How many sums of vectors can lie in a circle of radius $\sqrt{2}$

by

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INTRODUCTION

Let $a_1, a_2, \ldots, a_n$ be real numbers with the property $|a_i| < \sqrt{2}$.

Kneser [1] asked, what is the maximum number of sums $\sum_{i=1}^{k} a_i$, which can lie in an open interval of length $\sqrt{2}$, where $a_i = 0$ or 1. He proved, that the number is $n \log_2 n$, which is the largest possible number of ordered $n$-tuples with the property $\sum_{i=1}^{k} a_i < \sqrt{2}$.

Freiman [2] independently proved the same for two dimensional sets $\{ a_1, a_2, \ldots, a_n \}$.

THEOREM 2. If $a_1, a_2, \ldots, a_n$ are two-dimensional vectors with the property $|a_i| < \sqrt{2}$, then at least one of the $a_i$ can be expressed as a sum of the form $\sum_{j=1}^{k} a_{j,i}$, where $a_{j,i} = 0$ or 1.

In the proof we use a Halmos type theorem, which is summarized.

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DEFINITIONS AND THEOREMS

Let $\mathcal{G}$ be a partially ordered set with rank function. The $i$-th level in the set of elements of $\mathcal{G}$ with rank $i$ is a sequence $g_1, \ldots, g_n \in \mathcal{G}$ where $\text{rk}(g_j) = i$. A chain in $\mathcal{G}$ is of the form $g_1 \leq g_2 \leq \cdots \leq g_n$, where $n > 0$. We say that a partially ordered set is a representational chain set if we can split it into disjoint representational chains. A rank function $\text{rk}$ is defined as follows: for each element $x$ of $\mathcal{G}$, the rank of $x$ is the number of elements of $\mathcal{G}$ with rank less than or equal to $\text{rk}(x)$.

Theorem 1: Let $\mathcal{G}$ and $\mathcal{H}$ be partially ordered sets, and $r$ be a rank function on $\mathcal{G}$. If $\mathcal{G}$ is a representational chain set, then the subset of $\mathcal{G}$ consisting of all elements with rank $k$ is a representational chain set.

Theorem 2: Let $\mathcal{G}$ and $\mathcal{H}$ be partially ordered sets, and $r$ be a rank function on $\mathcal{G}$. If $\mathcal{G}$ is a representational chain set, then the subset of $\mathcal{G}$ consisting of all elements with rank $k$ is a representational chain set. Furthermore, if $\mathcal{G}$ is a representational chain set, then the subset of $\mathcal{G}$ consisting of all elements with rank $k$ is a representational chain set.
two different ways of these satisfy the conditions

$$E_3^2$$

$$E_4^2$$

as four different ways of these satisfy the conditions

$$E_5^2$$

then

$$E_6^2$$

where \( n \) denotes the number of elements of the 1st kind and 

$$n = \frac{1}{2} (n^+ - n^-)$$.

**Proofs**

The **proof of Theorem 2** follows the idea of the proof of the

**Theorem in [4] and of Theorem 3.3 in [2].**

By the definition of the symmetrical chain was, \( a \), and \( b \) are

divisible two distinct symmetrical chains. Defined by \( 1 \) and \( 2 \) the

given colored sets which have another solution only along these chains, that is

\( a^2 b^2 \) can only if \( a \) and \( b \) lie on the same chain. Thus, the set of

solutions in \( S' \) is a part of that in \( S(1) \). It follows that the set of co-

chains in \( S' \) is, as a part of that in \( S(1) \). So, it is sufficient to prove the

statement of the theorem for \( S' \) instead of \( S(1) \). However, the chosen sets

of the chains \( a_0, a_1, \ldots, a_n \) and \( b_0, b_1, \ldots, b_n \) is a rectangular lattice of pairs

\((a_i, b_i)\), where

\(a_i b_j = \frac{1}{2} (b_i a_j + a_i b_j)\) and \( \text{det} C = \text{det} E \).

We will prove in the Lemma, that the maximum number of the

elements of such a rectangular, under conditions \( E_1^2 \) and \( E_2^2 \) in the number
of elements of the two (n-1)-fold bundles, that is the rank of plane $(\mathcal{E}_h \mathcal{E}_h)$

\[(1) \quad \mathcal{E}_h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\]

and

\[(2) \quad \mathcal{E}_h = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\]

However, by the symmetry of the choice

\[(3) \quad \mathcal{E}_h = \mathcal{E}_h^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\]

and

\[(4) \quad \mathcal{E}_h = \mathcal{E}_h^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\]

follow, where $e = \frac{ac - bd}{2}$ and $f = \frac{ad - bc}{2}$.

Then, by (3), using (2) and (3) we get

\[\mathcal{E}_h \mathcal{E}_h = \mathcal{E}_h^{-1} \mathcal{E}_h^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathcal{E}_h^{-1} \mathcal{E}_h^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\]

Similarly, in the case (4)

\[\mathcal{E}_h \mathcal{E}_h = \mathcal{E}_h^{-1} \mathcal{E}_h^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\]

hence, let, if we choose the elements in a given (n-1)-fold way from every element, then we obtain elements of the $[\mathcal{E}_h \mathcal{E}_h]$ and $[\mathcal{E}_h^{-1} \mathcal{E}_h^{-1}]$ levels.

