1. Ancient theorems

1.0.

Let X be a finite set of n elements and $\mathcal{F}$ be a family of distinct subsets of X. Most of our book deals with the following type of problems. What is the maximal (minimal) size of the family $\mathcal{F}$ supposing that n is fixed and $\mathcal{F}$ satisfies certain conditions? The simplest conditions are that the members of $\mathcal{F}$ 1) do not contain another member and 2) are non-disjoint. This order of "simplicity" more or less coincides with the historical order. Therefore it is the most natural to start the present monograph with these problems.

The notation $|X|$ is used for the size, that is, the number of elements of the set X. The power set, that is, the family of all subsets of X is denoted by 2^X. A subset $\mathcal{F}$ of 2^X is called a family. Script upper case letters are used for families. Families are considered to be sets, that is, their members are distinct. The elements of the families are called members and are denoted by upper case letters. $\binom{X}{k}$ stands for the family of all k-element subsets of X. The underlying set of a family is denoted by X and has n elements, unless it is stated differently. $\lfloor x \rfloor$ is used for the integer part of x, while $\lceil x \rceil$ denotes the smallest integer $\geq x$.

1.1. Intersecting families

A family $\mathcal{F}$ is called intersecting if

(1.1) $F_1, F_2 \in \mathcal{F}$ imply $F_1 \cap F_2 \neq \emptyset$.

Theorem 1.1 (Erdős, Ko and Rado (1961))

(1.2) $\max |\mathcal{F}| = 2^{n-1}$

where the max is taken over all intersecting families $\mathcal{F}$.

Proof (Erdős, Ko and Rado (1961)). $\mathcal{F}$ is intersecting therefore at most one of the sets F and $X-F$ can be a member of it. Consequently, at most half of all the 2^n subsets can occur in $\mathcal{F}$: $|\mathcal{F}| \leq 2^n/2 = 2^{n-1}$. The family $\mathcal{F}=\{F: x \in F \subset X\}$ (x is a fixed element of X) provides equality in the theorem. □

There are many more extremal families, that is, families with equality in (1.2). Let e.g. $X = X_1+X_2$ be a partition of X (+ denotes the disjoint union) where $|X_1|=n_1$ is odd. The family

(1.3) $$\left\{F: |F \cap X_1| \geq \frac{n_1+1}{2}\right\}$$

is obviously intersecting: there is no room for two disjoint $\frac{n_1+1}{2}$ -element sets in an n_1 -element one. On the other hand, one of F and $X-F$ always belongs to the above family because $|F \cap X_1| \leq \frac{n_1-1}{2}$ and $|(X-F) \cap X_1| \leq \frac{n_1-1}{2}$ lead to the contradiction $|X_1| \leq n_1-1$. Therefore (1.3) has 2^{n-1} members as it was stated.

If n is odd we may choose $n_1=n$. Then (1.3) consists of all "big" sets of size $\geq \frac{n+1}{2}$. If n is even then we can choose $n_1=n-1$. Then (1.3) gives the family $\{F: |F| \geq \frac{n+1}{2}$ or $(|F|=n/2$ and $x \notin F)\}$ where x is a fixed element of X .

The number of extremal families is much larger. It is true that one can start to build an intersecting family in any way, it can be completed to be extremal:

Theorem 1.2 (Erdős, Ko and Rado (1961)). Let $\mathcal{F}$ be an intersecting family. There is another intersecting family $\mathcal{G}$ such that

$$\mathcal{F} \subset \mathcal{G} \quad \text{and} \quad |\mathcal{G}| = 2^{n-1} .$$

Proof. It is sufficient to prove that $|\mathcal{F}|<2^{n-1}$ implies that there is a set $G \notin \mathcal{F}$ such that $\mathcal{F} \cup \{G\}$ is intersecting. In this way we can enlarge $\mathcal{F}$, step by step, until $\mathcal{G}$ is obtained.

The number of partitions $A+(X-A)$ is obviously 2^{n-1}. As $\mathcal{F}$ is intersecting, A and $X-A$ cannot both belong to it, therefore the number of such partitions satisfying either $A \in \mathcal{F}$ or $X-A \in \mathcal{F}$ is exactly $|\mathcal{F}|$. Hence we have $2^{n-1}-|\mathcal{F}|>0$ partitions such that $A \notin \mathcal{F}$, $X-A \notin \mathcal{F}$.

Suppose that $A \notin \mathcal{F}$, $X-A \notin \mathcal{F}$. We shall see that one of them intersects all the members of $\mathcal{F}$. Indeed, if there were $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap A = \emptyset$, $F_2 \cap (X-A) = \emptyset$ then $F_1 \cap F_2 = \emptyset$ which yields a contradiction. This means that one of A and $X-A$ can be added to $\mathcal{F}$ retaining the intersecting property. □

1.2. Inclusion-free families

$\mathcal{F}$ is said to be inclusion-free if

(1.4) $\quad F_1, F_2 \in \mathcal{F}$, $F_1 \neq F_2$ imply $F_1 \not\subset F_2$.

These families are called antichains or Sperner families in the literature. The first non-trivial statement of the theory deals with such families.

Theorem 1.3. Sperner theorem (Sperner (1928)).

$$\max |\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1.5}$$

where the max is taken over all inclusion-free families.

It is easy to see that there is a family $\mathcal{F}$ with equality in (1.5). Take simply $\mathcal{F} = \binom{X}{\lfloor \frac{n}{2} \rfloor}$ or $\mathcal{F} = \binom{X}{\lceil \frac{n}{2} \rceil}$. These families are obviously inclusion-free and their size is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. The essential part of the proof of (1.5) is, therefore, to prove the inequality

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor} . \tag{1.6}$$

We give 3 different proofs of this statement in this Section, a fourth one in Exercise 1.10, and a fifth one in Chapter 7.

The first proof is not first in historical sense. It is the shortest. If the reader really wants to enjoy its brevity he should begin with the original proof of Sperner (third proof).

First proof of the Sperner theorem (Lubell (1966), Permutation method). A complete chain in an n-element X is a family $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ satisfying $|C_i| = i$ $(0 \le i \le n)$ and $C_0 \subset C_1 \subset \ldots \subset \subset C_n$. It is obvious that there is a one-to-one correspondence between the complete chains and the permutations of the elements of X. The total number of complete chains is therefore $n!$.

Let $\mathcal{F}$ be an inclusion-free family and let us count the number of pairs $(\mathcal{C}, F)$ where $\mathcal{C}$ is a complete chain, $F \in \mathcal{C}$, $F \in \mathcal{F}$. To a fixed $F \in \mathcal{F}$ there are $|F|!(n-|F|)!$ complete chains $\mathcal{C}$ satisfying $F \in \mathcal{C}$. Consequently, the number of these pairs is $\sum_{F \in \mathcal{F}} |F|!(n-|F|)!$. On the other hand, for any fixed $\mathcal{C}$ there is at most one $F \in \mathcal{F}$ with $F \in \mathcal{C}$ (otherwise there is an inclusion in $\mathcal{F}$). The number of such pairs $(\mathcal{C}, F)$ is therefore at most the number of complete chains. We obtained the inequality

(1.7)
$$\sum |F|!(n-|F|)! \le n! .$$

Hence

(1.8)
$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \le 1 .$$

$\binom{n}{|F|} \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and (1.8) imply

$$|\mathcal{F}| \cdot \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le 1$$

and (1.6). □

Inequality (1.8) has got an independent and important role in the theory. It has a long history. Yamamoto (1954) proved it first using the ideas of Sperner's original proof (see third proof). Bollobás (1964) proved a more general inequality (see) using an induction proof (see Exercise). Lubell found the above ingenious argument. Meshalkin (1967) proved another generalization (see Section) using Sperner's original argument. The literature quotes (1.8) as LYM inequality. We suggest the name <u>YBL inequality</u> which can be justified better.

Second proof of the Sperner theorem (Permutation method). A *cyclic permutation* π of the elements of $X=\{x_1,\dots,x_n\}$ is an ordering of the elements along a cycle (see Fig. 1.1). A set $A\subset X$ is called *consecutive* (along π) if its elements are

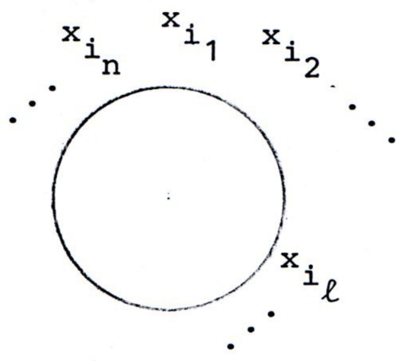

Fig. 1.1

consecutive along π.

Lemma 1.4. *An inclusion-free family* $\mathcal{A}$ *of consecutive sets along a fixed permutation has at most* n *members. If* $\mathcal{A}$ *has* n *members then they are equally sized*.

Proof. Suppose that F, $F'\in\mathcal{A}$ satisfy $|F|\le|F'|$ and have the same (clockwise) first element in π. The obvious consequence $F\subset F'$ is a contradiction. Any $x\in X$ occurs as the first element of a member of $\mathcal{A}$ at most once. Hence we have $|\mathcal{A}|\le n$.

Suppose now that $|\mathcal{A}|=n$ and its members are denoted by $F_1,\dots,F_n$ in this order along π. The first elements of F_1 and F_2 are neighbouring. Since $\mathcal{A}$ is inclusion-free $|F_1|\le|F_2|$ follows (see Fig. 1.2). We can deduce similarly

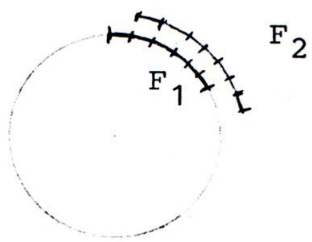

Fig. 1.2

$|F_2|\le|F_3|\le\ldots\le|F_n|\le|F_1|$. Hence we obtain $|F_1|=\ldots=|F_n|$ as desired. □

Turning back to the proof of the Sperner theorem let $\mathcal{F}$ be a given inclusion-free family and let us count the number of pairs (π,F) where π is a cyclic permutation, $F\in\mathcal{F}$ and F is consecutive along π . For any given F there are $|F|!(n-|F|)!$ cyclic permutations where the elements of F are consecutive: we can independently permute the elements of F and $X-F$, resp. The number of pairs is $\sum_{F\in\mathcal{F}} |F|!(n-|F|)!$. By Lemma 1.4 there are at most n appropriate sets F for any fixed π . Therefore the number of above pairs is at most n times the number of cyclic permutations. We obtained inequality (1.7) again:

$$(1.9)\qquad \sum_{F\in\mathcal{F}} |F|!(n-|F|)! \le n!$$

The proof can be completed as in the previous proof. □

The reader has probably noticed that the second proof is an over-complicated version of the first one. A chain in the first proof is nothing else but a permutation. A set is a member of such a chain iff its elements are consecutive at the beginning of the permutation. A cyclic permutation in the second proof is a class of n permutations from the first proof.

Why did we present both proofs? On the one hand we wanted to show the simplest proof, on the other hand, the second one offers an easy way to determine all the maximum sized inclusion-free families:

<u>Theorem 1.5</u> (Sperner (1928)) <u>If $\mathcal{F}$ is a maximum sized inclusion-free family then either</u>

$$F = \binom{X}{\lfloor \frac{n}{2} \rfloor}$$

or

$$F = \binom{X}{\lceil \frac{n}{2} \rceil}$$

holds.

Proof. We analyse the arguments of the second proof. The two sides of (1.6) can be equal only if the same holds for (1.9). That is, only if there are exactly n members of F being consecutive in π, for any π, for any π. It follows by Lemma 1.4 that any two members $F, F' \in F$ consecutive along the same π are equally sized. However, it is easy to see that there is a C for any two $F, F' \in F$ in which they are both consecutive: list first the elements of $F-F'$, then the elements of $F \cap F'$ and $F'-F$ and finish with $X-F-F'$. $|F|=|F'|$ is verified. We obtained that the members of a maximum sized inclusion-free family have the same size, say k. However $\binom{n}{k} = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ holds only in the cases listed in the theorem. That is, for odd n there are two extremal families, while for even n there is only one. □

For the third proof we need a complicated notation and a lemma. Let $A \subset 2^X$. The shadow $\sigma(A)$ of A is the family of all sets obtained from the members of A by omitting one element:

(1.10) $\quad \sigma(A) = \{B : B \subset A, \ |B|=|A|-1 \text{ for some } A \in A\}$

Of course, $A \subset \binom{X}{k}$ implies $\sigma(A) \subset \binom{X}{k-1}$.

Lemma 1.6 (Sperner (1928)). If $A \subset \binom{X}{k}$ $(1 \le k \le n)$ then

(1.11) $\quad |\sigma(A)| \ge \dfrac{k}{n-k+1} |A|$.

Proof. Let us count the number of pairs (A,B) where $A \in \mathcal{A}$, $A \supset B$, $|B|=k-1$. Fixing an $A \in \mathcal{A}$ there are exactly k $(k-1)$-element subsets $B \subset A$. Therefore the number of the above pairs is $k|\mathcal{A}|$. On the other hand, fixing a set $B \in \sigma(\mathcal{A})$ it can be a subset of at most $n-k+1$ $(=|X-B|)$ sets A. Consequently $|\sigma(\mathcal{A})|(n-k+1)$ is an upper estimate of the number of pairs. Hence $k|\mathcal{A}| \leq |\sigma(\mathcal{A})|(n-k+1)$ which is equivalent to (1.11).

