CONTINUOUS VERSIONS OF SOME EXTREMAL
HYPERGRAPH PROBLEMS

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INTRODUCTION

In [1] there is an application of the well-known generalization of
Turán's [2] problem to the probability theory. As [1] is written in Russian, it is
worthwhile to quote here briefly the statement and the idea of the proof.

Let us first recall the Erdős theorem in a special case:

If a graph has less than $n/4$ edges, then it must have a square.

In particular, if a graph has no empty triangles, then it must have a square.

Now let $G$ be a nonempty graph with $n$ vertices and $m$ edges.

We consider a random $G$ with $n$ vertices and $m$ edges.

The number of edges of $G$ is at least $\frac{1}{2} \frac{n^2}{4} - 1$.

We need to prove the conjecture of Erdős and Simonovits, namely that if $G$ has no
empty triangles, then the number of edges is at least $\frac{1}{2} \frac{n^2}{4} - 1/t$.

This inequality is obtained by considering the number of edges of the graph $G$.
Let $\xi$ and $\eta$ be independent and identically distributed random variables in a $d$-dimensional Euclidean space.

Then

$$\Pr(\xi + \eta \geq x) = \frac{1}{2} \Pr(\|\xi\|^2 \geq x).$$

Observe that for any 3 random $x_1, x_2, x_3$ with $|x_i| > x$ ($i = 1, 2, 3$) there is a path $\gamma$ such that $|x_1 + x_2 + x_3|$. Define a graph $G$ with the vertices $\{i(\gamma) = x_i, 1 \leq i \leq 3\}$, where $x_1$ and $x_2$ are connected if

$$\langle x_1, x_2 \rangle = x.$$ 

This graph, by the above remark, does not contain an empty triangle.

$$\Pr(\xi + \eta \geq x) = \Pr(\xi + \eta \geq x, \langle x_1, x_2 \rangle = x)$$

and this is the measure of the sets $\{x_1, x_2\}$ such that $x_1$ and $x_2$ are connected, $x_1 + x_2$ is tightly connected. For the discrete case we know that this number is at least the half of the total number of parts.

Thus we may expect the same for the measure of the parts $\{x_1, x_2\}$ in the direct product.

The aim of this paper is to investigate under what conditions can we translate the underlying discrete theorem to continuous case. It is a very new task if we suppose that the set of edges in the product space is "fin" (e.g. its boundary is a Jordan curve if the Euclidean space is $[0, 1]$), in contrast with the Langevin moment. However, our above example shows that usually we cannot suppose nothing else but measurability.

It should be mentioned that [11] already contains theorems of this type, but it is written in Russian and the results of the present paper go much farther.

The idea, that there is a need of continuous versions of combinatorial results is not new. E.g. North-Williams [1] suggested to apply this idea. M. Kati [2] worked out the analogue results of a certain combinatorial result of H. von St. Ball [3] without the existence of the limit. Very likely there are other similar papers and results producing the same results in terms of some given discrete result, but I think up to now there is no systematic treatment of this subject.
Let $G$ be a finite or infinite set. $G = (V, E)$ is called a directed graph (digraph) where $V \subseteq \mathbb{N}$, that is, $G$ consists of a non-empty set of vertices $V = \{v_1, \ldots, v_n\}$ and a set of directed edges $E = \{(v_i, v_j) \mid (i < j, v_i, v_j \in V)\}$. The elements of $V$ and $E$ are called vertices and edges, respectively. Multiple edges are excluded. The edges having the same $v_i$ are called loops if $v_i \in V$. If $E \subset V \times V$, then the induced subgraph $G_{v_i} = (E_v, (v_i, v_i))$ consists of all edges of $G$ which satisfy $v_i \in V$ for all $i$. If $J \subseteq V \times V$, then $(J, \subseteq V)$ is a subgraph of $(V, \subseteq V)$.

Let $G$ be a set of finite graphs. $G$ is transitive if $G$ is closed for the transitive closure of the vertices. If the vertices are not marked, this property automatically holds. It will be always negated without saying. We say that we choose a vertex $x \in V$. If $E = (x, y, z, \ldots) \in E$ the graph $G_{\{x, y, z, \ldots\}}$ contains the edges $(x, y), (y, z), (z, \ldots)$ if and only if $G$ contains the corresponding edges $(x, y), (y, z), (z, \ldots)$. $G$ is called transitive if for any number of $G_{v_i} \subseteq G$ holds.

$G$ is called boundary if for any spanned subgraph $G_{v_i}$ of $G$, $G_{v_i} \subseteq G$. If $G$ is not boundary, the boundary homot $G$ of $G$ can be produced in the following way: If $G$ is $G_{v_i}$ for some $v_i$, then $G_{v_i}$ is $G_{v_i}$ for any $v_i$. $G$ is $G_{v_i}$ if and only if all the spanned subgraphs (including $G_{v_i}$) are in $G$. It is easy to see that $G$ is always boundary. On the other hand, if $G$ is boundary, $G_{v_i}$ is also boundary.

