The number of triangles is more when they have no common vertex

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Abstract By the theorem of Mantel [5] it is known that a graph with \( n \) vertices and \( \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \) edges must contain a triangle. A theorem of Erdős gives a strengthening: there are not only one, but at least \( \left\lfloor \frac{n^2}{4} \right\rfloor \) triangles. We give a further improvement: if there is no vertex contained by all triangles then there are at least \( n - 2 \) of them. There are some natural generalizations when (a) complete graphs are considered (rather than triangles), (b) the graph has \( t \) extra edges (not only one) or (c) it is supposed that there are no \( s \) vertices such that every triangle contains one of them. We were not able to prove these generalizations, they are posed as conjectures.

1 Introduction

All graphs considered in this paper are finite and simple. Let \( G \) be such a graph, the vertex set of \( G \) is denoted by \( V(G) \), the edge set of \( G \) by \( E(G) \), the number of vertices in \( G \) is \( v(G) \) and the number of edges in \( G \) is \( e(G) \). We denote the degree of a vertex \( v \) by \( d(v) \), the neighborhood of \( v \) by \( N(v) \), the number of edges between vertex sets \( A \) and \( B \) by \( e(A, B) \) and the number of triangles in \( G \) by \( T(G) \). A triangle covering set in \( V(G) \) is a vertex set that contains at least one vertex of every triangle in \( G \). The triangle covering number, denoted by \( \tau_\Delta(G) \), is the size of the smallest triangle covering set. Let \( S \subset V(G) \) be any subset of \( V(G) \), then \( G[S] \) is the subgraph induced by \( S \).

Mantel [5] proved that an \( n \)-vertex graph with \( \left\lfloor \frac{n^2}{4} \right\rfloor + t \) \((t \geq 1)\) edges must contain a

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triangle. In 1941, Rademacher (unpublished, see [1]) showed that for even \( n \), every graph \( G \) on \( n \) vertices and \( \frac{n^2}{4} + 1 \) edges contains at least \( \frac{n}{2} \) triangles and \( \frac{n}{2} \) is the best possible. Later on, the problem was revived by Erdős, see [1], which is now known as the Erdős-Rademacher problem, Erdős simplified Rademacher’s proof and proved more generally that for \( t \leq 3 \) and \( n > 2t \) case. Seven years later, he [2] conjectured that a graph with \( \left\lfloor \frac{n^2}{4} \right\rfloor + t \) edges contains at least \( t \left\lfloor \frac{n}{2} \right\rfloor \) triangles if \( t < \frac{n}{2} \), which was proved by Lovász and Simonovits [4]. Motivated by earlier results, we give a further improvement for the case \( t = 1 \): if there is no vertex contained by all triangles then there are at least \( n - 2 \) of them in \( G \).

**Theorem 1** (Mantel [5]). The maximum number of edges in an \( n \)-vertex triangle-free graph is \( \left\lfloor \frac{n^2}{4} \right\rfloor \). Furthermore, the only triangle-free graph with \( \left\lfloor \frac{n^2}{4} \right\rfloor \) edges is the complete bipartite graph \( K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \).

**Theorem 2** (Erdős [1]). Let \( G \) be a graph with \( n \) vertices and \( \left\lfloor \frac{n^2}{4} \right\rfloor + t \) edges, \( t \leq 3 \), \( n > 2t \), then every \( G \) contains at least \( t \left\lfloor \frac{n}{2} \right\rfloor \) triangles.

Before presenting our main result, the following definitions, a theorem and a lemma are needed.

**Definition 1.** Let \( K_{i,n-i} \) denote a the complete bipartite graph on the vertex classes \( |X| = i \), \( |Y| = n - i \).

**Definition 2.** Let \( K^-_{i,n-i} \) denote a graph obtained from a complete bipartite graph \( K_{i,n-i} \) plus an edge in the class \( X \) with \( i \) vertices, see Figure 1.