□

We need this lemma because the method of the third proof is based on a transformation which replaces the largest (say k-element) members of $\mathcal{F}$ by some $(k-1)$-element ones. Therefore we have to know something about the number of the new members.

Third proof of the Sperner theorem. (Sperner (1928), Transformation method, Compressing method). Outline of the proof. Suppose that $\mathcal{F}$ satisfies the assumptions of the theorem. Consider the members of $\mathcal{F}$ with the maximum size k. This subfamily has, by Lemma 1.6, a large shadow among the $(k-1)$-element subsets. More precisely the size of the shadow is at least the size of the subfamily of the k-element members. It is obvious that the members of the shadow are not in $\mathcal{F}$. Let us replace this subfamily by its shadow. It is easy to see that the new family $\mathcal{F}'$ is inclusion-free. Iterating these steps we arrive to the position when the largest sizes are $\lfloor \frac{n}{2} \rfloor$. Then take the family of the complement sets of the members of $\mathcal{F}$ and repeat the procedure. The number of the members of $\mathcal{F}$ is not decreased at any step. Therefore the size of the original family $\mathcal{F}$ is at most the size of the last family $\binom{X}{\lfloor \frac{n}{2} \rfloor}$.

Formal proof. Suppose that $\mathcal{F}$ is extremal and let k denote the size of the largest members in $\mathcal{F}$: $k = \max\{|A| : A \in \mathcal{F}\}$. We introduce the notation

$$\mathcal{F}^i = \mathcal{F} \cap \binom{X}{i} .$$

Take the family $\mathcal{F}' = (\mathcal{F} - \mathcal{F}^k) \cup \sigma(\mathcal{F}^k)$ where the union is obviously disjoint. Choose any two members F_1 and F_2 of $\mathcal{F}'$. We have to show that $F_1 \not\subset F_2$. This is clear if they are both in $\mathcal{F} - \mathcal{F}^k$ or in $\sigma(\mathcal{F}^k)$, resp. If $F_1 \in \mathcal{F} - \mathcal{F}^k$, $F_2 \in \sigma(\mathcal{F}^k)$ and $F_1 \subset F_2$ were true then $F_2 \subset F_3$ would hold for some $F_3 \in \mathcal{F}^k \subset \mathcal{F}$. implying the contradiction $F_1 \subset F_3$, $F_1, F_3 \in \mathcal{F}$. Finally, if $F_1 \in \sigma(\mathcal{F}^k)$ $F_2 \in \mathcal{F} - \mathcal{F}^k$ then $|F_1| = k-1$ and $|F_2| \le k-1$ so $F_1 \subset F_2$ is impossible. Consequently $\mathcal{F}$ is an inclusion-free family.

If $k > \lceil \frac{n}{2} \rceil$ then $k > n-k+1$ therefore (1.11) gives $|\sigma(\mathcal{F}^k)| \ge \frac{k}{n-k+1} |\mathcal{F}^k| > |\mathcal{F}^k|$. Hence $|\mathcal{F}'| = |\mathcal{F}| - |\mathcal{F}^k| + |\sigma(\mathcal{F}^k)| > |\mathcal{F}|$ follows contradicting the maximality of $|\mathcal{F}|$. This proves $k \le \lceil \frac{n}{2} \rceil$ for any extremal inclusion-free family.

The complementary family

$$\mathcal{F}^c = \{F : X - F \in \mathcal{F}\}$$

is obviously inclusion-free, again. It is also extremal, therefore

$$n - \min_{F \in \mathcal{F}} |F| = \max_{F \in \mathcal{F}^c} |F| \le \left\lceil \frac{n}{2} \right\rceil$$

and hence

$$\min_{F \in \mathcal{F}} |F| \ge n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$$

hold. Combining our results, we obtain that the members F of an extremal F satisfy

(1.12) $$\left\lfloor \frac{n}{2} \right\rfloor \le |F| \le \left\lceil \frac{n}{2} \right\rceil .$$

If n is even, this proves (1.6) and the theorem. If n is odd then F can contain members of two different sizes, $\frac{n+1}{2}$ and $\frac{n-1}{2}$. The family $F' = \left(F - F^{\frac{n+1}{2}}\right) \cup \sigma\left(F^{\frac{n+1}{2}}\right)$ is inclusion-free, again. Its size is

$$|F'| = |F| - \left|F^{\frac{n+1}{2}}\right| + \left|\sigma\left(F^{\frac{n+1}{2}}\right)\right| \ge$$

$$\ge |F| - \left|F^{\frac{n+1}{2}}\right| + \frac{\frac{n+1}{2}}{n-\frac{n+1}{2}+1} \left|F^{\frac{n+1}{2}}\right| = |F|$$

by (1.11). On the other hand $F' \subset \binom{X}{\frac{n-1}{2}}$. We obtained $|F| \le |F'| \le \binom{n}{\frac{n-1}{2}}$ as desired. □

Although the names of the method given at the beginning of the proof are self-evident, we add some remarks. The name "Transformation method" denotes a method in which a certain transformation is applied for the considered family which does not decrease the size of the family and retains its properties. Usually we apply such a transformation many times. At the end, the family has some useful additional properties (here $\subset \binom{X}{\lfloor\frac{n}{2}\rfloor}$). Instead of applying the transformation many times we often suppose that the family is extremal. We show that if it does not have the desired property then a single application of the transformation enlarges the family contradicting its maximality. This latter way is formally shorter. The present special transformation is called "compressing".

1.3. Applications

A) Minimal keys in date bases. We describe the data base with the following model. Let B be a matrix. A row of B contains the data of a fixed object. A column of B contains the data of the same type (attribute) for all objects. So if we have m objects and n types of data (attributes, properties) then B is an $m \times n$ matrix. Let X denote the set of columns.

A set F of columns is called a key if the entries in these columns determine the entries of all other columns uniquely. More precisely if there are no two rows equal in the columns belonging to F but different in some other column then F is a key. It is obvious that if F is a key and $F \subset F'$ then F' is also a key. Therefore it is sufficient to consider the minimal keys: K is a minimal key if K is a key but no proper subset of it is a key. Let us denote the family of minimal keys by $\mathcal{K}$. By definition $\mathcal{K}$ is inclusion-free. Theorem 1.3 yields (Demetrovics (   ))

$$|\mathcal{K}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} . \tag{1.13}$$

It is proved in the above paper that one can find data bases with equality in (1.13).

The situation described for data bases is typical in many applications. A family $\mathcal{F}$ is given having the following property:

$$F_2 \subset F_1 \in \mathcal{F} \Rightarrow F_2 \in \mathcal{F} .$$

In such families the most important members are the minimal ones. They form an inclusion-free family therefore the Sperner theorem can be applied for their number.

B) Littlewood-Offord problem for the number of certain sums. Let $a_1,\dots,a_n$ be real numbers satisfying the condition $|a_i| \ge 1$ for all i. Let I be an open interval of length 1 and let $f(a_1,\dots,a_n,I)$ denote the number of different sums $\sum_{i\in A} a_i$ $(A\subset\{1,2,\dots,n\})$ lying in I. $f(a_1,\dots,a_n)$ denotes $\max f(a_1,\dots,a_n, I)$ over all open intervals of length 1. Finally,

$$f(n) = \max f(a_1,\dots,a_n)$$

over all a's satisfying the above conditions.

Theorem 1.7 (Erdős (1945)).

$$f(n) = \binom{n}{\lfloor \frac{n}{2}\rfloor} .$$

Proof. $f(1,\dots,1, (\lfloor\frac{n}{2}\rfloor-\frac{1}{2}, \lfloor\frac{n}{2}\rfloor+\frac{1}{2})) = \binom{n}{\lfloor\frac{n}{2}\rfloor}$ is obvious and this implies $f(n) \ge \binom{n}{\lfloor\frac{n}{2}\rfloor}$.

To prove the other direction let us verify

$$(1.14) \qquad f(a_1,a_2,\dots,a_n) = f(-a_1,a_2,\dots,a_n) .$$

The sums of $a_1,\dots,a_n$ are of the form $a_1 + \sum_{i\in A-\{1\}} a_i$ or $\sum_{i\in A} a_i$ where $A\subset\{1,\dots,n\}$ contains or does not contain 1, resp. We associate the sums $\sum_{i\in A-\{1\}} a_i$ and $-a_1+\sum_{i\in A} a_i$ with the above sums in this order. It is easy to see that the latter ones are all the possible sums of $-a_1,a_2,\dots,a_n$. The difference between the corresponding sums is a_1 , that is, the set of the latter sums can be obtained by shifting by $-a_1$, the set of the first type. This proves (1.14).

With repeated application of (1.14) we arrive to the situation where all a 's are non-negative. Suppose that this is the case, let I be an interval of length 1 and denote by F the family of sets $A \subset \{1,\dots,n\}$ satisfying $\sum_{i\in A} a_i \in I$. If $A \subsetneq B$, $A,B \in F$ then the difference $\sum_{i\in B} a_i - \sum_{i\in A} a_i =$ $= \sum_{i\in B-A} a_i \geq |B-A| \geq 1$. Both sums cannot be in I . This proves that F is inclusion free. The Sperner theorem yields $f(a_1,\dots,a_n, I) = |F| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Hence we have $f(a_1,\dots,a_n) \leq$ $\leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for any choice of a 's and this implies the statement of the theorem. □

C) Let $f(x_1,\dots,x_n)$ be a Boolean function of n variables, that is, both the variables and the function takes on the values 0 and 1 . The function is called monotonic if $x_i \leq y_i$ for all i , $(1\leq i\leq n)$ imply $f(x_1,\dots,x_n) \leq f(y_1,\dots,y_n)$.

A set of switches arranged like in Fig. 1.3 is called a θ -network.

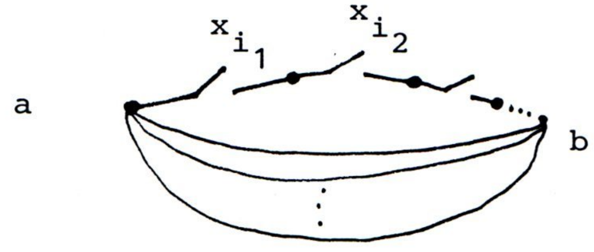

Fig. 1.3.

The switches labelled with the same variable are mechanically connected. If $x_i=0$ then all switches labelled with x_i are open and all switches labelled with $\bar{x}_i$ are closed. If $x_i=1$ then their positions are opposite. Any given θ-network defines a Boolean function: $f(x_1,\dots,x_n) = 1$ iff the points a and b are electrically connected at the positions $x_1,\dots,x_n$ of the switches. We also say that the θ-network represents the function $f(x_1,\dots,x_n)$.

Any f can be represented by a θ-network: take a branch for any system of values $x_1=\varepsilon_1,\dots, x_n=\varepsilon_n$ satisfying $f(\varepsilon_1,\dots,\varepsilon_n) = 1$, a branch contains n switches, the i th one is labelled with x_i if $\varepsilon_i=1$ and with $\bar{x}_i$ if $\varepsilon_i=0$. Let $\alpha(f)$ denote the minimum number of branches of a θ-network representing f . Similarly, $\beta(f)$ denotes the minimum number of switches.

Theorem 1.8.

(1.15) $$\alpha(f) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} ,$$

(1.16) $$\beta(f) \le \left\lceil \frac{n}{2} \right\rceil \binom{n}{\lceil \frac{n}{2} \rceil} ,$$

hold for any monotonic Boolean function f of n variables.

Proof. Let $f(x_1,\dots,x_n)$ be a monotonic Boolean function and fix a representation of it by a θ-network R . Suppose that there is a switch labelled with $\bar{x}_i$ for some $1\le i\le n$. Omit this switch and connect its endpoints. It is easy to see that the new θ-network R' also represents $f(x_1,\dots,x_n)$. Repeated application of this operation shows that $f(x_1,\dots,x_n)$ has always a minimum representation (in sense of α or β)

containing no switch labelled with $\bar{x}$.

A branch can be described with a set F of variables ($F \subset \{x_1, \ldots, x_n\}$) . A minimum representation cannot contain two branches with $F_1 \subset F_2$ because otherwise F_2 could be omitted. In this way the branches define an inclusion-free family $\mathcal{F}$, where $|\mathcal{F}|$ is the number of branches. Hence, applying the Sperner theorem, we obtain

$$\alpha(f) \leq |\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

proving (1.15).

The number of switches is $\sum_{F \in \mathcal{F}} |F|$. To prove (1.16) it is enough to show

$$\frac{\sum_{F \in \mathcal{F}} |F|}{\lceil \frac{n}{2} \rceil \binom{n}{\lceil \frac{n}{2} \rceil}} \leq 1 .$$

Since

$$i\binom{n}{i} = n\binom{n-1}{i-1} \leq n\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = \lceil \tfrac{n}{2} \rceil \binom{n}{\lceil \frac{n}{2} \rceil}$$

is obviously true, the inequality

$$\frac{\sum_{F \in \mathcal{F}} |F|}{\lceil \frac{n}{2} \rceil \binom{n}{\lceil \frac{n}{2} \rceil}} \leq \sum_{F \in \mathcal{F}} \frac{|F|}{|F| \binom{n}{|F|}} = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$$

follows from (1.8). □

For the analogous problems for non-monotonic functions see Tarján (1974).