It is easy to see, that we obtain every element of $[\mathcal{E}_h \mathcal{E}_h]$ and $[\mathcal{E}_h^{-1} \mathcal{E}_h^{-1}]$ in this way. This completes the proof.

Let $\mathbb{R}$ be the set of real $[\mathcal{E}_h \mathcal{E}_h]$ integrals, and $(\mathcal{E}_h \mathcal{E}_h)\ldots(\mathcal{E}_h \mathcal{E}_h)$ is a number of int.

in two different cases when the conditions.
\[
E''_1 = E_1, \quad E''_2 = E_2
\]

no different means of these variables satisfy the condition

\[
\begin{align*}
\sum_{i=1}^{m} & a_i x_i + b_i = 0, \\
\sum_{i=1}^{m} & a_i y_i + b_i = 0
\end{align*}
\]

case be exhibited as in given by the set of linear satisfying

\[
ax + by = c
\]

PROOF. If \( a, b, c \) are not 0, then \( E''_1 \) or \( E''_2 \) can lie to every point in the given set of elements on \( F \) with fixed second coordinate, then the equation must exist at least 20, however, it is easy to see that the equation will give the Line that has exactly its elements.

If \( a, b, c \) are not 0, the given mesh will have two linear elements. We have to prove that we can not have two elements. We prove it in an indirect way. Let \( a \neq 0 \) and \( b \neq 0 \) as the elements chosen of the first case. We have two elements in \( E''_1 \) or \( E''_2 \). Since we get \( \varphi_1 = \varphi_2 \), we conclude that \( \varphi_1, \varphi_2 \) are not 0. Therefore, \( \varphi_1, \varphi_2 \) have a configuration included in \( E''_1 \) and \( E''_2 \) is contradiction to our supposition. The proof is completed.

Let us move on to the PROOF OF THEOREM 1.

It is easy to see that we can reduce the problem to the case where the coordinates of the points are non-negative (transforming to \( a = 0 \)). Multiplying some \( a' = a \), and retransforming to \( a = 0 \), let \( S_1, S_2 \) be the set of variables, with non-negative and negative elements respectively. We shall use Theorem 3 for 0, a centrally ordered set of variables of \( S_1 \), and \( S_2 \) is partially ordered set of variables of \( S_2 \). Let \( S_1, S_2 \) be an open circle of lines \( S \) and consider the case lying at the circle. We may correspond with such cases a number of \( S_1, S_2 \), the subset of \( S_1 \) which have

\[
\text{Conditions } t
\]

Let us suppose to exactly, the density of these subsets satisfies \( E''_1 \) and \( E''_2 \).
Indeed, if two different vertices $(p_1, q_1) \neq (p_2, q_2)$ at $s_0$, then

$$s_0 = g_{p_1} \cdot g_{q_1} = g_{p_2} \cdot g_{q_2}$$

then for the corresponding sums

$$\left( \sum_{V \in V, V \neq V_0} s_{V} \cdot s_{V_0} \right) \left( \sum_{V \in V, V \neq V_0} s_{V} \cdot s_{V_0} \right)$$

holds. The number of the sums are summands with non-negative and negative second coordinates and with absolute value $1$. The number of summands is at least 3 by (5). It is easy to see, that the sum of such index functions always value $1$, which concludes the assumption that both terms in the square sum are disjunct.

The proof of the claim of (5) is completed.

In order to prove the claim for (5) we have to show that in the same

$$s_0 = g_{p_1} \cdot g_{q_1} = g_{p_2} \cdot g_{q_2}$$

at least two of the sums

$$\sum_{V \in V, V \neq V_0} s_{V} \cdot s_{V_0} = \sum_{V \in V, V \neq V_0} s_{V} \cdot s_{V_0} = \sum_{V \in V, V \neq V_0} s_{V} \cdot s_{V_0}$$

differ at least $g_2$. The difference of

$$\sum_{V \in V, V \neq V_0} s_{V} \cdot s_{V_0} \neq \sum_{V \in V, V \neq V_0} s_{V} \cdot s_{V_0}$$

The difference of the 0-th and 3-th sum is
Here, $v_1$ and $v_2$ are a (directed) ray of vectors lying in the same plane, with unit vector $n$. Thus, $|v_1| = 1$, $|v_2| = 1$. If the angle of $v_1$ and $v_2$ is $\frac{\pi}{2}$, then $|v_1\times v_2| = |v_1||v_2|$; conversely, if the angle is $\frac{\pi}{2}$ then $|v_1\times v_2| = |v_1||v_2|$. In other words, there are always two vectors in $V$ with identical direction and opposite orientation.

The constants $c_1$ and $c_2$ are easily satisfied in the case. We may apply Theorem 3, thus, the two-variable family of the partially ordered set of theorems $S_1 \subseteq S_2$ gives an ordered set. The two-variable family consists of the theorems with symbols: $|v_1||v_2|$ or $|v_1\times v_2|$. The theorem of each subset is $|v_1||v_2|$. The proof is completed.

CONCLUDING REMARKS
The conclusion of Theorem 6 is the best possible in the sense that the $k_{0,1}$ (154 kK) is the maximum or infimum.

It is obvious that the theorem does not hold for a larger number instead of $4$, since the vectors $(5,0)$ and $(7,0)$ would provide a counterexample. However, we have the following.

CONJECTURE. Theorem 1 holds with 2 instead of $4$ in the case $n=2$. 
REFERENCES