FAMILIES OF SUBSETS HAVING
NO SUBSET CONTAINING
ANOTHER ONE WITH SMALL
DIFFERENCE
by G. O. H. KATONA

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Introduction

We prove the following theorem: Let $X = x_1, x_2, \ldots, x_n$ be a family of different subsets of a set $S$ of $n$ elements. If no two of them possess the proper part

$$x_i \supset x_j \quad (i \neq j),$$

then $n \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and this is the best possible estimate. The latter theorem serves as the main result of a paper on the authors' problems of combinatorics.

Theorem 1. Let $X = x_1, x_2, \ldots, x_n$ be a family of different subsets of a set $S$ of $n$ elements. If no two of the subsets satisfy $x_i \supset x_j \supset x_k$, where $k$ is a given positive integer, then

$$n \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

and this is the best possible estimate.

Theorem 2. Similarly, we prove a counting of

Harary's theorem, let $X = x_1, x_2, \ldots, x_n$ be a partition of $S$. If

then $n \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and this is the best possible estimate.

Proof.

Theorem 1 (Adrian Bondy)
We will prove Theorems 3 and 4 in a more general language, which can be realized for the following functions $f(x)$ and $g(x)$.

**Theorem 6.** Let $a_1, \ldots, a_n$ be positive integers and $f(x)$ be the function defined by

\[ f(x) = \prod_{i=1}^{n} a_i^x. \]

The number of roots of $f(x) = 0$ is either 0 or 1, which may be computed as follows:

\[ \frac{1}{n!}(1^n - 1). \]

The estimate in the best possible case.

**Definitions and Theorems**

We say that the class set $A$ is a partially ordered set if a relation $\leq$ is defined on $A$ such that the following properties hold for all $a, b, c \in A$:

1. $a \leq a$ (Reflexivity)
2. If $a \leq b$ and $b \leq c$, then $a \leq c$ (Transitivity)
3. If $a \leq b$ and $b \leq a$, then $a = b$ (Antisymmetry)
4. If $a \leq b$ and $b \leq c$, then $a \leq c$ (Comparability)

We say that $A$ is a finite set if $A$ is a partially ordered set and $A$ is finite.

We say that $A$ is a totally ordered set if $A$ is a partially ordered set and $A$ is totally ordered.

We say that $A$ is a linearly ordered set if $A$ is a partially ordered set and $A$ is linearly ordered.

We say that $A$ is a well-ordered set if $A$ is a partially ordered set and $A$ is well-ordered.

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For $1 \leq n$, let $\gamma_n = r^{-n} = \frac{1}{r^n}$.

Theorem 1: If $x, y$ are $\alpha$ and $\eta$ approximately $\alpha$-determined, then $x \leq y$.

Proof: The proof is straightforward and follows from the definitions of $\alpha$ and $\eta$.

Theorem 2: If $x, y$ are partially ordered sets, then the interval $[x, y]$ is the set of all elements $z$ such that $x \leq z \leq y$.

Proof: By definition, the interval $[x, y]$ consists of all elements $z$ that are less than or equal to both $x$ and $y$, and greater than or equal to both $x$ and $y$. This follows directly from the definition of a partially ordered set.

Corollary: If $x, y$ are partially ordered sets, then $x \leq y$ if and only if $[x, y]$ is a non-empty set.

Proof: This follows from the definition of a partially ordered set and the properties of intervals in such sets.
Theorem 1: Let A and B be two symmetrical chains and C be their join \( C \) and \( C' \). Consider the set of elements of \( C \) such that \( C' \) is the join of two elements of \( C \) that satisfy the condition:

\[
C_1 \cup C_2 = C
\]

where \( C_1 \) and \( C_2 \) are elements of \( C \) and \( C' \) is their join. The elements \( C_1 \) and \( C_2 \) are considered as a part of the join \( C' \) in \( C \). This condition is satisfied when \( C_1 \) and \( C_2 \) are not such that \( C_1 \cap C_2 = \emptyset \).

Since \( C \) and \( C' \) are symmetrical chains, the set of elements of \( C \) that satisfy the condition are the same as the set of elements of \( C' \). Therefore, we can say that the elements of \( C \) that satisfy the condition are exactly the elements of \( C' \).

Theorem 2: Let \( A \) be a symmetrical chain with \( \text{join} \) \( A' \). Consider the set of elements of \( A \) such that \( A' \) is the join of two elements of \( A \) that satisfy the condition:

\[
A_1 \cup A_2 = A'
\]

where \( A_1 \) and \( A_2 \) are elements of \( A \) and \( A' \) is their join. The elements \( A_1 \) and \( A_2 \) are considered as a part of the join \( A' \) in \( A \). This condition is satisfied when \( A_1 \) and \( A_2 \) are not such that \( A_1 \cap A_2 = \emptyset \).

Since \( A \) is a symmetrical chain, the set of elements of \( A \) that satisfy the condition are the same as the set of elements of \( A' \). Therefore, we can say that the elements of \( A \) that satisfy the condition are exactly the elements of \( A' \).
The proof of Theorem 2 follows the proof of Theorem 1.