D) Search with subsets. Let x_0 be an unknown element of X $(|X|=n)$. The aim is to find x_0, but we are allowed to use only some special kind of information. A family $\mathcal{F}\subset 2^X$ is given and we may ask if $x_0 \in F$ or not for any $F\in\mathcal{F}$. x_0 should be uniquely determined by the answers to these queries. It is clear that this can be done iff for any $x,y\in X$, $x\neq y$ there is a member F of the family satisfying either $x\in F$ and $y\notin F$ or $x\notin F$ and $y\in F$. Such a family is called separating.

Theorem 1.9 (folklore).

$$\min |\mathcal{F}| = \lceil \log n \rceil$$

for separating families $\mathcal{F}$.

Proof. The incidence matrix M of a family $\mathcal{F}=\{F_1,\dots,F_m\}$ is an $m\times n$ 0,1 matrix in which the j th entry of the i th row is 1 iff $x_j\in F_i$. Let M^T denote the matrix obtained from M by changing the role of the rows and columns. The family whose incidence matrix is M^T is called the dual of $\mathcal{F}$ and is denoted by $\mathcal{F}^T$.

$\mathcal{F}$ is separating iff for any two columns there is a row containing different entries in these columns. More simply, $\mathcal{F}$ is separating iff its columns are distinct. This can be formulated in the way that $\mathcal{F}^T$ consists of distinct members. Hence $n\le 2^m$ follows and this implies $m\ge\lceil \log n\rceil$. It is easy to construct the incidence matrix of an extremal $\mathcal{F}$. □

A family $\mathcal{F}$ is called strongly separating iff for any $x,y\in X$, $x\neq y$ there is a member F satisfying $x\in F$, $y\notin F$. Dickson (1969) proved that a minimum strongly separating family has approximately $\log_2 n$ members. Later Spencer found the real way of solving the problem; he reduced it to the Sperner theorem:

<u>Theorem 1.10</u> (Spencer ( )). <u>The minimum cardinality</u> $|F|$ <u>of a strongly separating family in an</u> n <u>-element set is equal to the smallest integer</u> m <u>satisfying</u>

$$n \le \binom{m}{\lfloor \frac{m}{2} \rfloor} \tag{1.17}$$

<u>Proof</u>. It is easy to see that F is strongly separating iff F^T is inclusion-free. Theorem 1.5 implies (1.17) if m denotes the number of members of $|F|$, that is the size of the underlying set of F^T.

On the other hand, if m and n are integers satisfying (1.17) then it is easy to construct an inclusion-free family F^T on m elements with $|F^T|=n$. $(F^T)^T = F$ will be a strongly separating family on n elements and of size m. □

Exercises

1.1. Let $A \subset \binom{X}{k}$, $|X|=n$ and suppose that if $B \in \sigma(A)$ then it is contained in precisely $n-k+1$ different members of A. Prove that either $A=\emptyset$ or $A=\binom{X}{k}$.

1.2. Prove that equality in (1.11) implies either $A=\emptyset$ or $A=\binom{X}{k}$.

1.3. Prove Theorem 1.5 by the third proof of the Sperner theorem using Exercise 1.2.

1.4. Determine $\min|\sigma(A)|$ where $A \subset \binom{X}{2}$, $|X|=n$ and $|A|$ is given.

1.5. Make a guess on $\min|\sigma(A)|$ where $A \subset \binom{X}{k}$, $|X|=n$, $k \le a$, $|A|=\binom{a}{k}$ are fixed integers.

1.6. Prove that the YBL-inequality ((1.8)) may hold with equality iff $F=\binom{X}{k}$ for some $0 \le k \le n$.

1.7.* (Erdős, Ko and Rado (1961)) Prove that there is only one extremal family in Theorem 1.1 under the additional condition $F_1 \cap F_2 \cap F_3 \neq \emptyset$ $(F_1, F_2, F_3 \in \mathcal{F})$.

1.8.* (Bollobás ( )) Prove the YBL -inequality ((1.8)) by induction on n .

1.9. Determine $\max|\mathcal{F}|$ for inclusion-free families satisfying $|F| \leq k$ $(F \in \mathcal{F})$, where $k \leq \frac{n}{2}$ is a fixed integer.

1.10. Let S be a set of sequences of length n formed from the symbols) and (. Suppose that S has the following property: if a∈S and some of the symbols) in a are replaced by (then the so obtained sequence b cannot be in S . In any sequence, form pairs of the neighbouring brackets being in the right directions: ( ) . Then repeat this step not considering the brackets already paired, e.g.: (( )) . And so on ... . Prove that 1) the set of unpaired brackets form a sequence of symbols) followed by a sequence of symbols (; 2) $|S| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Hints to the exercises

1.1. Let $A \in \mathcal{A}$ and $B \in \binom{X}{k}$. Prove by induction on $|A-B|$ that $B \in \mathcal{A}$.

1.2. Use the argument of the proof of Lemma 1.6 and Exercise 1.1.

1.3. If $k > \frac{n+1}{2}$ then $|\sigma(\mathcal{F}^k)| > \mathcal{F}^k$ by (1.11). This is sufficient for the case if n is even. If n is odd then Exercise 1.2 should be used at $k = \frac{n+1}{2}$.

1.4. If $|\mathcal{A}| \geq \binom{a}{2}$ for some integer a $(2 \leq a \leq n)$ then we need at least a vertices. So $|\mathcal{A}| = \binom{a}{2}$ implies $|\sigma(\mathcal{A})| = a$ and $\binom{a}{2} < |\mathcal{A}| < \binom{a+1}{2}$ implies $|\sigma(\mathcal{A})| = a+1$. The constructions are $\binom{A}{2}$ with $|A|=a$ and its completion with $|\mathcal{A}|-\binom{a}{2}$ two-element sets containing a new $(a+1)$-st element.

1.5. See Theorem ...

1.6. See the proof of Theorem 1.5.

1.7. (Pósa ...) Take a smallest member $(\in \mathcal{F})$ of cardinality ℓ. An $(\ell-1)$-element subset of it is not in $\mathcal{F}$ therefore its complement $\in \mathcal{F}$. We found two sets with a one-element intersection.

1.8. The subfamily of the sets $F \in \mathcal{F}$ not containing a fixed $x \in X$ is inclusion-free. Write the YBL-inequality for $n-1$ elements and for these subfamilies with each x. Sum up these inequalities and take into account that $F \in \mathcal{F}$ occurs in $n-|F|$ inequalities.

1.9. $\binom{n}{k}$. Use the YBL -inequality.

1.10. Change the subsequence of unpaired brackets so that it contains $\lfloor \frac{n}{2} \rfloor$ symbols) followed by $\lceil \frac{n}{2} \rceil$ symbols (.

2. Intersecting families and shadows

2.1. Intersecting families of equally sized subsets

Theorem 1.1 answered the first question concerning intersecting families. This question was rather easy to answer. However it becomes considerably harder if we consider k-element subsets only:

Theorem 2.1 (Erdős, Ko and Rado (1961)). Let k $(1 \le k \le n/2)$ be a fixed integer. Then

$$\max|\mathcal{F}| = \binom{n-1}{k-1} \tag{2.1}$$

over all intersecting families $\mathcal{F} \subset \binom{X}{k}$.

If $k > n/2$ then (2.1) is not true. The problem becomes trivial in this case: any two members of $\binom{X}{k}$ meet, therefore $\max|\mathcal{F}| = \binom{n}{k}$. If $k \le n/2$, it is easy to construct a family giving equality in the theorem. Fix an element $x \in X$ and take all the k-element subsets of X containing x . This family is intersecting and its size is $\binom{n-1}{k-1}$, indeed. We have to prove only that

$$|\mathcal{F}| \le \binom{n-1}{k-1} \tag{2.2}$$

holds for any intersecting family of k-element sets. We give two proofs. The first one is much shorter, the second one is the historically first proof and is very illuminating: its ideas will be used several times.

First proof of the Erdős-Ko-Rado theorem. (Katona (1972), Permutation method.)

Outline of the proof. We use the ideas of the second proof of the Sperner theorem. A cyclic permutation of X will be considered and we first solve the analogous problem for intersecting families of consecutive sets (Lemma 2.2). We will see that their maximum size is k, that is, the proportion of the intersecting family in the whole family of consecutive sets is $\le k/n$. Then we derive by a counting argument $|F| \le \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$.

Formal proof

Lemma 2.2. If $\mathcal{A}$ is an intersecting family of k-element $(1 \le k \le n/2)$ consecutive subsets along a cyclic permutation π then

$$|\mathcal{A}| \le k .$$

Proof. Take an arbitrary member A of $\mathcal{A}$. Suppose that the elements of A are labelled in the following way: $A = \{x_1, \ldots, x_k\}$ (see Fig. 2.1).

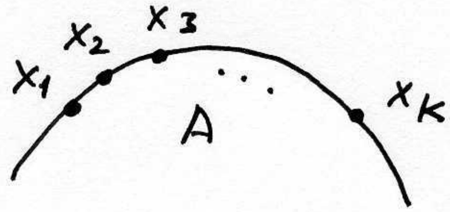

Fig. 2.1.

Any further member of $\mathcal{A}$ has its starting point or endpoint in A. Any point of A can serve at most once as a starting point and once as an endpoint. However, we cannot have a member B with endpoint x_i and a member C with starting point x_{i+1} $(1 \le i \le k-1)$ simultaneously in $\mathcal{A}$. They cannot meet at their other end since $2k \le n$. Therefore there is at most one member of $\mathcal{A}$ whose endpoint is x_i or starting point is

x_{i+1} . Their number is at most $k-1$. If A is included then at most k . □

Let $\mathcal{F}$ be an intersecting family of k-element subsets and let us count the number of pairs (π, F) , where π is a cyclic permutation of X and $F \in \mathcal{F}$ is consecutive in π . For a specified F there are $k!\,(n-k)!$ cyclic permutations in which F is consecutive. Therefore the number of the pairs is $|\mathcal{F}| \cdot k!\,(n-k)!$. Conversely, if we fix a π then there are at most k consecutive sets $F \in \mathcal{F}$ by Lemma 2.2. We obtained

$$|\mathcal{F}|k!(n-k)! \leq (n-1)!k$$

and this is equivalent to (2.2). □

It should be mentioned that the original proof of Katona was somewhat different. P. Erdős and E. C. Milner called his attention to the present simpler variant. It was published in Katona (1974). Later, independently, Biagioni also discovered this nicer way.

<u>Second proof of the Erdős-Ko-Rado theorem</u> (Modified version of Erdős, Ko and Rado (1961), Transformation method, Shifting method). First of all, let us introduce a transformation. Let x and y be different elements of X and $\mathcal{F}$ be a family. Define the families

$$\mathcal{F}' = \{F:\ F \in \mathcal{F},\ (x \notin F) \text{ or } (x,y \in F) \text{ or } (x \in F,\ y \notin F,\ F \cup \{y\} - \{x\} \in \mathcal{F})\}$$

(2.3)

$$\mathcal{F}'' = \{F:\ F \in \mathcal{F},\ x \in F,\ y \notin F,\ F \cup \{y\} - \{x\} \notin \mathcal{F}\} .$$

The <u>shift</u> $\tau_{x,y}(\mathcal{F})$ <u>of</u> $\mathcal{F}$ <u>from</u> x <u>to</u> y <u>is defined by</u>

(2.4) $\tau_{x,y}(\mathcal{F}) = \mathcal{F}' \cup \{F \cup \{y\} - \{x\}:\ F \in \mathcal{F}''\}$.

Outline of the proof. We will see (Lemma 2.3) that the shift of an intersecting family is also intersecting. Let moreover an ordering of the elements of X be fixed. Apply $\tau_{x,y}$ for any $y<x$ as long as it makes any changes, then the members of the obtained family will meet in the first $2k-1$ elements of X (Lemma 24). This observation makes an easy counting argument possible which uses induction on k.

Formal proof. First we need a lemma stating that $\tau_{x,y}(F)$ is intersecting if $F \subset \binom{X}{k}$ is intersecting. For further purposes the lemma will be stated in a more general form. F is called t-wise ℓ-intersecting $(1 \le t,\ 1 \le \ell)$ if

$$|F_1 \cap \ldots \cap F_t| \ge \ell$$

holds for any $F_1, \ldots, F_t \in F$. If $\ell = 1$ the name is simply t-wise intersecting while in the case $t=2$ we say that F is ℓ-intersecting

Lemma 2.3. Let $1 \le \ell \le k \le |X|$, $2 \le t$ be integers. If $F \subset \binom{X}{k}$ is a t-wise ℓ-intersecting family then $\tau_{x,y}(F)$ has the same size and the same properties.

