FUNCTIONS DEFINED ON A DIRECTED GRAPH

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D. Krich has formulated the following theorem [2] in a more general form:

Let G = (V, E) be a directed graph with a real function f defined on the vertices of G, where f(y) is the weight of the vertex y. Then the function f satisfies the following conditions:

1. f(x) = 0 for all x ∈ V.
2. f(x) ≤ 0 for all x ∈ V.
3. f(x) ∈ ℝ for all x ∈ V.

We will use the notation of C. F. Deuring [3].

Finite directed graphs

Let G = (V, E) be a finite directed graph without loops and multiple edges, where V = {v₁, v₂, ..., vₙ} is the set of vertices and E is the set of edges. Define

\[ f(x) = \frac{1}{|E|} \sum_{e \in E} g(e) \]

where g(e) is the weight of edge e. The function f satisfies the following conditions:

1. f(x) = 0 for all x ∈ V.
2. f(x) ≤ 0 for all x ∈ V.
3. f(x) ∈ ℝ for all x ∈ V.

We may formulate the following theorem:

Theorem 1: A finite directed graph G = (V, E) has the Krich property if and only if f(x) is the non-negative real value.

We prove the theorem by induction on the number of vertices in G. Base case: G is a graph with one vertex, and f(x) = 0.

Inductive step: Assume the theorem holds for graphs with fewer than n vertices, and let G = (V, E) be a graph with n vertices. Suppose G has a vertex x with f(x) < 0. Let y be a vertex such that (x, y) ∈ E. Then by the inductive hypothesis, f(y) ≥ 0. Since f(x) < 0, it follows that f(x) = f(y) - d(x, y) for some d(x, y) ≤ 0. Therefore, f(x) ≥ f(y) for all vertices y adjacent to x. This contradicts the assumption that f(x) < 0.

Hence, f(x) = 0 for all x ∈ V, and G has the Krich property.

We conclude that the theorem holds for all finite directed graphs G = (V, E).

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In this case, the right side is the arithmetic mean of values smaller than
are equal to \( g(x) \), but at least one of them is actually smaller. Similarly, \( B \) is a
subset of \( A \).

(1) In the case of two disjoint sets, we have shown that there are
contiguous elements of \( (g_1, g_2) \), if \( g_1 \neq g_2 \), \( g_1 < g_2 \), and
the minimum value \( g_1 \) is less than the maximum value \( g_2 \).

(2) In a homogeneous linear equation system, we must show that it has a
non-trivial solution. Consider the matrix \( N \) of coefficients of (3). Obviously, \( \alpha_1 = -\langle P, x \rangle \)

\( N = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} \)

Using the fact that \( B \) is a subset of \( X \), we obtain

(3) If \( x \leq c \) and \( y \leq c \),

\( n_{xy} = \begin{cases} 0 & \text{if } x \leq c \text{ and } y \leq c \\ 1 & \text{otherwise} \end{cases} \)

(4) If \( x \leq c \) and \( y \leq c \),

\( n_{xy} = \begin{cases} 0 & \text{if } x \leq c \text{ and } y \leq c \\ 1 & \text{otherwise} \end{cases} \)

(5) If \( x \leq c \) and \( y \leq c \),

\( n_{xy} = \begin{cases} 0 & \text{if } x \leq c \text{ and } y \leq c \\ 1 & \text{otherwise} \end{cases} \)

Similarly, for \( B \) a subset of \( X \), we obtain

\( n_{xy} = \begin{cases} 0 & \text{if } x \leq c \text{ and } y \leq c \\ 1 & \text{otherwise} \end{cases} \)

and for \( B \) a subset of \( X \), we obtain

\( n_{xy} = \begin{cases} 0 & \text{if } x \leq c \text{ and } y \leq c \\ 1 & \text{otherwise} \end{cases} \)

It is obvious. The matrix \( N \) has the form

\( N = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} \)

Put

\( (P(X, Y) + 3, n_{xy} = 0 \) for \( x \leq c \).

These values satisfy the last \( a \) equations because of (1), (4) and (5).