Example 1: Put $n = 3$. Let $G_{v_i}$ consist of the graphs $G$ having the property that $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ contains at least one non-loop edge for any 3 different vertices $x_{v_i}, y_{v_i}, z_{v_i}$. (This is $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ consists of all the graphs with 1 and 2 vertices) It is easy to see that $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ contains the graphs with 1 and 3 vertices if $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ is boundary. However, if $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ is not boundary, the subgraph $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ contains the graphs with 1 and 3 vertices if $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ is not in $G$. Thus, the subgraph spanned by $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$ contains no edge.

Example 2: Put $n = 3$ and let $G_{v_i}$ consist of the graphs $G$ containing all the possible loops and having the property that $G_{v_i}^{x_{v_i}, y_{v_i}, z_{v_i}}$
contains at least one non-loop edge for any 3 different vertices $x_1, x_2, x_3$.

It is easy to see that $G_v$ is hereditary and absolute.

Let $M = (E, v, a)$ be a measure space with a finite measure. In this paper we shall consider only finite measures. Furthermore, let $s \in S^m$ be a measurable set in the product space $(E, v, s^m) = (E, v, s_1, \ldots, s_m)$. We define the measure of a graph $G = (E, E^G)$ in the following way:

$\mu(G) = \mu_v(E^G)$. If $G$ is a family of finite (directed) graphs then introduce the notation

\begin{equation}
\mu_v(G) = \int \frac{\mu_v(E^G)}{\mu_v(f)}
\end{equation}

where the integrals is taken over all measurable $E \subseteq E$ satisfying the condition:

(1) all the finite subgraphs $G_v \subset G$ of G are measurable with respect to $\mu_v$.

It is easy to see that in condition (2) and in definition (1) $G_v$ can be used equivalently in place of $G$. Thus, in the future we can always assume that $G$ is hereditary without any loss of generality.

If there is no graph with $I(F)$ vertices containing all of its induced subgraphs from $E$, then $\mu_v(G)$ is undefined. We assume in this paper, that this is not the case, when $\mu_v(G)$ is used, it is always understood that there exists such a graph with restricted $I(F)$.

If $F$ is finite, $G$ will always be the family of all subsets of $X$.

$G_v$ denotes the measure space $(X, \mu_v, G_v)$, where $\mu_v(x) = \frac{1}{I(F)}$ for all $x \in X$, $I(F)$ is called an index of the set $E \subseteq E$, $\mu_v(x) > 0$ if there is a set $E \subseteq E$ such that $x \in E$.

Example 1. Let $G_v$ be as in Example 1. The theorem of Turán [1] says that if $G$ is a finite graph (without loops with $n$ vertices contains no empty triangle then the minimal number of edges is

\begin{equation}
\frac{n^2 - n}{2}
\end{equation}

for even $n$ and

\begin{equation}
\left\lfloor \frac{n^2 - n}{2} \right\rfloor + 1
\end{equation}

for odd $n$. If $H = G_v$ and $I (E, E^G)$ is a graph satisfying $\mu_v(G_v)$, then...
Using our relation (1) we can write

\[ \frac{1}{2^n} \leq \frac{1}{n} + \frac{1}{2^{n+1}} = \frac{1}{n} + \frac{1}{2^n} \]

if \( n \) is even

\[ \frac{1}{2^n} \leq \frac{1}{n} + \frac{1}{2^{n+1}} = \frac{1}{n} + \frac{1}{2^n} \]

if \( n \) is odd.

Observe, that for the class \( G_2 \) of Example 2 we can write

(4) \[ \frac{1}{2^n} > \frac{1}{n} + \frac{1}{2^{n+1}} \]

The next lemma will express that if we identify elements in \( X \), \( \mathcal{H}(X) \) is no longer a measurable set. Let \( \mathcal{M} = (X, \mathcal{A}, \mu) \) be an arbitrary measure space and \( A \subseteq X \) a measurable set. Define \( \mathcal{M}^\prime = (X', \mathcal{A}', \mu') \) in the following way: \( X' = X \setminus \{a| B \subseteq X \} \) and \( \mathcal{A}' = \{ E \in \mathcal{A}| E \not\subseteq B \} \) for some \( B \subseteq X \). Then \( \mu' \) is defined by \( \mu'(E) = \mu(B - E) + \mu(A) \) otherwise.

Lemma 1. Let \( \mathcal{H} \) be a measurable class of directed graphs. If \( X = (X, \mathcal{A}, \mu) \) be an arbitrary measure space and \( A \subseteq X \) a measurable set. Then

(1) \[ \mathcal{H}^\prime = \{ H | H \subseteq \mathcal{H}(X) \} \]

Proof. Let \( X' \) in \( \mathcal{M} \) be a measurable set in \( \mathcal{M}^\prime \) such that all the measurable subgraphs of \( G = (X', \mathcal{G}) \) are in \( \mathcal{G} \). Define the graph \( G = (X, \mathcal{G}) \) in the following way: \( (x_1, \ldots, x_n) \in X \) if \( (x_1, \ldots, x_n) \in X' \) and \( (x_1, \ldots, x_n) \in X' \) are adjacent. We prove that \( G \) also has the property that all of its finite spanned subgraphs are in \( \mathcal{G} \).