Proof. $\tau_{x,y}(F) \subset \binom{X}{k}$ and $|\tau_{x,y}(F)| = |F|$ are trivial, we have to prove only that $\tau_{x,y}(F)$ is t-wise ℓ-intersecting.

Let $G_1, \ldots, G_t \in \tau_{x,y}(F)$. The definition of $\tau_{x,y}$ shows that either

$$F_i = G_i \in F'$$

or

$$x \notin G_i,\ y \in G_i,\ F_i = G_i \cup \{x\} - \{y\} \in F''$$

holds.

(2.5) $\quad G_1 \cap \ldots \cap G_t \supset F_1 \cap \ldots \cap F_t - \{x\}$

is a trivial consequence. Here $|F_1 \cap \ldots \cap F_t| \geq \ell$ holds by the assumptions. If this size is $\geq \ell+1$ or $x \notin F_1 \cap \ldots \cap F_t$ then (2.5) leads to the desired inequality

(2.6) $\quad |G_1 \cap \ldots \cap G_t| \geq \ell$.

Thus we may suppose

(2.7) $\quad |F_1 \cap \ldots \cap F_t| = \ell$

and

(2.8) $\quad x \in F_1 \cap \ldots \cap F_t$.

If $y \in F_i$ holds for all i then we have $F_i = G_i$ for all i and hence (2.6) easily follows. We may assume

(2.9) $\quad y \notin F_1 \cap \ldots \cap F_t$.

In the case $y \in G_1 \cap \ldots \cap G_t$ (2.5) and (2.9) imply (2.6). Therefore

(2.10) $\quad x, y \notin G_1 \cap \ldots \cap G_t$

can be also assumed.

(2.8) and (2.9) imply that there are some j 's satisfying $x \in F_j$, $y \notin F_j$. We may suppose that one of them, say F_s , satisfies the additional condition $G_s = F_s$. Otherwise all G_i would to $\mathcal{F}'$.
Then, by the definition of $\tau_{x,y}$ $\mathcal{F}$ contains a member $\hat{F}_s = F_s \cup y - x$. $F_1 \cap \ldots \cap \hat{F}_s \cap \ldots \cap F_t$ is of size at least ℓ by the conditions. This intersection does not contain x . It cannot contain y otherwise $F_i = G_i$ holds for all $i \neq s$ and (2.6) is trivial. Hence

$$F_1 \cap \ldots \cap \hat{F}_s \cap \ldots \cap F_t = F_1 \cap \ldots \cap F_t - \{x\}$$

follows. This equation, combining with (2.7) and (2.8), implies $|F_1 \cap \ldots \cap \hat{F}_s \cap \ldots \cap F_t| = \ell-1$, a contradiction. □

Suppose that the underlying set is $X = \{1,2,\ldots,n\}$. Suppose that $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family. Take a pair $y<x$ $(1 \le y < x \le n)$ and consider $\tau_{x,y}(\mathcal{F})$. This new family is an intersecting family of the same size by Lemma 2.3. We repeat this transformation as long as $\tau_{x,y}(\mathcal{F}) \neq \mathcal{F}$ for some $y<x$. It is obvious that this procedure stops, because the transformation decreases the sum $\sum_{F\in\mathcal{F}} \sum_{x\in F} x$ and this sum has a minimum value. So after finitely many steps we arrive to a family $\mathcal{F}$ satisfying

(2.11) $$\tau_{x,y}(\mathcal{F}) = \mathcal{F}$$

for any $1 \le y < x \le n$. We say that $\mathcal{F}$ is shifted. We need a lemma stating that the members of a shifted intersecting family $(\subset\binom{X}{k})$ meet in $\{1,\ldots,2k-1\}$. For further purposes we prove a generalization:

Lemma 2.4 (Frankl ( )). Let $\mathcal{F}$ be a shifted t-wise ℓ-intersecting family on an n-element underlying set X where $n > \frac{t}{t-1}(k-\ell)+\ell-1$. Then the intersection of any t members of $\mathcal{F}$ contains at least ℓ elements in $T = \{1,\ldots,\lfloor\frac{t}{t-1}(k-\ell)\rfloor+\ell\}$.

Proof. Denote the intersection $F_1 \cap \ldots \cap F_t$ by R and suppose that

(2.12) $$|R \cap T| < \ell .$$

Choose $F_1,\ldots,F_t$ satisfying (2.12) and minimizing

(2.13) $$|R| = r \quad (\ge \ell) .$$

(2.12) and (2.13) imply $R-T \neq \emptyset$. Fix an $x \in R-T$. The sets $F_i - R$ are of size $k-r$. They may cover at least $t-1$ times at most $\frac{t(k-r)}{t-1}$ elements. However $\frac{t(k-r)}{t-1} + |R \cap T| \leq \frac{t(k-r)}{t-1} + r-1 \leq$ $\leq \frac{t(k-\ell)}{t-1} + \ell - 1 < |T|$ holds, therefore T contains an element y covered by at most $t-2$ of $F_i - R$'s, that is, F_i's. Suppose that $y \notin F_s$. Apply (2.11) for these x and y: $\hat{F}_s =$ $= F_s \cup \{y\} - \{x\} \in \mathcal{F}$. $y \notin F_1 \cap \ldots \cap \hat{F}_s \cap \ldots \cap F_t$ is obvious by the choice of y, therefore (2.12) remains true. On the other hand, $x \notin F_1 \cap \ldots \cap \hat{F}_s \cap \ldots \cap F_t$ shows that the size of this intersection is $<r$ contradicting the minimality of r. □

Let us turn back to the proof of the theorem. We may suppose that $\mathcal{F} \subset \binom{X}{k}$ is a shifted intersecting family. Lemma 2.4 states that the members of $\mathcal{F}$ meet in $\{1,\ldots,2k-1\}$. We use induction on k. The case $k=1$ is trivial. Suppose that $k>1$ and the theorem is proved for smaller values. Let a_i $(1 \leq i \leq k)$ denote the number of distinct i-element sets of the form $F \cap \{1,\ldots,2k\}$ where $F \in \mathcal{F}$.

$$a_i \leq \binom{2k-1}{i-1} \qquad (1 \leq i \leq k)$$

follows from the induction hypothesis in view of Lemma 2.4 for $1 \leq i \leq k-1$. In the case $i=k$ it also follows since at most one of the complementing pairs can occur, therefore $a_k \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$. The number of the j-element distinct sets of the form $F \cap \{2k+1,\ldots,n\}$ is at most $\binom{n-2k}{j}$. The obvious inequalities

$$|\mathcal{F}| \leq \sum_{i=1}^{k} a_i\, b_{k-i} \leq \sum_{i=1}^{k} \binom{2k-1}{i-1}\binom{n-2k}{k-i} = \binom{n-1}{k-1}$$

prove the theorem. □

The original proof of Erdős, Ko and Rado used only the idea that the members of a shifted intersecting family meet in $\{1,\ldots, n-1\}$. Then the calculation is somewhat easier but double induction is needed.

Let us say a few words about the connecitons to extremal graph theory. $F \subset \binom{X}{2}$ is called a graph (or simple graph). Generalizing this concept, many authors call a family $F \subset 2^X$ a hypergraph. If $F \subset \binom{X}{k}$ then F is a k-hypergraph or k-uniform hypergraph. Apparently, the present theory is a generalization of the theory of extremal graphs which is presented very well in Bollobás's book (Bollobás ( )). However, most of our theorems have no sense for graphs or are trivial. Theorems 1.1, 1.2, 1.3 and 1.4 are obviously non-generalizations of graph problems, while Theorem 2.1 is trivial for $k=2$.

2.2. Shadows

The next theorem deals with shadows. Actually, it gives the exact answer to the problem of Lemma 1.6 for some "nice" values of $|A|$. Later, in Section we will generalize it for any $|A|$. In a certain sense, this theorem stands behind both the Sperner and the Erdős-Ko-Rado theorems. First we need a lemma stating that $\tau_{x,y}$ does not increase the size of the shadow. For later purposes we formulate the lemma in a more general form, again. The s-shadow $(1 \leq s)$ of a family $A \subset 2^X$ is defined by

$$\sigma_s(A) = \{B: B \subset A, \ |B| = |A| - s \ \text{ for some } A \in A\} .$$

Lemma 2.5 (Katona (1964)). If $F \subset \binom{X}{k}$ then

(2.14) $$|\sigma_s(\tau_{x,y}(F))| \le |\sigma_s(F)| \quad (1\le s<k)$$

<u>for any distinct</u> x <u>and</u> y $(x,y\in X)$.

<u>Proof</u>. Let us define the function φ on $\sigma_s(\tau_{x,y}(F))-$ $-\sigma_s(F)$ by

(2.15) $$\varphi(A) = A\cup\{x\}-\{y\} .$$

We will see that φ is an injection into $\sigma_s(F)-\sigma_s(\tau_{x,y}(F))$.

First we show that $A\in\sigma_s(\tau_{x,y}(F))-\sigma_s(F)$ implies the existence of an $F\in F$ such that

(2.16) $$A\subset F\cup\{y\}-\{x\} ,\ x\in F ,\ y\notin F ,\ F\cup\{y\}-\{x\}\notin F .$$

Indeed, there is a G satisfying $A\subset G\in\tau_{x,y}(F)$, by definition. G is either a member of F or $G=F\cup\{y\}-\{x\}$ for an $F\in F$ satisfying (2.16). In the first case, however, $A\subset F\in F$ implies $A\in\sigma_s(F)$, a contradiction.

Now we show that $A\in\sigma_s(\tau_{x,y}(F))-\sigma_s(F)$ implies $x\notin A$, $y\in A$, that is, $|\varphi(A)|=k$. $x\notin A$ follows by (2.16). The assumption $y\notin A$ and (2.16) imply $A\subset F$ and $A\in\sigma_s(F)$. This contradiction proves $y\in A$.

Our next aim is to verify $\varphi(A)\in\sigma_s(F)-\sigma_s(\tau_{x,y}(F))$. (2.15) and (2.16) imply $\varphi(A)\subset F$, hence $\varphi(A)\in\sigma_s(F)$ is obvious. Suppose now that $\varphi(A)\in\sigma_s(\tau_{x,y}(F))$, that is, $x\in\varphi(A)\subset G\in\tau_{x,y}(F)$. Then $G\in F$ and either $x,y\in G$ or $G\cup\{y\}-\{x\}\in F$ hold. We have $A\subset G$ or $A\subset G\cup\{y\}-\{x\}$, resp. The conclusion is $A\in\sigma_s(F)$ in both cases. This contradiction shows $\varphi(A)\notin\sigma_s(\tau_{x,y}(F))$.

φ is obviously an injection, so

$$|\sigma_s(\tau_{x,y}(F))-\sigma_s(F)| \le |\sigma_s(F)-\sigma_s(\tau_{x,y}(F))|$$

holds. (2.14) easily follows. □

<u>Remark 2.6</u>. <u>Suppose that</u> $\tau_{x,y}(F) = \binom{A}{k}$ for some $A \subset X$, $x \notin A$, $y \in A$ <u>and</u> $F \neq \binom{A}{k}$, $F \neq \binom{A \cup \{x\} - \{y\}}{k}$. <u>Then strict inequality stands in</u> (2.14).

To prove this statement consider the families $F(x)$ (= the subfamily of the members of F containing x) and $F(y)$. It is easy to see that $F = F(x) + F(y) + \binom{A-\{y\}}{k}$ and that exactly one of the sets $G \cup \{x\}$ and $G \cup \{y\}$ is in F for any $(k-1)$-element subset G of $A-\{y\}$.

Take two sets F_1, F_2 satisfying $F_1 \in F(x)$, $F_2 \in F(y)$ and maximizing

$$|F_1 \cap F_2| = r \quad (\leq k-2) .$$

If $r < k-2$, one can find elements u and v such that $u \in F_1 - F_2 - \{x\}$ and $v \in F_2 - F_1 - \{y\}$ hold, resp. The set $F_1 \cup \{v\} - \{u,x\}$ has $k-1$ elements, therefore either $F_1 \cup \{v\} - \{u\}$ or $F_1 \cup \{y,v\} - \{u,x\}$ is in F. In the first case $(F_1 \cup \{v\} - \{u\}) \cap F_2$, in the second one $(F_1 \cup \{y,v\} - \{u,x\}) \cap F_1$ has a size $>r$. This contradiction proves $r = k-2$.

Take a $(k-s-1)$-element subset H_o of $F_1 \cap F_2$. $H_o \cup \{x\}$ and $H_o \cup \{y\}$ are both in $\sigma_s(F)$. Any $(k-s)$-element subset of $A-\{y\}$ is in $\sigma_s(F)$ while at least one of $H \cup \{x\}$ is also in $\sigma_s(F)$. Hence $|\sigma_s(F)| \geq \binom{|A-\{y\}|}{k-s} + \binom{|A-\{y\}|}{k-s-1} + 1 =$
$= \binom{|A|}{k-s} + 1$ follows. $\sigma_s(\tau_{x,y}(F)) = \binom{A}{k-s}$ proves the statement.

□

Our aim is to minimize $|\sigma(F)|$ given k and the size $|F|$ of the family $F \subset \binom{X}{k}$. Here X is the underlying set.