**Definition 3.** Let \( K^T_{i,n-i} \) denote a graph obtained from a complete bipartite graph \( K_{i,n-i} \)
minus an edge plus two adjacent edges in the class $X$ with $i$ vertices, one end point of the missing edge is the shared vertex of these two adjacent edges and the other one is in the class $Y$, see Figure 1.

**Lemma 3.** Let $G$ be a graph with $n$ vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, such that $\tau_{\Delta}(G) = 1$ and $T(G) \leq n - 3$. Then $G$ is one of the following graphs: $K_{\frac{n+1}{2}, \frac{n+1}{2}}$, $K_{\frac{n-1}{2}, \frac{n-1}{2}}$, or $K_{\frac{n+1}{2}, \frac{n-1}{2}}$.

**Theorem 4.** Let $G$ be a graph with $n$ vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, then either $\tau_{\Delta}(G) = 1$ or $T(G) \geq n - 2$.

## 2 Proofs of the main results

**Proof of Lemma 3.** Let $v_0$ be such a vertex that $G \setminus v_0$ contains no triangle. We distinguish two cases.

**Case 1.** $G \setminus v_0$ contains at least one odd cycle. Let $C_{2k+1}$ ($k \geq 2$) be the shortest odd cycle in $G \setminus v_0$ and $G'$ be the graph obtained from $G$ by removing the vertices of $C_{2k+1}$ and $v_0$, so $v(G') = n - 2k - 2$. Since $C_{2k+1}$ is the shortest cycle in $G \setminus v_0$, each vertex in $G'$ can be adjacent to at most 2 vertices in the $C_{2k+1}$, otherwise, we can find a shorter odd cycle. Since $G'$ is an $(n - 2k - 2)$-vertex triangle-free graph, by Theorem 1, $e(G') \leq \left\lfloor \left( \frac{n - 2k - 2}{2} \right)^2 \right\rfloor$.

Obviously, any two vertices of $C_{2k+1}$ are not adjacent, therefore

$$e(G \setminus v_0) \leq 2k + 1 + 2(n - 2k - 2) + \left\lfloor \left( \frac{n - 2k - 2}{2} \right)^2 \right\rfloor$$

$$= k^2 - nk + \left\lfloor \frac{n^2}{4} \right\rfloor + n - 2$$

$$\leq \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2 \ (k \geq 2).$$

Since $e(G) = d(v_0) + e(G') \leq (n - 1) + \left( \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2 \right) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$, the only possibility for $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ is that $d(v_0) = n - 1$ and $e(G \setminus v_0) = \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2$. In this case, we get $T(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2$, which contradicts $T(G) \leq n - 3$. 
Case 2. $G \setminus v_0$ has no odd cycles, then $G \setminus v_0$ is a bipartite graph and $e(G \setminus v_0) \leq \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor$. There are two subcases.

Case 2.1. $e(G \setminus v_0) = \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor$. Then $G \setminus v_0$ is $K_{\left\lceil \frac{n+1}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor}$ and $d(v_0) = e(G) - e(G \setminus v_0) = \left\lceil \frac{n}{2} \right\rceil + 1$. Let $d_1$ and $d_2$ be the numbers of neighbors of $v_0$ in classes $X$ and $Y$ of $K_{\left\lceil \frac{n+1}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor}$, respectively, then $d(v_0) = d_1 + d_2$ and $T(G) = d_1d_2$. So we need $d_1 + d_2 = \left\lceil \frac{n}{2} \right\rceil + 1$ and $d_1d_2 \leq n - 3$ hold true at the same time. When $n$ is even, we can see that the only solution is when $d_1 = 1$ and $d_2 = \frac{n}{2}$. The symmetric solution, $d_1 = \frac{n}{2}$, $d_2 = 1$ is not possible, since $d_1 \leq \frac{n}{2} - 1$ in this case. Therefore, we get that $G$ is $K_{\frac{n-1}{2}, \frac{n}{2}}$. Assume now that $n$ is odd, there are two possibilities,

(i) $d_1 = 1$ and $d_2 = \frac{n-1}{2}$, in the same way as in the even case, we get $T(G) = \frac{n-1}{2}$ and $G$ is $K_{\frac{n+1}{2}, \frac{n-1}{2}}$. When $d_1 = \frac{n-1}{2}$ and $d_2 = 1$, we also get $T(G) = \frac{n-1}{2}$ and $G$ is $K_{\frac{n+1}{2}, \frac{n-1}{2}}$.

