vertical lines, pattern color=gray!50] (-1,0) rectangle (0,1);
\fill[pattern=vertical lines, pattern color=gray!50] (0,-1) rectangle (1,0);
\fill[pattern=dots, pattern color=gray!50] (0,0) rectangle (1,1);
\end{tikzpicture}
\caption{Graph}
\end{figure}
On the other hand, it is easy to see that all such $P_s$ give the same value for $O_1$, maximizing $H_1$.

This result can be better visualized if the rows and the columns of the matrix are aligned according to the decreasing order of the numerical values. Figure 1 shows how to interpret the two opposite corners of each $2 \times 2$ shaded block and the $t = 1$ shaded one.

The proof of the next lemma is analogous.

Lemma 5. If $n$, and $t$, are both odd, and then $f$ is a partial numerical mapping (1) iff exactly one of the following two rows holds for each $k = 1, \ldots, n-1$, and $s = 1, 2, \ldots, t$.

\[
\begin{align*}
\left(\frac{1}{2} - \frac{1}{n} \right) &+ \left(\frac{1}{n} - \frac{1}{2} \right) s_1 & \leq f \\
\left(\frac{1}{2} + \frac{1}{n} \right) &+ \left(\frac{1}{n} - \frac{1}{2} \right) s_2 & \leq f.
\end{align*}
\]

The proof of the remaining case, when the parity is different, is again analogous. However, the formulation of the statement is less concise.

Lemma 7. If $n$, and $t$, are both odd, then $f$ is a partial numerical mapping (1) iff it contains exactly one element of the following sets for each $k$.

\[
\begin{align*}
\left\{ \left(\frac{1}{2} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{2} \right) s_1 \right\} &\leq f \\
\left\{ \left(\frac{1}{2} + \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{2} \right) s_2 \right\} &\geq f.
\end{align*}
\]
The statement is illustrated in Fig. 3. This is the correct one element of each shaded block (e.g., $2 \times 3$ or $2 \times 2$) for a particular constraint. It shows that each row of the matrix is indexed by $a_i$ and each column by $b_j$. The shaded blocks indicate the valid constraints.
By this we have finished the first part of our work: the extreme points minimizing \( |f| \) are characterized. In the rest of the paper we show that

\[
\sum_{\xi} A_\xi(a) \leq 1
\]

will equality only when \( p_j = 1 \) for some \( 0 \leq j < n, a \).

The next lemma is a similar statement for 2-part Spezner families.

Lemma 3. Let \( \mathcal{F} \) be a 2-part Spezner family on \( X = X_1 \times X_2 \)
\([x_1, x_2] = 1, \{x_1, x_2\} = \emptyset, \{x_1, x_2\} = 1\), and for \( p_j \) denote the number of the numbers \( \xi \) such that \( (x_1, x_2) = (x_1, x_2) \). Suppose that the following conditions hold for some indices \( 0 \leq x_1, x_2 \leq n \):

\[
\sum_{\xi} A_\xi(x_1) = 1, \quad (10)\]

\[
\sum_{\xi} A_\xi(x_2) = 1. \quad (11)
\]
Then
\[ p_n = \binom{a}{n} \binom{b}{n}. \] (12)

Proof. Introduce the following notation:
- \( A(B) = \{ F \subseteq X, x_n \in F \} \) (\( A = X_n \))
- \( B(B) = \{ F \subseteq X, x_n \notin F \} \) (\( B = X_n \))
- \( A(B) = \{ F \subseteq X, x_n \notin F \} \) (\( B = X_n \))
- \( B(B) = \{ F \subseteq X, x_n \in F \} \) (\( A = X_n \))

Observe that the above families are Spanner families for each \( A = X_n \), \( B = X_n \). Therefore
\[ \mathbb{E} \left[ \left( \sum_{i=0}^{a} \binom{b}{i} \right) \right] (13) \]
holds for any \( A = X_n \). Summing up for all sets \( d = X_n \), with \( |d| = a \) we obtain
\[ \sum_{d \in A_n} \mathbb{E} \left[ \left( \sum_{i=0}^{a} \binom{b}{i} \right) \right] (14) \]
which is equivalent to
\[ \sum_{d \in A_n} \mathbb{E} \left[ \left( \sum_{i=0}^{a} \binom{b}{i} \right) \right] (15) \]