Its size could have a role in the solution, but it does not have. (Except the trivial condition $|\mathcal{F}| \le \binom{|X|}{k}$.) Lemma 2.5 (with s=1) says that $\mathcal{F}$ be shifted without increasing $|\sigma(\mathcal{F})|$. One can feel that this is true in a stronger sense than just this shifting transformation $\tau_{x,y}$. Namely, if $|\mathcal{F}|$ has luckily the form $|\mathcal{F}| = \binom{a}{k}$ for some integer $a \ge k$ then $|\sigma(\mathcal{F})|$ is minimized for the family $\binom{A}{k}$ where $A \subset X$, $|A|=a$. The minimum would be then $\binom{a}{k-1}$. We will prove this statement here. The other values of $|\mathcal{F}|$ need more computations with binomial coefficients. We will do it in Section . We do not want to frighten the reader in such an early stage.

Theorem 2.7 (Special case of Kruskal (1963), Katona (1968)). If $\mathcal{F} \subset \binom{X}{k}$, $|\mathcal{F}| = \binom{a}{k}$ where $1 \le k \le a \le |X| = n$ are integers then

$$|\sigma(\mathcal{F})| \ge \binom{a}{k-1} \tag{2.17}$$

with equality if and only if $\mathcal{F} = \binom{A}{k}$ holds for some a-element subset A of X .

Proof (Frankl (1983), Transformation method, Shifting method). The family $\binom{A}{k}$ shows that (2.17) is the best possible estimate.

To prove (2.17) and the statement about equality we use induction on n . The statement is trivial for $n=1$. Suppose that it is true for $n-1$ and prove it for n . (2.17) is trivial if $k=a$, so we may suppose that $k<a$.

Let $X = \{1,2,\dots,n\}$ be the underlying set. The shift $\tau_{x,y}$ $(y<x)$ does not change the size of the family $\mathcal{F}$ (Lemma 2.3) on the other hand it does not increase the size of its

shadow (Lemma 2.5). Therefore it suffices to prove the theorem for a shifted family $\mathcal{F}$.

Introduce the notations

$\mathcal{A}(z) = \{A: z\in A\in\mathcal{A}\}$, $\mathcal{A}(\bar{z}) = \{A: z\notin A\in\mathcal{A}\}$,

$\mathcal{A}+z = \{A \cup \{z\}: A \in \mathcal{A}\}$

$\mathcal{A}-z = \{A-\{z\}: A\in\mathcal{A}\}$, where $z\in X$, $\mathcal{A}\subset 2^X$.

Let $\mathcal{F}_1 = \mathcal{F}(1)-1$ and $\mathcal{F}_2 = \mathcal{F}(\bar{1})$. Here

(2.18) $$|\mathcal{F}_1|+|\mathcal{F}_2| = |\mathcal{F}|$$

is obvious. If $A \in \sigma(\mathcal{F}_2)$ then there is an F satisfying $1 \notin F \in \mathcal{F}$ and $A\subset F$. Denoting $F-A$ by $\{y\}$ $(y\neq 1)$, $\tau_{y,1}(\mathcal{F}) = \mathcal{F}$ implies $F \cup \{1\}-\{y\} \in \mathcal{F}$. Hence $F-\{y\} = A \in \mathcal{F}_1$. We proved

$$\sigma(\mathcal{F}_2) \subset \mathcal{F}_1$$

what is the advantage of the shifted $\mathcal{F}$. Actually, we need only

(2.19) $$|\sigma(\ _2)| \le |\mathcal{F}_1| .$$

Suppose that $|\mathcal{F}_1| < \binom{a-1}{k-1}$. Then (2.18) implies $|\mathcal{F}_2| > \binom{a}{k}-\binom{a-1}{k-1} = \binom{a-1}{k}$. Take any subfamily $\mathcal{F}_3$ of $\mathcal{F}_2$ with $|\mathcal{F}_3| = \binom{a-1}{k}$. Its underlying set is smaller than n , we can use the induction hypothesis: $|\sigma(\mathcal{F}_2)| \ge |\sigma(\mathcal{F}_3)| \ge \binom{a-1}{k-1}$. By (2.19) we obtain

(2.20) $$|\mathcal{F}_1| \ge \binom{a-1}{k-1}$$

to be true in all cases.

Observe that $\mathcal{F}_1 \subset \sigma(\mathcal{F})$, $\sigma(\mathcal{F}_1)+1 \subset \sigma(\mathcal{F})$, $\mathcal{F}_1 \cap (\sigma(\mathcal{F}_1)+1 = \Phi$, imply our second crucial inequality

(2.21) $$|\sigma(\mathcal{F}_1)| + |\mathcal{F}_1| \le |\sigma(\mathcal{F})| .$$

Using (2.20), there is an $F_4 \subset F_1$ with $|F_4| = \binom{a-1}{k-1}$. We can use the induction hypothesis:

$$|\sigma(F_1)| \geq |\sigma(F_4)| \geq \binom{a-1}{k-2} . \tag{2.22}$$

The sum of (2.22) and (2.20) is

$$\binom{a}{k-1} = \binom{a-1}{k-2} + \binom{a-1}{k-1} \leq |\sigma(F_1)| + |F_1| .$$

(2.21) completes the proof of (2.17).

Equality in (2.17) implies equality in (2.22). Hence, by the induction hypothesis $F_4 = F_1$ consists of all $(k-1)$-element subsets of an $(a-1)$-element subset B of $X-\{1\}$. (2.18) implies $|F_2| = \binom{a-1}{k}$. This combined with $\sigma(F_2) \subset F_1$ gives rise to $F_2 = \binom{B}{k}$. Hence $F = (F_1+1) \cup F_2 = \binom{B\cup\{1\}}{k}$ follows. This proves that the shadow of the shifted family has exactly $\binom{a}{k-1}$ member only when $F = \binom{\{1,\dots,a\}}{k}$. Remark 2.6 proves that $\sigma(F) = \binom{a}{k-1}$ implies $F = \binom{A}{k}$, $|A|=a$ for any non-shifted family. □

The above theorem (in its general form) has a central role in the theory. The original proofs of it were very long. Now there are several shorter ones. In Section   we will say more about the history of the theorem and its different proofs.

Theorem 2.7 gives a much better inequality than Lemma 1.6 does. The new bound e.g. does not depend on n at all. So if this estimate is applied in the second proof of the Sperner theorem then sharper results can be obtained. Of course, the upper bound cannot be improved, but we can say something about the possible sizes of the members in an inclusion-free family (see Section   ).

On the other hand, the Erdős-Ko-Rado theorem is also an easy consequence of Theorem 2.7 (see Exercise 2.7).

The next theorem could be called the "intersecting Kruskal-Katona", because it tries to minimize the shadow for ℓ -intersecting families. Actually, it gives only an estimate. (For improvements see Section .) But it is sharp in the sense that it determines the minimum of the ratio $|\sigma(F)|/|F|$. Actually, for further use we need this result for s -shadows:

Theorem 2.8 (Katona (1964)). Let k, ℓ, s be positive integers such that $s \le \ell \le k \le n$. Suppose that $\emptyset \neq F \subset \binom{X}{k}$ is ℓ -intersecting. Then

$$|F| \frac{\binom{2k-\ell}{k-s}}{\binom{2k-\ell}{k}} \le |\sigma_s(F)| \tag{2.23}$$

holds, with equality only for $F = \binom{X'}{k}$ where $X' \subset X$, $|X'| = 2k-\ell$.

Proof (Katona (1964), modified in Frankl (1983), Transformation method, Shifting method). We distinguish two cases: 1. $n \le 2k-\ell$, 2. $n > 2k-\ell$. The proof of the first case uses a simple counting argument, while the basic line of the proof of the second case copies the second proof of the Erdős-Ko-Rado theorem.

1. $n \le 2k-\ell$. The number of pairs (F,G) where $F \in F$, $G \subset F$, $|G|=k-s$ is $|F|\binom{k}{k-s}$. On the other hand, any $G \in \sigma_s(F)$ is a subset of at most $\binom{n-k+s}{s}$ sets $F \in F$. Hence the inequality $|F|\binom{k}{k-s} \le |\sigma_s(F)|\binom{n-k+s}{s}$ follows. That is, we obtained the lower estimate $\binom{k}{k-s} / \binom{n-k+s}{s}$ for $|\sigma_s(F)|/|F|$. To prove (2.23) for this case we need the inequality

$$\frac{\binom{2k-\ell}{k-s}}{\binom{2k-\ell}{k}} \le \frac{\binom{k}{k-s}}{\binom{n-k+s}{s}} .$$

Carrying out the possible cancellations, this is equivalent to $\binom{k-\ell+s}{s} \ge \binom{n-k+s}{s}$ which is true when $2k-\ell \ge n$, with equality iff $2k-\ell = n$. The proof of (2.23) is complete for this case.

2. $n > 2k-\ell$. We use induction on k. The case $k=1$ is trivial. Suppose that $k>1$ and that (2.23) holds for smaller values of k. A shift $\tau_{x,y}$ does not change the size of F and it remains ℓ-intersecting (Lemma 2.3). On the other hand it does not increase the size of the shadow (Lemma 2.5). Therefore we may suppose that F is shifted. Lemma 2.4 implies that any two members of F intersect each other in at least ℓ elements of $T = \{1,\dots, 2k-\ell\}$.

Let F_A denote the family $F_A = \{F: F\in F, F-T=A\}$ for any $A \subset \{1,\dots,n\}-T$. Introduce the notations $A-B = \{A-B: A\in A\}$ and $A+B = \{A\cup B: A\in A\}$ for any $B\subset X$, $A\subset 2^X$. The family F_A-A is an ℓ-intersecting family of $(k-|A|)$-element subsets of T, by Lemma 2.4. If $A\neq 0$ the induction hypothesis can be applied:

$$(2.24) \qquad |\sigma_s(F_A-A)| \ge \frac{\binom{2(k-|A|)-\ell}{k-|A|-s}}{\binom{2(k-|A|)-\ell}{k-|A|}} |F_A-A| .$$

If $A=\emptyset$, then (2.24) follows by case 1. We prove that the coefficients in (2.24) are $\ge$ than the coefficient in (2.23).

$$(2.25) \qquad \frac{\binom{2(r-1)-\ell}{r-1-s}}{\binom{2(r-1)-\ell}{r-1}} \geq \frac{\binom{2r-\ell}{r-s}}{\binom{2r-\ell}{r}} \qquad (r>\ell\geq s\geq 1)$$

reduces to $(r-s)(r-\ell+s) \geq r(r-\ell)$ after carrying out the possible cancellations and this is equivalent to $s(\ell-s) \geq 0$ which is true. This proves (2.25), with equality only for $\ell=s$. By repeated application of (2.25) we obtain

$$(2.26) \qquad \frac{\binom{2(k-|A|)-\ell}{k-|A|-s}}{\binom{2(k-|A|)-\ell}{k-|A|}} \geq \frac{\binom{2k-\ell}{k-s}}{\binom{2k-\ell}{k}} \qquad (k-|A| \geq \ell \geq s)$$

with equality only for $\ell=s$. (2.24) and (2.26) imply

$$(2.27) \qquad |\sigma_s(F_A-A)| \geq \frac{\binom{2k-\ell}{k-s}}{\binom{2k-\ell}{k}} |F_A-A| .$$

Let us observe that

$$\sigma_s(F) \supset \sum_{A\subset\{1,\ldots,n\}-T} (\sigma_s(F_A-A)+A) .$$

Hence we have the desired inequality

$$|\sigma_s(F)| \geq \sum_{A\ \{1,\ldots,n\}-T} |\sigma_s(F_A-A)+A| =$$

$$= \sum_{A\subset\{1,\ldots,n\}-T} |\sigma_s(F_A-A)| \geq \frac{\binom{2k-\ell}{k-s}}{\binom{2k-\ell}{k}} \sum_{A\subset\{1,\ldots,n\}-T} |F_A-A| =$$

$$= \frac{\binom{2k-\ell}{k-s}}{\binom{2k-\ell}{k}} \sum_{A\subset\{1,\ldots,n\}-T} |F_A| = \frac{\binom{2k-\ell}{k-s}}{\binom{2k-\ell}{k}} |F|$$

by (2.27) and $F = \sum_{A\subset\{1,\ldots,n\}-T} F_A$.

We have to check the cases of equality, only. In case 1 a necessary condition of the equality in (2.23) is $n = 2k-\ell$ as it was remarked above. On the other hand any G $(|G|=k-s)$ must be a subset of exactly $\binom{n-k+s}{s}$ sets $F \in \mathcal{F}$, that is, $\mathcal{F} = \binom{T}{k}$ where $|T| = 2k-\ell$.

In case 2 we consider first a shifted $\mathcal{F}$. Then $\mathcal{F}$ contains $\{1,\dots,k\}$ as a member. Hence $\mathcal{F}_\emptyset \neq \emptyset$ follows. The equality in (2.23) implies equality in (2.27) with $A=\emptyset$. This belongs to case 1: $\mathcal{F}_\emptyset = \binom{T}{k}$. Let $F \in \mathcal{F}$ satisfy the condition $|F \cap T| \le k-1$. Then there is an $F' \in \binom{T}{k}$ satisfying $|F \cap F'| < \ell$. This contradiction proves $\mathcal{F} = \mathcal{F}_\emptyset = \binom{T}{k}$. Remark 2.6 shows that we may have equality in (2.23) for a non-shifted $\mathcal{F}$ only when $\mathcal{F} = \binom{B}{k}$ for some $|B| = 2k-\ell$. □

The original proof used only the idea that the members of a shifted family meet in $\{1,\dots, n-1\}$ in at least ℓ elements. Then the calculation is somewhat easier but double induction is needed.