(ii) $d_1 = 2$ and $d_2 = \frac{n-3}{2}$, then $T(G) = 2(\frac{n-3}{2}) = n - 3$ and $G$ is $K_{\frac{n+1}{2}, \frac{n-3}{2}}$. Similarly, when $d_1 = \frac{n-3}{2}$ and $d_2 = 2$, we get the same result.

Case 2.2. $e(G \setminus v_0) = \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor - t$. Then $d(v_0) = \left\lceil \frac{n}{2} \right\rceil + 1 + t, 1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 2$. Let $G \setminus v_0$ be the bipartite graph with partitions $X'$ and $Y'$, where $|X'| = i'$, then we have

$$i'(n-1-i') \geq \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor - t$$

$$\Rightarrow \begin{cases} \frac{n-1-\sqrt{4t+1}}{2} \leq i' \leq \frac{n-1+\sqrt{4t+1}}{2}, & \text{n is even,} \\
-\frac{n-1-2\sqrt{t}}{2} \leq i' \leq \frac{n-1+2\sqrt{t}}{2}, & \text{n is odd.} \end{cases}$$  

(1)

Suppose $v_0$ has $d_1$ ($\geq 1$) neighbors in $X'$ and $d_2$ ($\geq 1$) neighbors in $Y'$. Since $G \setminus v_0$ is bipartite, if $d_1d_2 = 0$, then $G$ contains no triangle which contradicts the fact that $\tau_\Delta(G) = 1$. In this situation, $d_1d_2 \geq T(G) \geq d_1d_2 - t = d_1(\left\lfloor \frac{n}{2} \right\rfloor + 1 + t - d_1) - t = -d_1^2 + (\frac{n}{2} + 1 + t)d_1 - t \geq -d_1^2 + (\left\lfloor \frac{n}{2} \right\rfloor + 1)d_1$.

When $n$ is even, we know that the solutions of $n - 3 \geq T(G) = d_1(\frac{n}{2} + 1 - d_1)$ is exactly one of $d_1 = 1$ or $d_2 = 1$ holds like in Case 2.1. However, when $d_2 = 1$, since $d_1 + d_2 = \frac{n}{2} + 1 + t$, we have $d_1 = \frac{n}{2} + t$, which contradicts (1) namely $i' \leq \frac{n-1+\sqrt{4t+1}}{2}; (1 \leq t \leq \frac{n}{2} - 2)$ because $d_1 \leq i'$. The case $d_1 = 1$ and $d_2 = \frac{n}{2} + t$ can be settled in the same way.
When $n$ is odd, $n - 3 \geq T(G) = d_1\left(\frac{n}{2}\right) + 1 - d_1$ implies that one of $d_1 = 1$, $d_2 = 1$, $d_1 = 2$ or $d_2 = 2$ holds. By symmetry we can consider the cases $d_1 = 1$ and $d_1 = 2$. We check the details of the following 3 subcases.

(i) $t = 1$ and $d_1 = 1$. We get $d_2 = \frac{n+1}{2}$ because $d_1 + d_2 = \frac{n-1}{2} + 1 + t$. Since $d_2 \leq |Y'| = n - 1 - i' \leq \frac{n-1+2\sqrt{t}}{2} = \frac{n+1}{2}$, we get $|Y'| = \frac{n+1}{2}$ and $|X'| = \frac{n-3}{2}$. Since $e(G \setminus v_0) = \frac{n-1}{2} \frac{n-1}{2} - 1$, we see that $G \setminus v_0$ is $K_{\frac{n-3}{2}, \frac{n+1}{2}}$. Thus, $G$ is $K_{\frac{n-1}{2}, \frac{n+1}{2}}$ and $T(G) \leq d_1d_2 = \frac{n+1}{2}$.