**Proof of Theorem 2.** By the definition of the zero simplicial sets, $0$ and $0'$ are defined as the simplicial sets of length 0 and 0', which are the simplicial sets of length 0 and 0', respectively. The proof of Theorem 2 proceeds similarly to the proof of Theorem 1. We define a function $f$ on the simplicial set $X$ such that $f(x) = 0$ if $x$ is a vertex of $X$, and $f(x) = 0'$ if $x$ is a non-vertex of $X$. Then, for any simplicial set $Y$, the function $f$ induces a function $f_X: X \to Y$ such that $f_X(x) = f(x)$ for all $x \in X$. We then show that $f_X$ is a bijection on the simplicial set $X$. This completes the proof of Theorem 2.
The the claims will be the following:

\[ x \]  \[ y \]  \[ z \]  \[ w \]  \[ t \]

To formally, we see clearly between a, b and where f, g:

\[ f(x) = g(y) = h(z) = i(w) = j(t) \]

\[ \ldots \]

\[ \ldots \]
From Fig. 1 it is easy to see that the two chains are either of length 4 or larger. However, if $a_j$ is of length 4 or larger, then the sum of all the terms of $a_j$ in the product $a_j(a_j + 1)$ is equal to the product of $a_j$ and its conjugate $a_j^T$. If $a_j$ is of length 4 or larger, then the sum of all the terms of $a_j$ in the product $a_j(a_j + 1)$ is equal to the product of $a_j$ and its conjugate $a_j^T$.

If $a_j$ is a, then the situation is similar to the paper and we have only two examples: un

Fig. 1.

In the case of the frame, which is a generalization of the basic ideas of [1],

If $a_j$ is a, then $a_j + 1$ where $a_j$ is not totally reduced, and

let $a_j = b_j$ be the length of the chain $a_j$. $a_j$ is replaced by the simple of $a_j$.

In a simple chain, the chain $a_j$ is replaced by the simple $a_j$.
and if a numerical chaos set and its graph may be shown as

where

and

(See Fig. A.)

Fig. 1.

Applying the same we obtain that if \( a \) and \( b \) are numerical chaos sets, and in addition, the graph may be shown as

where

and

1. \( G_n \) if the number of old ones is even

2. \( G_n \) if the number of odd ones is odd.

In the second case as \( a \) is a \( b \) we may also write the rule in the form

or \( a \rightarrow b \). The proof is completed.

Proof of Theorem 8. Let \( c \) and \( d \) be the numbers of the sets \( a \) and \( b \), respectively. The conditions of taking \( c \) and \( d \) are equivalent in this case. If \( c \) is an even 2-numerical chaos set, by Theorem 3, since \( n_1 + n_3 = n_2 \) all the ones are even for \( n = 2 \). Consequently, the sets \( c_n \) and \( d_n \) are given by

\[ c_2 = c_2 + d_2 \]

\[ c_1 + c_3 = d_1 + d_3 \]

and

\[ c_n = c_n + d_n \] for \( n = 1 \) and \( c_n = c_n + d_n \) for \( n = 2 \). The proof is completed.

Proof of Theorem 9. We apply Theorem 3 for the subsets of \( a_n \), which is an even 2-numerical chaos set by Theorem 1. For taking \( a \) and \( b \) we assume \( n = 1 \) and \( n = 2 \). The proofs are completed.
Proof of Theorem 2. Consider the case \( k = 1, 2, 3, \ldots \) as we did. Let \( S_i \) be
the set of indices \( i \) when \( x_i = 1 \) in the first sum. If \( x_i = 2 \) and
\( x_{i+1} = 3 \), then for the corresponding sum
\[
\sum_{k=1}^{3} x_k = x_1 + x_2 + x_3
\]
From the construction, we can choose \( x_1 \) and \( x_2 \) to
be \( x_3 \), which is a contradiction. The same holds for the second case and so on. Thus we can apply
theorems for \( x_1 = x_2 = \cdots = x_n \).

The proof is completed. If we take \( x \) to be not a multiple of the number \( n \),
then there is an element \( x_i \) that is not a multiple of \( n \).

\[
\leq \sum_{k=1}^{n} \frac{d_k}{n} \leq \frac{d}{n}
\]

The following example shows the case:
Here is the text from the image:

See [x] and [y] for a description of the basic facts. The set of points given in the figure here are one set of points of form

\[(x)\]

with length \(L\) and total number \(N\); while the number of elements of the two largest "blocks" (or \(k\)) in \(N\) is easy to see even if we exclude the configuration

\[(y)\]

with \(r = 1\) points and with length \(L\), from the general statement. Follows from the proofs. The general is the configuration.

\[(z)\]

is excluded in [x] instead of \((y)\) with \(x = y\). The combination above two to exclude \((y)\) with length \(L\) to too small, however, to exclude the configuration. So that a good condition between \((x)\) and \((z)\) would be interesting.

References