Suppose, in the contrary, that \( G \) contains a (finite) spanned subgraph \( G_2 \notin \mathcal{G} \). There are two possibilities:
\[ \lvert T \cap A \rvert \leq 1, \]
\[ \lvert T \cap A \rvert > 1. \]

In case (a) put \( T = (T \setminus A) \cup \{a\} \). Then \( G^T a \) is isomorphic to \( G_0 \), consequently \( G^T a \) has a spanning subgraph isomorphic to \( G \), what is a contradiction. In case (b) we can obtain \( G^T A \), from \( G_0 \), by identifying the vertices in \( T \setminus A \). It means that \( G_0 \) can be obtained from \( G^T a \)

by repeated applications of shifting the vertex \( a \). As \( G \) is admissible \( \lvert G^T A \rvert = 0 \) follows from \( G_0 \not\subseteq G \), and this is a contradiction again.

Choose any \( \Delta^T_a \) satisfying (1) in \( 3^{\Delta^T} \). We just proved that the corresponding \( \Delta \) satisfies (2) for \( \Delta \subseteq \Delta^T_a \), too. Consequently for each \( \Delta^T \)

\[ \sum_{w \in \Delta^T} \Delta^T_w \cap \Delta^T_a \times \Delta^T_a \]

holds by (1). It is easy to see, that \( \Delta^T_a \circ \Delta^T_a = \Delta^T_a \) and \( \Delta^T_a \circ \Delta^T_a = \Delta^T_a \), from which we obtain by (1) that

\[ \Delta^T_a \circ \Delta^T_a = \Delta^T_a \circ \Delta^T_a = \Delta^T_a \circ \Delta^T_a \]

holds for any \( \Delta^T \) satisfying (2). Hence (2) follows using (1), again. The proof is completed.

**Lemma 2.** Let \( G \) be a countable class of directed graphs. Then the limit

\[ \lim_{n \to \infty} (9) \]

exists if furthermore \( M \) is an arbitrary nonzero ratio then

\[ (9) \quad M(G) = \lim_{n \to \infty} (9). \]

**Proofs.**

1. From Lemma 1 it follows by induction that

\[ \lim_{n \to \infty} (9) \]

where
\[ N^* = (V', d', a'), \quad V' = \{ v_1, \ldots, v_{d'} \}, \quad a'(v_j) = b_j \]

Consider, \( 1 \leq i \leq d \), \( \sum b_i = 1 \).

2. Let \( H = (V, a, p) \) be an arbitrary normed space and \( G = (E, \mathcal{P}) \) be a graph satisfying \( \mathcal{C} \subseteq (E, \mathcal{P}) \) (maximal). The relations

\[
\mu_0^* \geq \sum_{x \in V} \sum_{y \in V} [a(\mathcal{P}, x, y)]^2
\]

will be proved next. Define the function \( f \) on \( N^* \) in the following way:

\[
H(V_1, \ldots, V_{d'}) = \sum_{x \in V} \left( \prod_{1 \leq i \leq d'} a(x, v_i) \right)
\]

(1)

\[
H(V_1, \ldots, V_{d'}) = \sum_{x \in V} \left( \prod_{1 \leq i \leq d'} a(x, v_i) \right)
\]

(2)

Assume that \( v_1, \ldots, v_{d'} \) are all different. In the case \( H(V_1, \ldots, V_{d'}) \)

is simply the number of signs of \( \prod_{1 \leq i \leq d'} a(x, v_i) \). The signed subgraphs of

\( G(V_1, \ldots, V_{d'}) \) are signed subgraphs of \( G \), consequently \( G(V_1, \ldots, V_{d'}) \) satisfies condition (2).

(2)

\[
\mu_0^* \geq \sum_{x \in V} \left( \prod_{1 \leq i \leq d'} a(x, v_i) \right)
\]

(3)

It is easy to see that
\[ x^2 \Omega_{\alpha} \leq \int_{\Omega} x^4 \, dx \]

(12) yields

\[ x^2 \Omega_{\alpha} \leq \int_{h} x^2 \, dx \]

We need some more functions,

\[ \alpha_{i_1, \ldots, i_k} = \begin{cases} 1 & \text{if } (i_1, \ldots, i_k) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \]

(1 \leq i \leq n, 1 \leq k \leq d).

The equality

\[ \alpha_{i_1, \ldots, i_k} = \sum_{i=1}^{\infty} \alpha_{i_1, \ldots, i_k} \]

is obvious and leads to

\[ \int_{\Omega} x^2 \, dx = \sum_{i=1}^{\infty} \int_{\Omega} x^2 \, dx \]

Assume that \( \alpha_{i_1, \ldots, i_k} \) are different. Then

\[ \int_{h} x^2 \, dx = \sum_{i=1}^{\infty} \int_{h} x^2 \, dx \]

The inequality

\[ \int_{\Omega} x^2 \, dx \leq \]

(13) holds in this case, too; it follows from (9), (13) and (14).
follows from (14), (17) and the trivial estimation $\int f \, d\mathcal{H}^n \leq \mu \mathcal{H}^n (E)$.