(ii) $t \geq 2$ and $d_1 = 1$. By $d_1 + d_2 = \frac{n-1}{2} + 1 + t$, we have $d_2 = \frac{n-1}{2} + t > \frac{n-1+2\sqrt{t}}{2}$, which contradicts $d_2 \leq |Y'| = n - 1 - i' \leq \frac{n-1+2\sqrt{t}}{2}$.

(iii) $t \geq 1$ and $d_1 = 2$. By $d_1 + d_2 = \frac{n-1}{2} + 1 + t$, we have $d_2 = \frac{n-1}{2} + t - 1$. However, $T(G) \geq d_1d_2 - t = 2\left(\frac{n-1}{2} + t - 1\right) - t \geq n - 2$, which contradicts $T(G) \leq n - 3$.

In conclusion, when $n$ is even, $G$ is $K_{\frac{n-1}{2}, \frac{n+1}{2}}$. When $n$ is odd, $G$ is either $K_{\frac{n-1}{2}, \frac{n+1}{2}}$ or $K_{\frac{n+1}{2}, \frac{n-1}{2}}$.

Using Lemma 3, we are able to give the proof of Theorem 4.

Proof of Theorem 4. We prove our result by induction on $n$. The induction step will go from $n - 2$ to $n$, so we check the bases when $n = 3$ and $n = 4$, obviously, our statement is true for these two cases. Suppose Theorem 4 holds for $k = n - 2$ ($n \geq 5$), we separate the rest of the proof into 2 cases.

Case 1. Every edge in $G$ is contained in at least one triangle. Then $T(G) \geq \left\lceil \frac{n^2}{4} \right\rceil + 1 \geq n - 2$.

Case 2. There exists at least one edge $uv$ which is not contained in any triangle. Then $u$ and $v$ cannot have common neighbor in $G \setminus \{u, v\}$, which implies that $e(\{u, v\}, G \setminus \{u, v\}) \leq n - 2$. Therefore, $e(G \setminus \{u, v\}) \geq \left\lceil \frac{n^2}{4} \right\rceil - (n - 2) = \left\lceil \frac{(n-2)^2}{4} \right\rceil + 1$. In this point, we split the rest of the proof into 3 subcases.

Case 2.1 $e(G \setminus \{u, v\}) \geq \left\lceil \frac{(n-2)^2}{4} \right\rceil + 3$. By Theorem 2, we get $T(G \setminus \{u, v\}) \geq 3 \left\lceil \frac{n-2}{2} \right\rceil$, which implies that $T(G) \geq 3 \left\lceil \frac{n-2}{2} \right\rceil \geq n - 2$.

Case 2.2. $e(G \setminus \{u, v\}) = \left\lceil \frac{(n-2)^2}{4} \right\rceil + 2$. When $n$ is even, by Theorem 2, we get
\(T(G \setminus \{u, v\}) \geq n - 2\), since \(T(G) \geq T(G \setminus \{u, v\})\), we are done. When \(n\) is odd, we have \(e(\{u, v\}, G \setminus \{u, v\}) = n - 3\), then there exists \(w \in V(G \setminus \{u, v\})\) such that edges \(vw, uw \notin E(G)\). If \(e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 1\), then the number of triangles which contains \(u\) or \(v\) is at least 1. By Theorem 2, \(T(G \setminus \{u, v\}) \geq n - 3\) holds, thus, \(T(G) \geq n - 2\). Otherwise, \(G \setminus \{u, v, w\}\) is bipartite and all triangles in \(G \setminus \{u, v\}\) are adjacent to \(w\). since \(e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 0\), no triangle contains \(u\) or \(v\). Therefore, \(\tau_\Delta(G) = \tau_\Delta(G \setminus \{u, v\}) = 1\) and all triangles in \(G\) are adjacent to \(w\).