Observe that the conditions $\ell \le k \le n$ and $s \le k$ are quite natural in the above theorem but the problem has a meaning also in the case $\ell < s$. However, in this case (2.23) does not hold, moreover $|\sigma_s(\mathcal{F})|/|\mathcal{F}|$ is not bounded from zero (see Exercise 2.8).

2.3. ℓ-intersecting families

The generalization of Theorem 1.1 for ℓ-intersecting families is non-trivial:

<u>Theorem 2.9</u> (Katona (1964)).

$$\max |\mathcal{F}| = \begin{cases} \displaystyle\sum_{i=\frac{n+\ell}{2}}^{n} \binom{n}{i} & \text{if } n+\ell \text{ is even,} \\ \displaystyle\sum_{i=\frac{n+\ell+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+\ell-1}{2}} & \text{if } n+\ell \text{ is odd} \end{cases}$$

for ℓ-intersecting families $F \subset \binom{X}{k}$ $(2 \le \ell \le n)$. There is only one extremal family: $F = \sum_{i=\frac{n+\ell}{2}}^{n} \binom{X}{i}$ if $n+\ell$ is even and

$F = \sum_{i=\frac{n+\ell+1}{2}}^{n} \binom{X}{i} + \binom{X-\{x\}}{\frac{n+\ell-1}{2}}$ if $n+\ell$ is odd, where $x \in X$ is a fixed element.

Proof (Method of complementing families).

Outline of the proof. Observing that the complement of an $(n-i+\ell-1)$-element member of F cannot be a subset of an i-element set in F and using Theorem 2.8 we infer that the total number of $(n-i+\ell-1)$-element and i-element members of F is bounded by $\binom{n}{n-i+\ell-1}$. Then we just have to add up the obtained inequalities.

Formal proof. Let $\ell \le i < \frac{n+\ell-1}{2}$. The members of F^i and $F^{n-i+\ell-1}$ meet in at least ℓ elements. Therefore, if $A \subset B \in F^i$ $|A| = i-(\ell-1)$, $C \in F^{n-i+\ell-1}$ then $A \cap C \neq \emptyset$. Hence $\bar{A} \neq C$, that is, $\sigma_{\ell-1}(F^i)^c \cap F^{n-i+\ell-1} = \emptyset$ follows. Consequently:

$$(2.27) \qquad |\sigma_{\ell-1}(F^i)| + |F^{n-i+\ell-1}| \le \binom{n}{n-i+\ell-1}.$$

Apply Theorem 2.8 to F^i with $s = \ell-1$. The coefficient in (2.23) is $\binom{2i-\ell}{i-\ell+1} / \binom{2i-\ell}{i} = \binom{2i-\ell}{i-1} / \binom{2i-\ell}{i}$. This is greater than 1 for $\ell \ge 2$ since $i-1 \ge \frac{2i-\ell}{2}$. Substituting this result into (2.27),

$$(2.28) \qquad |F^i| + |F^{n-i+\ell-1}| \le \binom{n}{n-i+\ell-1}$$

is obtained where equality holds only if $F^i = \emptyset$.

The case $i = \frac{n+\ell-1}{2}$ is somewhat different. (2.28) can be obtained in the same way, however Theorem 2.8 should be applied in a sharper form. The coefficient in (2.23) is $\binom{n-1}{\frac{n+\ell-3}{2}} \Big/ \binom{n-1}{\frac{n+\ell-1}{2}} = \frac{n+\ell-1}{n-\ell+1}$. Taking into account that $F^i = F^{n-i+\ell-1}$ for this i, (2.27) results in

$$\left|F^{\frac{n+\ell-1}{2}}\right| \left(\frac{n+\ell-1}{n-\ell+1} + 1\right) \le \binom{n}{\frac{n+\ell-1}{2}}$$

and this is equivalent to

$$\left|F^{\frac{n+\ell-1}{2}}\right| \le \binom{n-1}{\frac{n+\ell-1}{2}} \tag{2.29}$$

where only the family of all $\frac{n+\ell-1}{2}$ -element sets of an $(n-1)$ - element subset gives equality. $F^i = \emptyset$ is obvious for $i<\ell$ by the ℓ -intersecting property. $|F^n| \le \binom{n}{n}$ is trivial. Hence the statement of the theorem follows by (2.28) and (2.29):

$$|F| = \sum_{i=0}^{n} |F^i| = \sum_{i=\ell}^{n-1} |F^i| + \binom{n}{n} =$$

$$= \begin{cases} \displaystyle\sum_{i=\ell}^{\frac{n+\ell-2}{2}} \left(|F^i| + |F^{n-i+\ell-1}|\right) + \binom{n}{n} \le \sum_{i=\frac{n+\ell}{2}}^{n} \binom{n}{i} & \text{if } n+\ell \text{ is even} \\ \displaystyle\sum_{i=\ell}^{\frac{n+\ell-3}{2}} \left(|F^i| + |F^{n-i+\ell-1}| + \left|F^{\frac{n+\ell-1}{2}}\right| + \binom{n}{n}\right) \le \sum_{i=\frac{n+\ell+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+\ell-1}{2}} & \text{if } n+\ell \text{ is odd.} \end{cases}$$

Equality is possible only when (2.28) and (2.29) are equalities, that is, $F^i = \emptyset$ $\left(\ell \le i \le \frac{n+\ell-2}{2}\right)$ and $F^{\frac{n+\ell-1}{2}} = \binom{X-\{x\}}{\frac{n+\ell-1}{2}}$ for some $x \in X$. □

2.4. Intersecting families without common points

$x \in X$ is called a <u>common point</u> of $\mathcal{F}$ is $F \in \mathcal{F}$ implies $x \in F$. We showed after Theorem 1 that there are many extremal intersecting families. Only one of them has a common point. The situation is very different for equally sized subsets. If we do not allow the existence of a common point then the maximum size of an intersecting family is considerably reduced:

<u>Theorem 2.10</u> (Hilton and Milner ( )).

$$\max|\mathcal{F}| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$$

<u>for intersecting families</u> $\mathcal{F} \subset \binom{X}{k}$, $2 \le k \le n/2$ <u>satisfying</u>

$$\bigcap_{F \in \mathcal{F}} F = \emptyset . \tag{2.30}$$

<u>Proof</u>. (Frankl and Füredi (1983), Transformation method, Shifting method). Let us first give the construction of the extremal family. Fix $x \in X$ and K satisfying $|K|=k$, $x \notin K$. The family $\mathcal{F}$ consisting of K along with all k-element sets containing x and intersecting K is obviously intersecting. A common point y should be in K, however (by $k \ge 2$ and $n \ge 2k$) $\mathcal{F}$ has a member containing x and meeting K in an element different from y. Thus $\mathcal{F}$ has no common point and $|\mathcal{F}|$ is obviously equal to the value given in the theorem.

To prove that an intersecting family $\mathcal{F}$ satisfying (2.30) can have at most

$$|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \tag{2.31}$$

members we use induction on k . A family of intersecting 2-element subsets contains either a common point or forms a triangle. This proves (2.31) for $k=2$. Suppose that $k>2$ and that (2.31) is proved for smaller values. Moreover let us suppose that F is maximal under the above conditions. Set $X=\{1,\dots,n\}$ and apply the transformations $\tau_{x,y}$ $(x>y)$ for F . It cannot be applied without limitations, because it may cause the appearance of a common point. We distinguish two cases:

Case 1. The repeated applications of $\tau_{x,y}$ never produce a common point. After some steps we arrive at a shifted intersecting family $H\subset\binom{X}{k}$ satisfying $|H|=|F|$, by Lemma 2.3. In addition, H has no common point.

Case 2. After repeated applications of $\tau_{x,y}$ we arrive at a family G without a common point but $\tau_{x_1,y_1}(G)$ $(x_1>y_1)$ has a common point (y_1) . Permute the elements of X moving x_1 and y_1 to 2 and 1 , resp. The family obtained from G is denoted by G' . It is obvious $G\cap\{1,2\}\neq 0$ for all $G\in G'$. Therefore any k-element subset of X containing $\{1,2\}$ can be added to G' . Therefore, by the maximality of $|F|=|G'|$, all these sets of in G' :

$$G\in G' \quad \text{for any} \quad \{1,2\}\subset G\subset X\ ,\ |G|=k\ . \tag{2.32}$$

Apply the transformations $\tau_{x,y}$ $(3\le y<x)$ for G' as long as it makes any changes. We cannot obtain a common point because the sets listed in (2.32) are unchanged during these transformations and any point $y\ge3$ is missing from at least one of them. The final family H clearly possesses the following properties:

$\mathcal{H} \subset \binom{X}{k}$, $|\mathcal{H}|=|\mathcal{F}|$, $\mathcal{H}$ is intersecting (Lemma 2.3), $\mathcal{H}$ has no common point,

(2.33) $$H \cap \{1,2\} \neq 0 \quad \text{for all} \quad H \in \mathcal{H}$$

and

(2.34) $$\tau_{x,y}(\mathcal{H}) = \mathcal{H} \quad \text{for all} \quad 3 \le y < x \ .$$

In Case 1 we may simply apply Lemma 2.4 ($t=2$, $\ell=1$) to obtain

(2.35) $$H_1 \cap H_2 \cap \{1,\dots,2k\} \neq \emptyset \text{ for all } H_1,H_2 \in \mathcal{H} \ .$$

However, in Case 2 we have to repeat the argument of Lemma 2.4 because $\mathcal{H}$ is not shifted. Suppose that (2.35) is not true. Choose such a pair H_1 , H_2 violating (2.35) and minimizing $|H_1 \cap H_2|$. (2.33) implies that H_1 and H_2 intersect $\{1,2\}$ in distinct elements. Thus $H_1 - \{1,2\}$ and $H_2 - \{1,2\}$ are $(k-1)$ -element sets. They are non-disjoint, therefore there exist $x \in H_1 \cap H_2 \cap \{2k+1,\dots,n\}$ and $y \in \{3,\dots,2k\}-H_1-H_2$. Then (2.34) implies $H_3 = H_1 \cup \{y\}-\{x\} \in \mathcal{H}$. However, here $H_2 \cap H_3 \cap \{1,\dots,2k\} = \emptyset$ and $|H_2 \cap H_3| < |H_1 \cap H_2|$ hold, contradicting the minimality of $|H_1 \cap H_2|$. This contradiction proves (2.35) for both cases.

Let $\mathcal{A}$ be defined by $\mathcal{A} = \{H \cap \{1,\dots,2k\} : H \in \mathcal{H}\}$. Then $\mathcal{A}^i$ denotes the family of the i -element intersections of the members of $\mathcal{H}$ with $\{1,\dots,2k\}$. We prove now the inequality

(2.36) $$|\mathcal{A}^i| \le \binom{2k-1}{i-1} - \binom{k-1}{i-1} \quad (2 \le i \le k-1) \ .$$

Assume $2 \le i \le k-1$ and suppose that (2.36) fails: $|\mathcal{A}^i| > \binom{2k-1}{i-1} - \binom{k-1}{i-1} \ge \binom{2k-1}{i-1} - \binom{2k-i-1}{i-1} + 1$. $\mathcal{A}^i$ is intersecting by

(2.35). Thus the induction hypothesis yields that $\mathcal{A}^i$ has a common point, say $x \in \bigcap_{A \in \mathcal{A}^i} A$. As x is not a common point of $\mathcal{H}$, we may choose an $H \in \mathcal{H}$ with $x \notin H$. The total number of i-element subsets of $\{1,\ldots,2k\}$ containing x is $\binom{2k-1}{i-1}$. The members of $\mathcal{A}^i$ intersect H by (2.35). The number of i-element subsets of $\{1,\ldots,2k\}$ containing x but disjoint to H is $\binom{|\{1,\ldots,2k\}-H|-1}{i-1} \geq \binom{k-1}{i-1}$. This proves (2.36), contradicting our assumption.

(2.36) also holds for $i=1$. Indeed, if $|\mathcal{A}^1| > 0$, then $H_1 \cap \{1,\ldots,2k\}$ contains exactly one element x for some $H_1 \in \mathcal{H}$. (2.35) implies that all $H \in \mathcal{H}$ contain x. This contradiction proves $|\mathcal{A}^i| = 0$.

Finally

$$(2.37) \qquad |\mathcal{A}^k| \leq \binom{2k-1}{k-1} = \frac{1}{2}\binom{2k}{k}$$

is a consequence of the fact that A and $\{1,\ldots,2k\}-A$ cannot be simultaneously in $\mathcal{A}^k \subset \mathcal{H}$ as $\mathcal{H}$ is intersecting.