4. As (15) holds for any $E$ satisfying (2), the inequality

$$\mathcal{H}^n (\mathcal{M}_n, G) \leq \frac{C}{n \cdot n!} \mathcal{H}^n (\mathcal{M}_n, G) = \mathcal{H}^n (\mathcal{B}, G)$$

follows from (1). Hence (15) holds if $n \to \infty$, consequently

(15) $\lim_{n \to \infty} \mathcal{H}^n (\mathcal{M}_n, G) = 0$.

If we prove (2), (16) is equivalent to (4). Moreover (16) is valid for $M = \mathcal{M}_n$, too.

Hence

$$\lim_{n \to \infty} \mathcal{H}^n (\mathcal{M}_n, G) = 0$$

and (7) holds. The lemma is proved.

Example 6. Let $G_{x}$ be chosen as in Examples 2 and 3. Choose $M = (0, \pi, a, a^2)$ to be the Lebesgue-measurable set on $(0,1)$ interval. Condition (2) on $E$ $\mathcal{B}^n$ can be interpreted as follows: any rectangle having one corner on the diagonal of the cube $[0,1]^n$ and exactly one corner on the diagonal must have another corner in $E$ (see Fig. 1a), and $E$ contains the diagonal.

![Fig. 1a](image1.png) ![Fig. 1b](image2.png)
Example 5. If we use $G_+ \in \mathcal{G}$ in the previous lemma, it gives the same result. In general, we shall see that the condition of doobility is unnecessary when $M$ is strong. However, this is not the case when $M$ just takes strong. Let, for instance, $M = (x, \alpha, \mu)$ be defined in the following way: $E = (x_i, x_i')$, $\alpha_i(x_i') = x_i'$, $\alpha_i' = 1 - \alpha_i$ where $\epsilon$ is a small positive number. Chosen the set $E = (x_i, x_i')$. Obviously $E$ satisfies (2) with $G_+$, but $\mu(E)^\varepsilon = 1$ if $x_i < x_i' <$ 

Consequently, if $M$ is not doobable, we are not able to give such a point-taking lower estimation for $(M, G, \mu)$ as (6). However, it is possible, even in this case, to give a lower estimation by doing $\mu(E)$, where the $\alpha$'s are not affected. First we present two easy lemmas, what we need here.

Lemma 3. Let $M = (f, x, \mu)$ and $M' = (f, x, \mu)$ be two measure spaces, assume $\mu(E(x)) \geq \mu(f(x))$ and suppose that

$$\sum_{i=1}^n \mu(E(x)_i) \geq \mu(E(x))$$

for any collection of disjoint sets $E(x)_i$. Then

$$\sum_{i=1}^n \mu(E(x)_i) \geq \mu(E(x))$$

for any collection of disjoint sets $E(x)_i$. Then

Proof. (Warning, it is easier to prove, than to read.)

It follows from (1) that for any $\lambda > 0$ there exists an $E \in \mathcal{E}$ satisfying (2) and

$$\mu(E(x)) \geq \frac{\mu(E(x))}{\lambda}.$$
\[ (a + m \oslash t_{x_1} - E_t) \circ [\phi] - E_t < 1 \]

This inequality obviously implies

\[ (a + m \oslash t_{x_1} - E_t) \circ [\phi] - E_t < 0 \]

and

\[ (a + m \oslash t_{x_1} - E_t) \circ [\phi] - E_t < 0. \]

Without loss of generality, we can suppose that there is a partition \( X = \{ x_1, \ldots, x_n \} \) (where \( x_i \in A \setminus \{ \mu \} \)) such that \( E_t \) is a union of rectangles of the form \( A_{x_1} \circ \cdots \circ A_{x_n} \). Then we can write

\[ (a + m \oslash t_{x_1} - E_t) \circ [\phi] - E_t = \]

\[ \sum_{x_1} \left( (a + m \oslash t_{x_1} - E_t) \circ [\phi] - E_t \right) < 0 \]

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\[
\begin{align*}
&\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \sum_{n=1}^{n} \left| \lambda_{ij}^{kl} \lambda_{mn}^{ij} \right| \\
&\left( \left| \lambda_{ij} \right| = \left| \lambda_{mn} \right| \right) \\
&\left( \left| \lambda_{ij} \right| = \left| \lambda_{mn} \right| \right)
\end{align*}
\]

Using (29). Observe that this inequality holds for \( \mathbb{Z}^n \) in place of \( \mathbb{Z} \). It follows that

\[
\begin{align*}
&\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \sum_{n=1}^{n} \left| \lambda_{ij}^{kl} \lambda_{mn}^{ij} \right| \\
&\left( \left| \lambda_{ij} \right| = \left| \lambda_{mn} \right| \right) \\
&\left( \left| \lambda_{ij} \right| = \left| \lambda_{mn} \right| \right)
\end{align*}
\]