**Case 2.3.** \(e(G \setminus \{u, v\}) = \left\lceil \frac{(n-2)^2}{4} \right\rceil + 1\), then \(e(\{u, v\}, G \setminus \{u, v\}) = n - 2\). When \(e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 0\), \(G \setminus \{u, v\}\) is bipartite, it has at most \(\left\lceil \frac{(n-2)^2}{4} \right\rceil\) edges, contradicting the assumption of the case.

Suppose \(e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 1\). Since \(|N(u) \setminus v \cup N(v) \setminus u| = n - 2\), we have \(e([N(u) \setminus v], [N(v) \setminus u]) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor\). Thus, \(e(G \setminus \{u, v\}) = \left\lceil \frac{(n-2)^2}{4} \right\rceil + 1\) implies that \(G \setminus \{u, v\}\) is obtained from \(K_{\left\lceil \frac{n-2}{2} \right\rceil, \left\lceil \frac{n-2}{2} \right\rceil}\) plus an edge, say \(\{j, k\}\), in one class. Therefore, all triangles in \(G\) contain \(\{j, k\}\) and hence \(\tau_\Delta(G) = 1\) follows.

Now we assume that \(e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 2\), then the number of the triangles containing \(u\) or \(v\) is at least 2. It is easy to check that if \(v(G) = 5\) then \(G \setminus \{u, v\}\) is a triangle and either \(\tau_\Delta(G) = 1\) or \(T(G) = 4\). Therefore, we may assume \(n \geq 6\). Since \(e(G \setminus \{u, v\}) = \left\lceil \frac{(n-2)^2}{4} \right\rceil + 1\), by the induction hypothesis, either \(\tau_\Delta(G \setminus \{u, v\}) = 1\) or \(T(G \setminus \{u, v\}) \geq n - 4\). When \(T(G \setminus \{u, v\}) \geq n - 4\), we have \(T(G) \geq T(G \setminus \{u, v\}) + 2 \geq n - 2\). Otherwise, \(\tau_\Delta(G \setminus \{u, v\}) = 1\) and \(T(G \setminus \{u, v\}) \leq n - 5\) hold. By Lemma 3, we see that when \(n\) is even, \(G \setminus \{u, v\}\) is \(K_{\frac{n-2}{2}, \frac{n-2}{2}}\), when \(n\) is odd, \(G \setminus \{u, v\}\) is either \(K_{\frac{n-3}{2}, \frac{n-1}{2}}\) or \(K_{\frac{n-3}{2}, \frac{n-1}{2}}\) or \(K_{\frac{n-1}{2}, \frac{n-3}{2}}\). Let us check what will happen in these cases.

We first give the following technical lemma:

**Lemma 5.** Let \(f(a, b) = ab + (A - a)(B - b)\), where \(A\) and \(B\) are integers, \(1 \leq a \leq A, 1 \leq b \leq B\), then \(f(a, b) \geq \min\{A, B\}\).

**Proof of Lemma 5.** Obviously, when \(AB = \max\{A, B\}\), \(f(a, b) \geq 1 = \min\{A, B\}\). Otherwise, we have \(A, B \geq 2\). Without loss of generality, fix \(b\), then \(f(a, b)\) is a linear function
of variable $a$. Since $\frac{\partial f}{\partial a} = b - (B - b)$, thus, $f(a, b)$ is decreasing when $b < \frac{B}{2}$ and $f(a, b)$ is increasing when $b > \frac{B}{2}$. Therefore,

$$f(a, b) \geq \begin{cases} f(A, b) = Ab, & b \leq \frac{B}{2}, \\ f(1, b) = b + (A - 1)(B - b), & b > \frac{B}{2}. \end{cases}$$

It is easy to check that $Ab \geq A$, when $b \leq \frac{B}{2}$, and $b + (A - 1)(B - b) = B(A - 1) + b(2 - A) \geq B$ when $b > \frac{B}{2}$. Hence, we get $f(a, b) \geq \min\{A, B\}$. Obviously, if $\min\{A, B\} = A$, the equality holds only when $a = A$ and $b = 1$, if $\min\{A, B\} = B$, the equality holds only when $a = 1$ and $b = B$.