It remains to solve the equation system

\( \sum_{x} n_{xy} (P(x)) = 3, n_{xy} = 0 \) for \( x \leq c \).
We will show in an indirect way that \( W \rightarrow h \) where \( M \) denotes the matrix ofcoefficients of this equation. Assume the contrary for simplicity and set

\[
p(n) = \frac{\text{1}}{\text{2}} \sum_{i=0}^{\text{2}} x_i n_i, \\
p(n) < 0.
\]

Now since the negative elements are isolated from each row, \( M(n) \neq 0 \).

In the above matter, \( x_i \) are linearly dependent, that is, there are \( x_{i_1}, \ldots, x_{i_r} \) not zero such that

\[
\sum_{i=1}^{r} x_i = 0.
\]

We may assume there is a positive number among \( x_{i_1}, \ldots, x_{i_r} \). We may assume \( x_{i_1} > 0 \). Let us choose the indices in such a manner that

\[
x_{i_1}, \ldots, x_{i_k} > 0 \quad \text{and} \quad x_{i_{k+1}}, \ldots, x_{i_r} < 0
\]

By condition (a) there are \( n_{i_1}, n_{i_2} \leq 0 \) such that

\[
m_n = 1.
\]

We separate two cases:

1. \( k > 1 \).
2. \( k = 1 \).

In the case (1) \( n_{i_1} < 0 \), and in the case (2) we obtain from (g) and (h) the inequality

\[
\frac{\text{1}}{\text{2}} x_i n_i < 0.
\]

Instead of (1), we will show that (2) cannot hold for the same reason. The following inequality is trivial:

\[
\frac{\text{1}}{\text{2}} x_i n_i = \frac{\text{1}}{\text{2}} (x_i y_i + x_i z_i) \leq \frac{\text{1}}{\text{2}} (x_i y_i + x_i y_i) \leq \frac{\text{1}}{\text{2}} (x_i y_i + x_i y_i) + \frac{\text{1}}{\text{2}} (x_i y_i + x_i y_i) + \frac{\text{1}}{\text{2}} (x_i y_i + x_i y_i)
\]

and is the case (1) strict inequality holds. Then, in the case (2) (11) and (12)

\[
\frac{\text{1}}{\text{2}} x_i n_i < 0.
\]

In the case (1) (12) follows from (11) and (12). The proof is completed.

Conclusion. Every homomorphism and strongly normal finitely presented has the abelian property.
The proof is as follows:

Let $G$ be an undirected graph. If we orient the directions by the edges of $G$ in an arbitrary manner, then the resulting graph $\hat{G}$ is called an orientation of $G$. A graph $\tilde{G}$ is called a tournament if every pair of distinct vertices is connected by a directed edge in one of the orientations $\hat{G}$. The following theorem states that if $G$ is a tournament, then $\tilde{G}$ is the Kneser property.

Theorem 2.4: If $G$ is a finite undirected non-complete graph, then $\tilde{G}$ has no two disjoint edges.

Proof: If $G$ is complete, we have trivially the desired result. If $G$ is not complete, then there exist two distinct vertices $u$ and $v$ which are not connected by an edge. If we add an edge between $u$ and $v$, then we get a complete graph $G'$. By the previous result, if $G'$ is a tournament, then $\tilde{G'}$ is the Kneser property. Hence, $\tilde{G'}$ has no two disjoint edges, and therefore, $\tilde{G}$ has no two disjoint edges.

General solution of (1)

If $G$ is a directed graph with the Kneser property, every orientation of the system of $G$ has one or more edges in a complete graph $G'$. In this case, the graph $G'$ is a tournament.

The following proposition is a directed graph $G'$ and known to be a complete graph $G'$. If $G'$ is a complete graph, then $G'$ has the Kneser property.

Proposition 2.5: If $G'$ is a complete graph with the Kneser property, then $G'$ has the Kneser property.

Let $G'$ be the complete graph with the Kneser property. Then $G'$ has no two disjoint edges. Let $G'$ be a tournament with the Kneser property. Then $G'$ has no two disjoint edges. Therefore, $G'$ is a tournament.
Consider all the minimal sub-arrays \( \lambda_1, \lambda_2, \ldots, \lambda_m \) of \( \Xi \). Each \( \lambda_i \) is an array of \( \lambda_i \) elements of \( \Xi \). The order of \( \lambda_i \) is determined as usual. The order of \( \lambda_i \) is determined in the proof of Theorem 2. Only then do we have more than a sub-array.

REFERENCES