From the inequality (29) and (29) we obtain

\[
\left| \lambda_{ij}^{kl} \right| \leq \left| \lambda_{ij} \right| \left| \lambda_{mn} \right|
\]
\[
\left| \frac{A(2)}{A(1)} - \frac{B(2)}{B(1)} \right| < \frac{1}{2} \left( \frac{1}{2(1)} - \frac{1}{2(2)} \right) = 2^{-1}\ln \frac{1}{\epsilon}.
\]

Consequently, letting \( \epsilon \) tend to 0 with 0, \( A(2) - B(2) \) tends to 0.

\[ A(2) - B(2) < 2^{-1}\ln \frac{1}{\epsilon}. \]

holds. We know from (1) that
\[ \frac{A(2)}{A(1)} > \frac{B(2)}{B(1)} \]
as \( \epsilon \) tends to 0. This implies by (20) that
\[ A(2) - B(2) < 2^{-1}\ln \frac{1}{\epsilon}. \]

Similarly,
\[ A(2) - B(2) < 2^{-1}\ln \frac{1}{\epsilon}. \]

and (22a) and (22b) are equivalent to (23). The proof is completed.

Lemma 6. Let \( \mathcal{X} = (X, \mathcal{B}, \mu) \) be a measure space, where \( X = \{x_1, \ldots, x_n\} \) (\( n \geq 2 \)), \( \mathcal{B} \) is a Borel \( \sigma \)-algebra, \( \mu \) is a positive measure. Let \( \mathcal{A} \) denote the set of subsequences \( \{x_{i_1}, \ldots, x_{i_k}\} \in \mathcal{X} \) containing a pair \( \{x_i, x_j\} \) (\( i, j \in \mathcal{X} \)) such that \( \mu(x_i, x_j) > 0 \). Then \( \mathcal{A} \) is a Borel set.

Proof. Let \( B \) denote the set of sequences \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \in \mathcal{A} \). \( B \) is a Borel set, where \( \mu_i = \mathbb{E} \). Obviously,
\[ \mu_i(x_i, x_j) = \mathbb{E} \mu(x_i, x_j). \]

It is sufficient to prove \( \mathcal{A} \mu = 0 \). If \( C \) denotes the set of pairs
Proof.

1. For sets of simplicity assume first that $x_i = 0$ when $i > r$ for some integer $r > 0$. The first assertion of the proof of Lemma 2 does not work last, because Lemma 2 holds only for countable $\mathcal{W}$'s. However, the second assertion works even with a modification. (13) does not hold, so (12) is not true, when the values $y_j$ are not necessarily different.

In this case we have to use the law of the large numbers. For any $x_0 > 0$, $x_1 > 0$, there are $x_i > 0$, such that the numbers $x_i$ of the $y_i$'s of the elements of $\mathcal{W}$ in the sequence $y_0, \ldots, y_n \in \mathcal{W}$ satisfy the conditions

$$
\left| \frac{x_i}{2^n} - \frac{x_0}{2^n} \right| < \varepsilon \quad (0 \leq i 

$$

with $\varepsilon$ small.

2. On the other hand, the maximum of the sequences $y_0, \ldots, y_n \in \mathcal{W}$ having $x_i = y_i \neq 0$ for $i > r$, as in Lemma 3, then, A holds with a maximal $\mu^{\mathbb{F}^*}$. Note that (18) and the $\varepsilon$ maximum being in $\mathcal{F}$ are sufficient. It means that $\mathbb{F}^{\mathbb{W}_r} \in \mathcal{W}$ (13) can be bounded from below (apart from a set of measure 0 in $\mathbb{F}^*$) by an $\omega(\mathcal{W}, \mathcal{W})$, where $\mathcal{W}$ is a discrete measure space with measures $\{\omega_0, \ldots, \omega_n\}$ (the number of $\omega_j$'s is 2, of some $\mathbb{F}^{\mathbb{W}_r}$ satisfying (18)). This is what we have instead of (13).

$$
\mu^{\mathbb{F}^*} \leq \omega_{0,0,0,\ldots,0}
$$

(19)
which holds with a measure $\mu(\mathbf{x}) = A$. It is easy to see that

(30) \[ M(W, G) = R(W, G) \]

If $M'$ is the normalized version of $M$ (this holds the constants by $\alpha$. So we can write

(31) \[ \frac{1}{\alpha} M(W, G) \cdot \frac{1}{\alpha} M(W, G) < \sum_{i=1}^{k} \frac{1}{\alpha^2} \]

If $M'$ runs over the constant space with constants $\frac{1}{\alpha}, \ldots, \frac{1}{\alpha} \cdot \frac{1}{\alpha}$ satisfying (28). However, $M'$ differs from $M(W, G)$ only by a sign. It follows from (26) that (27) is satisfied if $\frac{1}{\alpha^2}$ is the measure $M'$ and $M(W, G)$. Using Lemma 1 we obtain