**Case 2.3.1.** $G \setminus \{u, v\}$ is $K_{\left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil - 1}$, which implies that when $n$ is even, $G \setminus \{u, v\}$ is $K_{\frac{n}{2} - 1, \frac{n}{2} - 1}$ and when $n$ is odd, $G \setminus \{u, v\}$ is $K_{\frac{n-3}{2}, \frac{n-1}{2}}$. Let $X$ and $Y$ be the two classes of $K_{\left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil - 1}$ and $(j, k)$ be the extra edge in $X$, where $|X| = \left\lfloor \frac{n}{2} \right\rfloor - 1$, see Figure 2. Since 

$$e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 2, \quad |N(u) \setminus v \cup N(v) \setminus u| = n - 2 \quad \text{and} \quad N(u) \setminus v \cap N(v) \setminus u = \emptyset,$$

we see that either $N(u) \setminus v$ or $N(v) \setminus u$ contains at least one vertex in both classes $X$ and $Y$. Without loss of generality, say at least $N(u) \setminus v$ has this property.

Let $|N(u) \setminus v \cap X| = a$ and $|N(u) \setminus v \cap Y| = b$, where $1 \leq a \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $1 \leq b \leq \left\lceil \frac{n}{2} \right\rceil - 1$. Then the number of triangles which are adjacent to $u$, containing one vertex in $X$ and one in $Y$ is $ab$ while the number of triangles which are adjacent to $v$, containing one vertex in $X$ and one in $Y$ is $(A - a)(B - b)$. Hence, we get $T(G) \geq ab + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 - a\right)\left(\left\lfloor \frac{n}{2} \right\rfloor - 1 - b\right) + \left\lfloor \frac{n}{2} \right\rfloor - 1. \quad$ 

By Lemma 5, we see $T(G) \geq \left\lfloor \frac{n}{2} \right\rfloor - 1 + \left\lfloor \frac{n}{2} \right\rfloor - 1 = n - 2.$
Case 2.3.2. $n$ is odd and $G \setminus \{u, v\}$ is $K_{\frac{n-1}{2}, \frac{n+3}{2}}$. Let $X$ and $Y$ be the two classes of $K_{\frac{n-1}{2}, \frac{n+3}{2}}$ and $\{j, k\}$ be the extra edge in $X$, where $|X| = \frac{n-1}{2}$. Similarly as in the previous case, either $N(u) \setminus v$ or $N(v) \setminus u$ contains at least one vertex in both classes $X$ and $Y$. Without loss of generality, say at least $N(u) \setminus v$ has this property.

Let $|N(u) \setminus v \cap X| = a$ and $|N(u) \setminus v \cap Y| = b$, where $1 \leq a \leq \frac{n-1}{2}$ and $1 \leq b \leq \frac{n-3}{2}$, then $T(G) \geq ab + \left(\frac{n-1}{2} - a\right)\left(\frac{n-3}{2} - b\right) + \frac{n-3}{2}$. By Lemma 5, we get $T(G) \geq \frac{n-3}{2} + \frac{n-3}{2} \geq n - 3$, the equality holds only if $a = 1$ and $b = \frac{n-3}{2}$. Let $s \in X$ and $\{u, s\} \in E(G)$, $a = 1$ and $b = \frac{n-3}{2}$ implies that either $s \in \{j, k\}$ then $\tau_\Delta(G) = 1$, or $s \notin \{j, k\}$ then there exists one more triangle $\{v, j, k\}$, thus $T(G) \geq n - 3 + 1 = n - 2$.