We count now the number of members of $\mathcal{H}$ using (2.36) and (2.37). For a fixed $A \in \mathcal{A}^i$ there are at most $\binom{n-2k}{k-i}$ "continuations", that is, members $H \in \mathcal{H}$ satisfying $H \cap \{1,\ldots,2k\} = A$. We infer

$$|\mathcal{H}| \leq \sum_{i=1}^{k} |\mathcal{A}^i| \binom{n-2k}{k-i} \leq \sum_{i=1}^{k-1} \left(\binom{2k-1}{i-1} - \binom{k-1}{i-1}\right)\binom{n-2k}{k-i} + \binom{2k-1}{k-i} =$$

$$= \sum_{i=1}^{k} \left(\binom{2k-1}{i-1} - \binom{k-1}{i-1}\right)\binom{n-2k}{k-i} + 1 =$$

$$= \sum_{i=1}^{k} \binom{2k-1}{i-1}\binom{n-2k}{k-i} - \sum_{i=1}^{k} \binom{k-1}{i-1}\binom{n-2k}{k-i} + 1 = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 .$$

□

Let us make a final remark comparing the order of magnitudes of the bounds of the Erdős-Ko-Rado and the Hilton-Milner theorems, resp. Let $f(n)$ and $g(n)$ be positive real-valued functions defined on the set of positive integers. The notation $g(n) = \mathcal{O}(f(n))$ denotes the fact that for some constant C $g(n) \le Cf(n)$ holds for all n. Similarly, $g(n) = o(f(n))$ means that $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

It is easy to see that

$$\binom{n-1}{k-1} = \frac{(n-1)\dots(n-k+1)}{(k-1)!} = \frac{n^{k-1}}{(k-1)!} + \mathcal{O}(n^{k-2})$$

for any fixed k. On the other hand $\frac{n^{k-1}}{(k-1)!}$ cancels from $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Therefore it is $\mathcal{O}(n^{k-2})$. In conclusion, the order of magnitude of the maximum intersecting family of equally sized subsets becomes definitely smaller if the family cannot have a common point.

2.5. Applications

A) Search with qualitatively independent sets. We consider the same modul of search as in Section 1.3 D). Let F and G be two members of the family $\mathcal{F}$ of subsets of possible questions. If $F \subset G$ and we know the answer $x \in F$ then it also determines the answer $x \in G$. Similarly, if $\bar{F} \subset G$ then the answer $x \notin F$ determines $x \in G$. The situation is analogous when $F \subset \bar{G}$ or $\bar{F} \subset \bar{G}$. If these cases are excluded then knowing the answer for the question "$x \in F$ or $x \notin F$" the same question for G is not answered in advance. We call a family $\mathcal{F}$ qualitatively independent if

(2.38) $$F \not\subset G \ , \ \bar{F} \not\subset G \ , \ \bar{F} \not\subset \bar{G} \ , \ F \not\subset \bar{G}$$

hold for any distinct members F, G of $\mathcal{F}$. This can be formulated so that F and G divide X into 4 non-empty parts. The following theorem answers a question of Rényi:

<u>Theorem 2.11</u> (Katona (1973), Bollobás JCTA 15 (1973), 363-366, Daykin ( )).

$$\max|\mathcal{F}| = \binom{n-1}{\lceil \frac{n}{2} \rceil}$$

<u>for qualitatively independent families</u>.

<u>Proof</u> (Katona (1973), Transformation method, Compressing method). The family $\binom{X-\{x\}}{\lceil \frac{n}{2} \rceil}$ is qualitatively independent for any $x \in X$. Thus we have to prove

(2.39) $$|\mathcal{F}| \le \binom{n-1}{\lceil \frac{n}{2} \rceil}$$

only.

The family $\mathcal{F}+\mathcal{F}^c$ is an inclusion-free family by (2.38). Hence $|\mathcal{F}+\mathcal{F}^c| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$ follows by the Sperner theorem. That is, $|\mathcal{F}| = |\mathcal{F}^c| \le \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor}$. If n is even, this yields the desired inequality (2.39). The case of odd n's remains only.

$F \in \mathcal{F}$ can be substituted by $\bar{F}$ without violating (2.38). Therefore we may suppose that $|F| \ge \frac{n+1}{2}$ holds for all $F \in \mathcal{F}$. Apply now the idea of the third proof of the Sperner theorem. Let $k > \frac{n+1}{2}$ be the maximum size in $\mathcal{F}$. The family $\mathcal{F}' = (\mathcal{F}-\mathcal{F}^k) \cup \sigma(\mathcal{F}^k)$ is inclusion-free and satisfies $|\mathcal{F}'| > |\mathcal{F}|$

by the arguments described there. $\bar{F} \not\subset G$ for $F, G \in \mathcal{F}'$ follows also easily from the same property of $\mathcal{F}$. $F \not\subset \bar{G}$ is a consequence of $|F|, |G| \geq \frac{n+1}{2}$. $\mathcal{F}'$ satisfies (2.38) and has more members than $\mathcal{F}$ has. This proves that a maximally sized $\mathcal{F}$ must consist of $\frac{n+1}{2}$-element subsets and $\mathcal{F}^c \subset \binom{X}{\frac{n-1}{2}}$. By (2.38) $\mathcal{F}^c$ is intersecting, Theorem 2.1 can be applied to obtain

$$|\mathcal{F}| = |\mathcal{F}^c| \leq \binom{n-1}{\frac{n-3}{2}} = \binom{n-1}{\lceil \frac{n}{2} \rceil} . \qquad \square$$

The proof using the permutation method (see Exercise 2.9) is probably easier, but the present proof shows more directly the connection to the Erdős-Ko-Rado theorem. The reader can find further results along this line in Kleitman-Spencer (Discrete Math. 6 (1973), 255-262).

B) <u>Reconstruction of graphs and hypergraphs from their line-graph</u>. A family $\mathcal{H} \subset \binom{X}{k}$ is called sometimes k <u>-graph</u> (for k=2 : <u>graph</u>). The <u>line-graph</u> $L(\mathcal{H})$ of $\mathcal{H}$ is a graph with vertex-set $\mathcal{H}$ in which two vertices $H_1, H_2 \in \mathcal{H}$ are connected iff $H_1 \cap H_2 = \emptyset$. An old question (see e.g. Lovász (book)) is the following one: Under which conditions does $L(\mathcal{H})$ determine $\mathcal{H}$ uniquely or, equivalently, when can $\mathcal{H}$ be reconstructed from $L(\mathcal{H})$? The next theorem deals with this problem. Before formulating it we need some definitions. We say that $\mathcal{H} \subset \binom{X}{k}$ and $\mathcal{H}' \subset \binom{X'}{k}$ $(|X| = |X'|)$ are <u>isomorphic</u> iff there is a bijection $\varphi: X \to X'$ such that $H \subset \mathcal{H}$ iff $\varphi(H) = \{\varphi(X): x \in H\} \in \mathcal{H}'$. $\mathcal{H}(x)$ is the subfamily of the members containing the fixed element x, $\deg_{\mathcal{H}}(x) = |\mathcal{H}(x)|$ and

$$\deg(\mathcal{H}) = \min_{x \in X} \deg_{\mathcal{H}}(x) .$$

<u>Theorem 2.12</u> (Erdős Pé. - Füredi (1980)). <u>Suppose that</u> $\mathcal{H} \subset \binom{X}{k}$, $\mathcal{H}' \subset \binom{X'}{k}$, $|X| = |X'| = n$, $n \geq 2k \geq 4$ <u>and</u>

$$(2.40) \qquad \deg(\mathcal{H}) > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$$

<u>hold. Then</u> $\mathcal{H}$ <u>and</u> $\mathcal{H}'$ <u>are isomorphic iff</u> $L(\mathcal{H})$ <u>and</u> $L(\mathcal{H}')$ <u>are isomorphic</u>.

<u>Proof</u>. If $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic then $L(\mathcal{H})$ and $L(\mathcal{H}')$ are trivially isomorphic. Suppose, conversely, that $L(\mathcal{H})$ and $L(\mathcal{H}')$ are isomorphic, that is, there is a bijection $\varphi: \mathcal{H} \to \mathcal{H}'$ such that

$$\varphi(H_1) \cap \varphi(H_2) \neq \emptyset \quad \text{iff} \quad H_1 \cap H_2 \neq 0$$

holds for all $H_1, H_2 \in \mathcal{H}$.

$\mathcal{H}(x)$ is an intersecting family. Consequently $\varphi(\mathcal{H}(x)) = \{\varphi(H) : H \in \mathcal{H}(x)\}$ is also intersecting. Theorem 2.10, $|\varphi(\mathcal{H}(x))| = |\mathcal{H}(x)| = \deg_{\mathcal{H}}(x) \geq \deg(\mathcal{H})$ and (2.40) imply that $\varphi(\mathcal{H}(x))$ has a common point $\psi(x)$. It cannot have two common points because otherwise its size is $\leq \binom{n-2}{k-2} < \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$ contradicting (2.40), i.e. $\psi(x)$ is unique.

Suppose that $\psi(x_1) = \psi(x_2)$. Then $\varphi(\mathcal{H}(x_1))$ and $\varphi(\mathcal{H}(x_2))$ have the same common point. Therefore $\mathcal{H}(x_1) \cup \mathcal{H}(x_2)$ is intersecting, consequently it has a common point x_3 by Theorem 2.10 and (2.40). Like $\varphi(\mathcal{H}(x))$, $\mathcal{H}(x_1)$ has also exactly one common point, hence $x_1 = x_3$ follows. Similarly we have $x_2 = x_3$. We obtained $x_1 = x_2$. This proves that $\psi(x)$ is a bijection between X and X' .

Finally, $x \in H$ implies $\psi(x) \in \varphi(H)$. Since $|H| = |\varphi(H)| = k$ holds, $\varphi(H) = \{\psi(x) : x \in H\}$ proves that $x \in H$ iff $\psi(x) \in \varphi(H)$, i.e., $\mathcal{H}$ and $\mathcal{H}'$ are really isomorphic. □

If $k=2$, (2.40) has the form $\deg(\mathcal{H}) > 3$ and Theorem 2.12 reduces to an old theorem of Whitney (1932).

C) A problem of Erdős and Sárközy. Let n be an even number and $a_1, \ldots, a_n$ be reals such that $|a_1| \le 1$ $(1 \le i \le n)$. We form all the sums $\sum_{i=1}^{n} \varepsilon_i a_i$ where $\varepsilon_i = +1$ or -1 $(1 \le i \le n)$. Denote by $f(n)$ the minimum number of such sums satisfying $\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| \le 1$ over all possible choices of a_i .

Theorem 2.13 (Erdős and Sárközy (1978)).

$$f(n) = \binom{n}{\frac{n}{2}} \qquad (n \text{ is even}).$$

Proof. Changing the sign of an a_i it does not change the set of these sums, therefore we may suppose that $a_i \ge 0$ for all i . Introduce the notation $\sum_A = \sum_{i \in A} a_i - \sum_{i \notin A} a_i$ where $A \subset \{1, \ldots, n\}$ and $\bar{A} = \{1, \ldots, n\} - A$.

Suppose that $A, B \subset \{1, \ldots, n\}$ and $|A \cap B| \le 1$. Then

$$\sum_A + \sum_B = \sum_{i \in A} a_i - \sum_{i \in \bar{A}} a_i + \sum_{i \in B} a_i - \sum_{i \in \bar{B}} a_i =$$

$$= 2 \sum_{i \in A \cap B} a_i - 2 \sum_{i \in \overline{A \cup B}} a_i \le 2$$

proves that either $\sum_A$ or $\sum_B$ is at most 1 . This implies the sets A corresponding to the sums $\sum_A > 1$ form a 2 -intersecting family. By Theorem 2.9 we know that this family has at most $\sum_{i=\frac{n}{2}+1}^{n} \binom{n}{i}$ members. That is, this is the maximum number of sums satisfying $\sum_A > 1$.

It can be proved that $\sum_A + \sum_B \geq -2$ holds if $|\overline{A \cup B}| \leq 1$, that is, $|\bar{A} \cap \bar{B}| \leq 1$. This implies that the family of sets satisfying $\sum_A < -1$ satisfies the condition $|\bar{A} \cap \bar{B}| \geq 2$. Hence the number of these sets is, again by Theorem 2.9, at most

$$\sum_{i=\frac{n}{2}+1}^{n} \binom{n}{i} .$$

The number of sums $-1 \leq \sum_A \leq 1$ is therefore at least $2^n - 2 \sum_{i=\frac{n}{2}+1}^{n} \binom{n}{i} = \binom{n}{\frac{n}{2}}$. The setting $a_1 = \ldots = a_n = 1$ shows that this estimate is sharp. □

Theorem 2.13 has a very natural generalization. Let $a_1, \ldots, a_n$ be r-dimensional vectors of length ≤ 1 . Beck (1983, EJC) proved that the number of sums $\sum_{i=1}^{n} \varepsilon_i a_i$ $(\varepsilon_i = \pm 1)$ satisfying $\left| \sum_{i=1}^{n} \varepsilon_i a_i \right| \leq \sqrt{r}$ is at least $c(r) \frac{2^n}{\sqrt{n}}$. "Unfortunately" he used analitical methods.