(32) \[ |R(W, G) - R(W, G)| \leq 2 \cdot \frac{1}{\alpha^2} \]

(31) and (32) result in

(33) \[ M(W, G) - M(W, G) = 2 \cdot \frac{1}{\alpha^2} + 2 \cdot \frac{1}{\alpha^2} \]

For a set $A \subseteq X'$ with a measure at least $\mu(A) = \frac{1}{\alpha^2}$ and hence

(34) \[ \int A \alpha d\mu = \int A \alpha d\mu \]

\[ \alpha d\mu = 2 \cdot \frac{1}{\alpha^2} + 2 \cdot \frac{1}{\alpha^2} \]

follows. This is the substitute of (13). Then $\alpha$ remains unchanged. From (33) and (34) we obtain

(35) \[ \mu(\mathbf{x}^*) - \mu(\mathbf{x}) \leq 2 \cdot \frac{1}{\alpha^2} + 2 \cdot \frac{1}{\alpha^2} \]

instead of (30). This inequality holds with arbitrary small $\varepsilon$ and $\delta$ if $\alpha = \alpha_0$. On the other hand $\alpha = \alpha_0$.
holds for any $\alpha > 0$ if $n$ is large enough, and $F$ satisfies (2).

(7) $\mu Q_a(n, \beta_0) - \alpha < \mu H(n)$

follows from (6), consequently

(8) $\sum_{n=0}^{\infty} \mu H(n, \beta_0) - \alpha < \mu H(n, \beta_0)$

holds.

2. Repeat the proof with $\mu H(n, \beta_0)$ in place of $\mu$ (or in place of $\mu H(n, \beta_0)$, since $F = F_1, \ldots, F_{n-1}$, where $F = F_1$).

3. If the measure of the set of sequences $(x_1, \ldots, x_n)$ with non-zero elements from $X$ is $\mu(X^n - X)$, then $\mu(X^n - X)$ holds for a set of sequences with measure $\mu(X^n - X) = 0$. Consequently, (30) and (35) are true with $\mu(X^n - X) = 0$ on the plane of $\mu(X^n - X)$.

3.1. It is easy to see that this implies that

$\mu H(n, \beta_0) - \alpha < \mu H(n, \beta_0)$

exists. The case where $\alpha = 0$ if $n > 0$ is proved.

3.2. Prove that the limit (2) exists in the general case, when $n$ has infinitely many measure terms. Denote the sequence $\{x_1, x_2, \ldots, x_n\}$ by $x_n$. Prove that $\mu H(x_n, \beta_0)$ converges if $\alpha > 0$. Fix an $\alpha > 0$ and choose $n_0 = n_0(\alpha)$ so that $\sum_{n=n_0}^{\infty} \alpha < \varepsilon$ holds. Such an $n_0$ exists since the measure is finite. The measure and
in $W_p^{1,2}(\Omega)$ and $W_p^{1,2}(\Omega)^\perp$ by at most $\varepsilon$. Let $m_1, m_2 \geq m_0$. Then, we can use Lemma 3.

(37) $\|\nabla(u - v)\|_{L^2(\Omega)} \leq \frac{1}{2} \|u - v\|_{L^2(\Omega)}$.

(38) $\|\nabla(u - v)\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|u - v\|_{L^2(\Omega)}^2$. On the other hand, choose an $s = \frac{1}{2} \|u_0 - v_0\|_{L^2(\Omega)}$ so that

(39) $\|u_0 - v_0\|_{L^2(\Omega)}^2 - 2\|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 < \varepsilon$.

Subsequently, both (38), (40) and (41) it easily follows that

(40) $\frac{1}{2} \|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 < \varepsilon$.

(41) $\frac{1}{2} \|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 < \varepsilon$.

which is absurdly small if $m_0$ is large enough. Consequently, the limit

$\lim_{i \to \infty} \int_\Omega \nabla u_i \cdot \nabla v \, dx$ exists by definition.

(42) $\|u_0 - v_0\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 < \varepsilon$.

(43) $\|u_0 - v_0\|_{L^2(\Omega)}^2 - 2\|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 < \varepsilon$.

and hence $\frac{1}{2} \|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2$. Finally, the distributions of $W_p^{1,2}(\Omega)$ and $W_p^{1,2}(\Omega)$ differ by at most $\varepsilon$. Let $m_1, m_2 \geq m_0$. Then, we can use Lemma 3.