By (10) we have equality here and in (14), consequently (12) holds with equality for all \( A = X_n \), \( |d| = a \). By Lemma 8, one of the numbers \( p_2, p_3 \) and \( p_2(A), p_3(A) \) is equal to 0 only if \( A = X_n \). Then, the other one is zero as well. Therefore, \( p_2(A) = p_3(A) \), and hence \( \sum_{d \in A_n} \mathbb{E} \left[ \left( \sum_{i=0}^{a} \binom{b}{i} \right) \right] (15) \)

All sets \( S \) satisfying \( F \subseteq X_n \), \( F \subseteq X_n \) are counted in. Choose one of them, its complement in \( X_n \) will be denoted by \( S' \). Therefore, \( S \cap S' = \emptyset \), \( (S' \subseteq X_n \), \( S' \subseteq X_n \), \( S' \subseteq X_n \), \( S' \subseteq X_n \), and
\[ \mathbb{E} \left[ \left( \sum_{i=0}^{a} \binom{b}{i} \right) \right] (15) \]
all field. \( \mathcal{F}(R) \) is a Spencer family, it satisfies
\[
\sum_{\mathcal{F}(R)} \frac{p_j}{n_j} < 1
\]

The sum of these inequalities for all \( R \in \mathcal{F}(R) \) leads to
\[
\sum_{\mathcal{F}(R)} \frac{p_j}{n_j} < 1
\]

Because \( \sum_{\mathcal{F}(R)} \frac{p_j}{n_j} \geq \sum_{R \in \mathcal{F}(R)} \frac{p_j}{n_j} \). The equality in (111) implies that we must have equality in (111) for all \( R \in \mathcal{F}(R) \), excluding \( \mathbb{R}^n \). By Lemma 5.8, \( \sum_{R \in \mathcal{F}(R)} \frac{p_j}{n_j} 

Therefore all sets \( \mathcal{R} \), \( \mathcal{S} \) are \( \mathcal{R} \)-sets. \( \mathcal{R} \)-sets are a pair of \( \mathcal{R} \)-sets, that is, \( \mathcal{R} \cap \mathcal{S} \neq \mathcal{R} \), \( \mathcal{R} \cup \mathcal{S} \neq \mathcal{R} \). Therefore \( \mathcal{R} \) includes all sets \( \mathcal{R} \), \( \mathcal{S} \), where \( \mathcal{R} \cap \mathcal{S} = \emptyset \). Hence \( p_j = \frac{1}{2} \).

Theorem 8. For \( X = \{x_1, \ldots, x_n \} \), \( x_i \in \{0, 1\} \), \( i = 1, \ldots, n \). The maximally joint Craig Spencer families are of the form
\[
\mathcal{S} = \{ (x, i) \mid 0 \leq x_i \leq 1 \}, \quad i \in \{0, 1\}
\]

where \( \mathcal{S} \) is a partial transversal described in Lemma 5.5 (Fig. 3–4).

Proof. Lemma 5.5 determined the extreme points maximizing \( \mathcal{F}(R) \) for the 2-spacer, Spencer families. To prove the theorem we will have to show that no proper convex linear combination of these maximum extreme points can be the profile matrix of a 2-spacer, Spencer family.

Suppose that \( M \) is the profile matrix of a 2-spacer family and \( M \) is a convex linear combination of extreme points described in Lemma 5.5.

\[
H_{\text{Max}} = \sum_{i=1}^{n} A_i \delta_i
\]

where \( A_i \) is the extreme point determined by the partial transversal \( \delta_i \).

and
\[
\delta_i = \left\{ \begin{array}{ll}
1 & \text{if } \delta_i \\
0 & \text{otherwise}
\end{array} \right.
\]
Consider first the case when \( a \) and \( b \) are of equal parity. By symmetry we may assume that \( a < b \). It is obvious from Lemma 1 that the determinant of the \( a \times a \) matrix satisfies the equation
\[
\frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) (b-a) = \frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) x_1
\]
and
\[
\frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) (b-a) = \frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) x_1
\]
These inequalities imply
\[
\frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) (b-a) = \frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) x_1
\]
and
\[
\frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) (b-a) = \frac{1}{2} \left( \sum_{i=1}^{a} x_i \right) x_1
\]
On the other hand, all entries \( x_i \), with \( i \leq b \), are non-negative. Therefore, for any \( b \times b \) matrix, there is a row with exactly one non-zero entry. If this row contains \( b \) non-zero entries, then the determinant of the matrix is non-zero. Conversely, if the determinant of the matrix is non-zero, then there is a row with exactly one non-zero entry. This row cannot have more than one non-zero entry, since all entries are non-negative.