Case 2.3.3. $n$ is odd and $G \setminus \{u, v\}$ is $K_{\frac{n-1}{2}, \frac{n+3}{2}}$. Since $\frac{n-1}{2} \geq 3$, we get $n \geq 7$. Let $X$ and $Y$ be the classes of $K_{\frac{n-1}{2}, \frac{n+3}{2}}$, $\{j, z\}$ and $\{z, k\}$ be the two extra edges in $X$ and $\{z, w\}$ be the missing edge in $K_{\frac{n-1}{2}, \frac{n+3}{2}}$, see Figure 2.

Let $|N(u) \setminus v \cap X| = a$ and $|N(u) \setminus v \cap Y| = b$. Since $|N(u) \setminus v \cup N(v) \setminus u| = n - 2$ and $N(u) \setminus v \cap N(v) \setminus u = \emptyset$, when $a = 0$, we have $X \subseteq N(v) \setminus u$. If $N(v) \setminus u = X$, clearly, all triangles in $G$ contain $z$ and hence $\tau_\Delta(G) = 1$. Otherwise, $|(N(u) \setminus u) \cap Y| \geq 1$. It is easy to check that $T(K_{\frac{n-1}{2}, \frac{n+3}{2}}) = n - 5$, therefore, in this case we get $T(G) \geq n - 5 + 2 + \frac{n-1}{2} - 1 \geq n - 1$ ($n \geq 7$). When $b = 0$, then $Y \subseteq N(v) \setminus u$. If $N(v) \setminus u = Y$ then $N(u) \setminus v = X$, we see that all triangles in $G$ contain $z$ and hence $\tau_\Delta(G) = 1$. Otherwise, $|(N(v) \setminus u) \cap X| \geq 1$. When $|(N(v) \setminus u) \cap X| = 1$, if $(N(v) \setminus u) \cap X = \{z\}$, obviously, all triangles in $G$ contain $z$, hence $\tau_\Delta(G) = 1$. If not, then clearly $T(G) \geq n - 5 + 1 + \frac{n-3}{2} \geq n - 2$ ($n \geq 7$). It is easy to check that $T(G)$ reaches the lower bound when $|(N(v) \setminus u) \cap X| = 1$ for $n \geq 9$ and when $n = 7$, $T(G) \geq 5$ holds in all cases. Therefore, we get either $\tau_\Delta(G) = 1$ or $T(G) \geq n - 2$.

Now suppose that, $1 \leq a \leq \frac{n-1}{2}$ and $1 \leq b \leq \frac{n-3}{2}$. Then $T(G) \geq ab + \left(\frac{n-1}{2} - a\right)\left(\frac{n-3}{2} - b\right) + n - 5$, by Lemma 5, we get $T(G) \geq \frac{n-3}{2} + n - 5 \geq n - 2$ ($n \geq 9$). Since $T(G) \geq 5$ when $n = 7$, we see that $T(G) \geq n - 2$ holds in this case.

This completes the proof. \qed
3 Open problems

Let $V_1, V_2, \ldots, V_r$ be pairwise disjoint sets where $\left\lceil \frac{n}{2} \right\rceil \geq |V_1| \geq |V_2| \geq \ldots \geq |V_r| \geq \left\lfloor \frac{n}{2} \right\rfloor$ and $\sum |V_i| = n$ hold. Define the graph $T_r(n)$ with vertex set $\bigcup V_i$ where $\{u, v\}$ is an edge if $u \in V_i$, $v \in V_j (i \neq j)$, but there is no edge within a $V_i$. The number of edges of the graph $T_r(n)$ is denoted by $t_r(n)$. The following fundamental theorem of Turán is a generalization of Mantel’s theorem.

**Theorem 6 (Turán [6]).** If a graph on $n$ vertices has more than $t_{k-1}(n)$ edges then it contains a copy of the complete graph $K_k$ as a subgraph.