(44) $\|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 - 2\|\nabla(u_0 - v_0)\|_{L^2(\Omega)}^2 < \varepsilon$. Summarizing (34), (43) and (40) we obtain

(45) $\|\nabla(u - v)\|_{L^2(\Omega)}^2 - 2\|\nabla(u - v)\|_{L^2(\Omega)}^2 < \varepsilon$. Summarizing (42), (43) and (40) we obtain

(46) $\|\nabla(u - v)\|_{L^2(\Omega)}^2 - 2\|\nabla(u - v)\|_{L^2(\Omega)}^2 < \varepsilon$. Summarizing (42), (43) and (40) we obtain

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(60) $\|\nabla(u - v)\|_{L^2(\Omega)}^2 < \varepsilon$. Summarizing (42), (43) and (40) we obtain
6. Since (37) is a special case of Lemma 3, we have that:

\[ \sum \int M(a, b, c) \, dx < \sum \int f(x) \, dx + \epsilon. \]

Remark 1. This theorem is a special case of Lemma 3. We have proved it separately to show the basic idea of the proof, which is contained in the following technical details:

Remark 2. This result is a special case of Lemma 3. We have proved it separately to show the basic idea of the proof, which is contained in the following technical details:

Remark 3. The existence of the limit (38) cannot be considered to be new. Although it has never been formulated in such a general form, the result can be seen as a special case of (38) can be essentially used here, too (see also [105, 113]).
To the application (see [11] and a forthcoming paper) we need the
formulations of (6) and (7). All of our information concerning this is
included in Lemma 5. However, the key is in understanding when
there is equality in (7). I conclude that it holds under the conditions of
Lemma 5, but I was not able to prove it.

Theorem 1. Let G be a class of digraphs and

\[ H = \{S, T\} \]

be a spanning graph, where \( S = (x_1, x_2, \ldots, x_n) \) and \( T = (y_1, y_2, \ldots, y_m) \)

\( (x_i, y_i) \in E \) and \( x_i = y_i \). Then \( H \) is a subgraph of \( G \).

(40) \( \langle H, G \rangle = \langle V(G), H, G \rangle \).

Proof. Lemma 5 gives one side of (40), thus we have to prove

(40) \( \langle H, G \rangle \leq \langle V(G), H, G \rangle \).

By (5) by (4) we know that there is a partition

\[ A_1 \cup A_2 \cup \ldots \cup A_p \]

of \( H \) such that \( A_i \) are measurable with \( m(A_i) \leq \frac{1}{p} \). By

(41) \( \langle H, G \rangle \leq \langle V(G), H, G \rangle \).

(42) \( \langle H, G \rangle \leq \langle V(G), H, G \rangle \).

where \( \langle H, G \rangle \leq \langle V(G), H, G \rangle \) is a distribution with

\[ \sum_{i=1}^{p} \frac{1}{p} \leq \sum_{i=1}^{p} \frac{1}{p} \]

Consequently, by Remark 3, the limit of (41) is

(43) \( \langle H, G \rangle \leq \langle V(G), H, G \rangle \).

This proves (40) and the theorem.

I conjecture that the theorem holds for non-directed graphs, too,

However I was not able to prove it in the general case. A very special

case follows from the following theorem of Bourbaki, Erdős and Marcus...
[15] A class $G$ of directed $2$-graphs is called strongly hereditary if $G \subseteq H$ implies $G \subseteq H$ for any $H$ (not necessarily spanned) subgraph of $G$.

Theorem 6.1. Let $G$ be a strongly hereditary class of directed $2$-graphs. Then there exists an $r \times r$ matrix $A=(a_{ij})$ with the following properties. For any $k$, let $\mathcal{G}$ be a class of directed $2$-graphs $G$ of size $k$ such that $\mathcal{G} \subseteq G$. Then $\mathcal{G}$ is strongly hereditary if and only if $\sum_{i,j} a_{ij} = 1$ for all $i,j$.

Proof. We have to prove:

(15) $\sum_{i,j} a_{ij} = 1$ \iff $\mathcal{G}$ is strongly hereditary.

We do this by constructing a graph $(X,E)$ on $X$. If $A_1,A_2,\ldots,A_k$ are such that $\sum_{i} A_i = (X,E)$ then there is a partition $X = C_1 \cup \cdots \cup C_k$ with $\mathcal{A}_i \subseteq C_i$ where $\mathcal{A}_i$ is defined in (14). Let $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k$ denote the sets of vertices in $X$ for which the vertex $x \in C_i$ are connected with all the vertices in $C_j$. If $a_{ij}=1$ then $x \in C_i$ is not connected to $x \in C_j$. If $a_{ij}=0$, we use a disjoint partition $C_i \cap C_j = \emptyset$. $\mathcal{A}_i \subseteq C_i$, $\mathcal{A}_j \subseteq C_j$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$. It follows from Theorem 6.1 that all the finite spanned subgraphs of $G(A_1,\ldots,A_k)$ belong to $G$ (assuming that it is strongly hereditary).
For the other hand it is easy to see that $F_{\alpha}(1, \ldots, K)$ is measurable. Using (1) we obtain
\begin{equation}
\frac{\mu^{R}(\alpha)\delta_{\alpha}(1, \ldots, K)}{\mu^{R}(\alpha)} = \frac{\mu^{R}(\alpha)\delta_{\alpha}(1, \ldots, K)}{\mu^{R}(\alpha)}
\end{equation}

Let now $\alpha_{n}$ be equal to $\frac{1}{n^{2}}$ Then,
\begin{equation}
\left| \frac{\mu^{R}(\alpha)\delta_{\alpha}(1, \ldots, K)}{\mu^{R}(\alpha)} - \frac{1}{n^{2}} \right| \leq \frac{1}{n^{2}} \leq \frac{1}{n^{2}}
\end{equation}

only holds for any $n$. Suppose $N = \infty$, (32) and the theorem follows from (31), (34) and (35). The proof is complete.

Remark 4. Let us define $HR(G, \delta)$ or $HR(G)$ in (1) but using any rather than $\delta$. Let further $G$ denote any set of complements of the elements of $G$. Obviously, all the spanned subgroups of $(E, F)$ belong to $G$ of all the spanned subgroups of $(E, F')$ being to $G$. Consequently,
\begin{equation}
HR(G, \delta) = HR(G) = 1.
\end{equation}

It follows that Theorems 1 and 2 hold for $HR(G, \delta)$ and Lemma 5 holds with the opposite direction of the inequality sign. Lemma 1 also holds if $G$ is $\delta$-independent.

Remark 5. Throughout, we considered the case of directed graphs. In case of undirected graphs we usually consider any undirected edge by the set of all connected subsets of it. Let us see an example.

Example 6. We try to be in the continuous version of a theorem of Bollobás [12], meaning that in an undirected graph $G$ we allow the loops we allow with in vertices containing no 2 different edges $a, b, c$ with $(x - y)^{2} < e^{2}$ the number of edges is $g^{2}$.

Let now $G$ be the set of directed graphs not containing edges $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$, where the letters with the same indices denote edges with internal (1 different) vertices but with different
directions, and those (3 different) sets satisfy \( (x - 3) \cup (y - 3) - (x - y) \subseteq c \). What is \( M(G, G) \) in this case? Failing any directions for all the targets we obtain from Solikh’s theorem that we can choose at most \( \left( \frac{3}{2} \right) \) edges from these \( \{x, y\} \). This is true for all the 6 cases. On the other hand we can choose all the \( H(n, k) \) for \( n \geq 2 \). This is,

\[
\frac{3n - \frac{3}{2}n + 1}{2} > \text{max} \{n, k\}.
\]

From Lemma 5 and Remark 5: \( M(G, G) \leq \frac{3}{2} \) for absolute measure spaces \( M \). It is now to construct a graph \( (G, G) \) with measure \( \frac{3}{2}(G, G) \). Divide \( G \) into 3 disjoint classes \( E_1, E_2, E_3 \), with equal measures and take all the edges having their coordinates in different classes. This shows that \( M(G, G) \leq \frac{3}{2} \) for absolute measures.

Remark 5. In [11], Erdős and Simonovits proved that

\[
\max M(G, G) = \frac{1}{p - 1},
\]

where \( G \) is a class of undirected graphs and \( p \) is the minimum of the chromatic numbers of the graphs in the class. \( G \) and \( p > 1 \). (The latter condition simply expresses that \( G \) contains all the complete graphs).

Lemma 5 and a trivial construction imply that \( M(G, G) = \frac{1}{p - 1} \) for an absolute measure \( M \) under the above conditions.

A GENERALIZATION

For some applications (see [11]) and a forthcoming paper we need a generalization. It does not differ too much from the previous results, neither in form nor in proving. Thus we give here briefly the necessary definitions and completely omit the proof.

Let \( GO \) denote a class of directed graphs, where the vertices of the graphs are colored by \( f \) colors (no assumption on the coloring).

We say that \( GO \) is available if doubling any vertex of \( G \in GO \) and
given the same old label to both then for the new graph $G_t \equiv G(0)$, again.

Let $M = (X, \mathcal{E}, \mu)$ be a measure space and $X = X_1 \cup \ldots \cup X_N$ a
disjoint partition. Colour $X_t$ by the $t$th colour. Then define

$$H(M, X_1, \ldots, X_N; G(0) = \frac{1}{N} \sum_{x \in X} \mathcal{E}^x_{\mu}(x)$$

where the infinitesimal measure of measurable $X \in \mathcal{E}$ such that every finite
spanned subgraph of $G \equiv (X, \mathcal{E})$ is isomorphic and colored identically
with an element of $G(0)$. $\mathcal{E}^x_{\mu}$ denotes the finite measure space with
$\frac{\mu(x)}{\sum \mu(x)}$ elements, with uniform distribution $\frac{1}{N}$ and
with partition $\{I_j\} = x$. 

Theorem 3. Let $G(0)$ be a set of embedded directed graphs, and
$M = (X, \mathcal{E}, \mu)$ be a measure space with a disjoint partition $X_1, \ldots, X_N$.
Suppose further that either (i) is countable or $M$ is atomless. Then

$$H(M, X_1, \ldots, X_N; \mu) = \inf H(M, X_1, \ldots, X_N; G(0))$$

holds if $x_1, \ldots, x_N$ are arbitrary satisfying

$$\frac{\mu(x)}{\sum \mu(x)} \leq \frac{1}{N} \quad (1 \leq i \leq N).$$

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Remark added in August 1977. Recently B. Bollobás [10] has
shown me that he can prove Lemma 2 under the condition
that $G$ is strongly hereditary.
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