The most natural construction is to add one edge to $T_{k-1}(n)$ in the set $V_1$. This graph is denoted by $T_{k-1}^e(n)$. It contains not only one copy of $K_k$ but $|V_2| \cdot |V_3| \cdots |V_{k-1}|$ of them. [3] proved that this is the least number. Observe that the intersection of all of these copies of $K_k$ is a pair of vertices (in $V_1$). If this is excluded, the number of copies probably increases. This is expressed by the following conjecture. Take $T_{k-1}(n)$, add an edge $\{x, y\}$ in $V_1$, an edge $\{u, v\}$ in $V_2$ and delete the edge $\{u, x\}$. This graph is denoted by $T_{k-1}^e$. It contains almost the double of the number of copies of $K_k$ in $T_{k-1}^e(n)$.

**Conjecture 1.** If a graph on $n$ vertices has $t_{k-1}(n) + 1$ edges and the copies of $K_k$ have an empty intersection then the number of copies of $K_k$ is at least as many as in $T_{k-1}^e$:

$$|V_2| - 1)|V_3| \cdot |V_4| \cdots |V_{k-1}| + (|V_1| - 1)|V_3| \cdot |V_4| \cdots |V_{k-1}| = (|V_1| + |V_2| - 2)|V_3| \cdot |V_4| \cdots |V_{k-1}|.$$

Of course this would be a generalization of our Theorem 4. Now we try to generalize it in a different direction. What is the minimum number of triangles in an $n$-vertex graph $G$ containing $\left\lceil \frac{n^2}{4} \right\rceil + t$ edges if $\tau_\Delta(G) \geq s$ is also supposed. The problem is interesting only when $0 < t < s$. Otherwise, if $t \geq s$ then $\tau_\Delta(G) = t$ is allowed. By Lovász-Simonovits’ theorem [4], we know that the number of triangles is at least $t \left\lfloor \frac{n}{2} \right\rfloor$ with equality for the following graph. Take $K_{\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor}$ where the two parts are $V_1(|V_1| = \left\lceil \frac{n}{2} \right\rceil)$ and $V_2(|V_2| = \left\lfloor \frac{n}{2} \right\rfloor)$, respectively. Add $t$ edges to $V_1$. Here all triangles contain one of the new added edges, therefore $\tau_\Delta(G) \leq t$ and the extra condition on $\tau_\Delta(G)$ is not a real restriction.

Hence we may suppose $0 < t < s$. Choose $2(s - 1)$ distinct vertices in $V_1$ (of $K_{\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor}$):
Let $x_1, x_2, \ldots, x_{s-1}, y_1, y_2, \ldots, y_{s-1}$ and two distinct vertices in $V_2 : u_1, u_2$. Add the edges $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{s-1}, y_{s-1}\}, \{u_1, u_2\}$ to $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and delete the edges $\{x_1, u_1\}, \ldots, \{x_{s-t}, u_1\}$. Let $K_{s,t}^{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ denote this graph. It is easy to see that it contains $\left\lceil \frac{n^2}{4} \right\rceil + t$ edges. On the other hand it contains $s$ vertex disjoint triangles if $\left\lceil \frac{n}{2} \right\rceil \geq 2(s-1)+1$ and $\left\lfloor \frac{n}{2} \right\rfloor \geq s+1$.

Therefore, $\tau_\triangle(K_{s,t}^{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) = s$ holds if $n$ is large enough. We believe that this is the best possible construction.

**Conjecture 2.** Suppose that the graph $G$ has $n$ vertices and $\left\lceil \frac{n^2}{4} \right\rceil + t$ edges, it satisfies $\tau_\triangle(G) \geq s$ and $n \geq n(t,s)$ is large. Then $G$ contains at least as many triangles as $K_{s,t}^{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ has, namely $(s-1)\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor - 2(s-t)$.

In the case $t = 1, s = 2$ our Theorem 4 is obtained. There is an obvious common generalization of our two conjectures.

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**References**