D) <u>Linear combination of</u> 0, 1 <u>valued random variables</u>. Let $\xi_1, \ldots, \xi_n$ be independent random variables such that $P(\xi_i = 1) = p$, $P(\xi_i = 0) = 1-p$. Let further $\alpha_1, \ldots, \alpha_n$ be non-negative real numbers which sum to one.

<u>Theorem 2.14</u> (Liggett JCTA 23 (1977) 15-21).
$P\left(\sum_{i=1}^{n} \alpha_i \xi_i \geq \frac{1}{2} \right) \geq p$, <u>provided that</u> $p \geq \frac{1}{2}$.

Proof. Suppose first that $\sum_{i\in A} \alpha_i \neq \frac{1}{2}$ for all $A \subset X = \{1,\dots,n\}$. For $0 \le k \le n$, let N_k be the number of subsets A of X of size k for which $\sum_{i\in A} \alpha_i > \frac{1}{2}$. These sets form an intersecting family, therefore the Erdős-Ko-Rado theorem (Theorem 2.1) gives $N_k \le \binom{n-1}{k-1}$ for $k \le n/2$. We have

$$P\left(\sum_{i=1}^{n} \alpha_i \xi_i \ge \frac{1}{2}\right) = \sum_{k=0}^{n} p^k (1-p)^{n-k} N_k .$$

As $N_k + N_{n-k} = \binom{n}{k}$, the above row is equal to

$$p^{n/2}(1-p)^{n/2}\binom{n-1}{\frac{n}{2}-1} + \sum_{k<n/2} p^k (1-p)^{n-k} N_k + \sum_{k<n/2} p^{n-k}(1-p)^k\left(\binom{n}{k}-N_k\right) =$$

$= N_k + N_{n-k} = \binom{n}{k}$, the above row is equal to

$$p^{n/2}(1-p)^{n/2}\binom{n-1}{\frac{n}{2}-1} + \sum_{k<n/2} p^k (1-p)^{n-k} N_k +$$

$$+ \sum_{k<n/2} p^{n-k}(1-p)^k\left(\binom{n}{k}-N_k\right) =$$

$$= p^{n/2}(1-p)^{n/2}\binom{n-1}{\frac{n}{2}-1} + \sum_{k<n/2} p^k (1-p)^{n-k}\left(N_k - \binom{n-1}{k-1}\right) +$$

$$+ \sum_{k<n/2} p^k (1-p)^{n-k}\binom{n-1}{k-1} + \sum_{k<n/2} p^{n-k}(1-p)^k\left(\binom{n-1}{k-1}-N_k\right) +$$

$$+ \sum_{k<n/2} p^{n-k}(1-p)^k\binom{n-1}{k} =$$

$$= \sum_{k<n/2}\left(\binom{n-1}{k-1}-N_k\right)\left(p^{n-k}(1-p)^k - p^k(1-p)^{n-k}\right) +$$

$$+ \sum_{k=0}^{n} p^k (1-p)^{n-k}\binom{n-1}{k-1} ,$$

where the term with exponent $n/2$ occurs only for even n. The first sum of the above expression is non-negative by $N_k \leq \binom{n-1}{k-1}$ and $p \geq \frac{1}{2}$. The second sum is equal to p. Therefore the investigated probability is $\geq p$, indeed. □

E) <u>Proportion of triangles of a certain shape</u>. Burnashev (SIAM Review 24 (1982) 477-478 Problem 82-18) asked if the following statement is true in Hilbert spaces. For any $\delta>0$ there is an $\varepsilon(\delta)>0$ such that for any set of points $x_1, \ldots, x_N$ there is a subset $x_{i_1}, \ldots, x_{i_M}$ of size $M \geq \varepsilon(\delta)N$ inducing no triangle having two sides ≤ 1 and one side $\geq 1+\delta$. These triangles are called δ-<u>bad</u>. The answer is negative.

The following counterexample will be given in the Hilbert space of infinite sequences $(y_1, \ldots, y_n, \ldots)$ of real numbers such that $\sum_{i=1}^{\infty} y_i < \infty$ with the usual norm $\sum_{i=1}^{\infty} y_i$. Actually, only finite sequences will be used in the construction, but the dimension will be unbounded. It will be shown that one can construct $N = \binom{n}{2}$ sequences of length n not satisfying the above statement with $\delta = \sqrt{2}-1$. That is, for any given ε, $\varepsilon\binom{n}{2}$ of these sequences always induces a $(\sqrt{2}-1)$-bad triangle, providing that n is large enough.

<u>Theorem 2.15</u> (R. Ahlswede; P. Erdős; F. Chung, A. Odlyzko, L. Shepp, SIAM Review 25 (1983) 574-575). <u>There are</u> $\binom{n}{2}$ <u>sequences</u> $(y_1, \ldots, y_n)$ <u>of real numbers such that any</u> $n+1$ <u>of them induce a</u> $(\sqrt{2}-1)$-<u>bad triangle</u>.

Proof. Consider all the sequences $(y_1,\ldots,y_n)$ having two components $\frac{1}{\sqrt{2}}$ and $n-2$ components . There is a natural way to associate these sequences with the edges of the complete graph K_n on n vertices $(K_n = \binom{X}{2},\ |X|=n)$. The distance of two distinct sequences are 1 or $\sqrt{2}$ iff the corresponding edges of K_n meet or do not meet, resp. Three sequences form a $(\sqrt{2}-1)$ -bad triangle iff the corresponding 3 edges in K_n form a (non-circuit) path. So, if we choose a subset of the above sequences inducing no 1-bad triangle then the graph G formed from the corresponding sequences cannot contain a (non-circuit) path of 3 edges.

Determine the maximum number of edges of G under this condition. A connected component of G cannot contain two disjoint edges. Let n_i denote the number of vertices of the i-th component $\left(\sum_{i=1}^{n} n_i = n\right)$. If $n_i \geq 4$, then the very special case (k=2) of the Erdős-Ko-Rado theorem can be applied: the number of edges in the i-th component is at most n_i-1 . If n_i equals 1, 2 or 3 then the number of edges is obviously at most 0, 1 or 3, resp. We have in all cases at most n_i edges. The total number of edges of G is at most $n = \sum_{i=1}^{r} n_i$, proving the theorem. □

In the above proof, we could have used a theorem of Erdős and Gallai (Acta Sci. Math. Hung. 10 (1959) 337-357) stating, in a special case, that if a graph on n vertices has $n+1$ edges then it contains a path of 3 edges. Here we quoted Erdős-Ko-Rado since we list its applications. For a generalization see Exercise .

Exercises, problems

2.1. Let $\mathcal{F} \subset \binom{X}{j}$ and define

$\sigma^1(\mathcal{F}) = \{B : |B|=j+1 \text{ and } B \supset A \text{ for some } A \in \mathcal{F}\}$.

Prove $\sigma^1(\mathcal{F}) = (\sigma(\mathcal{F}^c))^c$.

2.2. Suppose $\mathcal{F} \subset \binom{X}{j}$. Prove

$$|\sigma^1(\mathcal{F})| \ge |\mathcal{F}|\frac{n-j}{j+1} .$$

2.3. Erdős-Ko-Rado (1961). Prove the following generalization of Theorem 2.1. (It was its original version.) Let $\mathcal{F} \subset \binom{X}{0}+\binom{X}{1}+\ldots+\binom{X}{k}$ $(1 \le k \le n/2)$ be an intersecting inclusion-free family. Then $|\mathcal{F}| \le \binom{n-1}{k-1}$.

2.4. Let $\mathcal{A}$ be an intersecting inclusion-free family of at most k -element $(1 \le k \le n/2)$ consecutive subsets along a cyclic permutation. Prove $|\mathcal{A}| \le \min\{|A| : A \in \mathcal{A}\}$.

2.5.* Suppose that $\mathcal{F} \subset \binom{X}{k}$ has the property that $F_1,\ldots,F_t \in \mathcal{F}$ implies $|\bigcup_{i=1}^{t} F_i| \le m$. Prove that $\tau_{x,y}(\mathcal{F})$ $(x,y \in X)$ has the same property.

2.6.* Suppose that $\mathcal{F} \subset \binom{X}{k}$ has the property that $\sum_{1\le i_1<i_2\le t} |F_{i_1} \cap F_{i_2}| \ge \ell$. Prove that $\tau_{x,y}(\mathcal{F})$ $(x,y \in X)$ has the same property.

2.7.* (Daykin, Katona). Deduce Theorem 2.1 from Theorem 2.7 or Theorem 2.8.

2.8 (Katona). Show that, using the notations of Theorem 2.8, $|\sigma_s(\mathcal{F})|/|\mathcal{F}|$ is not bounded by a positive constant from below if s , ℓ and k are fixed in the following way:

$1 \leq \ell \leq k$, $\ell < s \leq k$.

2.9.* Prove Theorem 2.11 by the permutation method.

2.10 (Liggett JCTA 23 (1977) 15-21). Let F be a family of k-element subsets of an n-element set $(k \leq n/2)$. Define $\tilde{F}$ as the family of all k-element sets disjoint to some member of F . Prove the following sharpening of Theorem 2.1:

$$\frac{|F|}{|F| \ |\tilde{F}|} \leq \frac{k}{n} .$$

2.11 (Ahlswede). Let $F \subset \binom{X}{k}$ where $3 \leq k \leq n/2$. Suppose that $F_1, F_2, F_3 \in F$, $F_1 \cap F_2 \neq \emptyset$ and $F_2 \cap F_3 \neq 0$ imply $F_1 \cap F_3 \neq \emptyset$. Prove $|F| \leq \binom{n-1}{k-1}$ what is a sharpening of Theorem 2.1.

2.12 Conjecture. Let $F \subset \binom{X}{k}$ where $3 \leq k \leq n/2$. Suppose that F contains no 4 members satisfying $F_1 \cap F_2 \neq \emptyset$, $F_2 \cap F_3 \neq \emptyset$, $F_3 \cap F_4 \neq \emptyset$, $F_1 \cap F_3 = F_1 \cap F_4 = F_2 \cap F_4 = \emptyset$. Then $|F| \leq \binom{n-1}{k-1} + 1$ for $n > n_o(k)$. (The case $k=2$ is entirely different; see Erdős-Gallai.)

Hints

2.1. Trivial consequence of the definitions.

2.2. Either use Exercise 2.1 and Lemma 1.6, or apply the method of the proof of Lemma 1.6.

2.3. Use the compressing method with Exercise 2.2 as long as the family consists of merely k-element members.

2.4. Repeat the method of the proof of Lemma 2.2 starting with a minimum sized member of $\mathcal{A}$.

2.5. Follow the proof of Lemma 2.3: By $\bigcup_{i=1}^{t} G_i \subset \bigcup_{i=1}^{t} F_i \cup \{y\}$ we may suppose that $|\bigcup_{i=1}^{t} F_i| = m$ and $y \notin \bigcup_{i=1}^{t} F_i$. Moreover, $x \in \bigcup_{i=1}^{t} G_i$ holds. On the other hand, there is an $F_j = G_j$ satisfying $x \in F_j$, $y \notin F_j$. Then $\hat{F}_j = F_j \cup \{y\} - \{x\}$ leads to a contradiction.

2.6. Let $G_1, \ldots, G_t \in \tau_{x,y}(\mathcal{F})$ and suppose that they are transformed from $F_1, \ldots, F_t \in \mathcal{F}$. Denote by I_1, I_2, I_3, I_4 and I_5 the set of indices i satisfying

$$x, y \in G_i ,$$

$$x \notin G_i , \ y \in G_i , \ F_i = G_i$$

$$x \notin G_i , \ y \in G_i , \ F_i = G_i \cup \{x\} - \{y\} ,$$

$$x \in G_i , \ y \notin G_i \quad (\Rightarrow G_i \cup \{y\} - \{x\} \in \mathcal{F})$$

$$x \notin G_i , \ y \notin G_i ,$$

resp. Define

$$\hat{F}_i = \begin{cases} G_i & \text{if } i \in I_1 \\ G_i & \text{if } i \in I_2 \\ G_i \cup \{x\}-\{y\} & \text{if } i \in I_3 \\ G_i \cup \{y\}-\{x\} & \text{for } a = \min(|I_3|,|I_4|) \text{ elements of } I_4 \\ G_i & \text{for the other elements of } I_4 \\ G_i & \text{if } i \in I_5 . \end{cases}$$

Observe the $\hat{F}_i \in \mathcal{F}$ for all i. As $G_i \cap G_j$ and $\hat{F}_i \cap \hat{F}_j$ can differ only in x and y, $\sum_{1 \le i_1 < i_2 \le t} |G_{i_1} \cap G_{i_2}| \ge \sum_{1 \le i_1 < i_2 \le t} |\hat{F}_{i_1} \cap \hat{F}_{i_2}|$ can be proved by comparing the total number of occurrences of x and y in the intersections.

2.7. Suppose that $\mathcal{F} \subset \binom{X}{k}$ $(k \le n/2)$ is intersecting but has more than $\binom{n-1}{k-1}$ members. Observe that $\mathcal{F}$ and $\sigma_{n-2k}(\mathcal{F}^c)$ are disjoint, therefore $|\mathcal{F}| + |\sigma_{n-2k}(\mathcal{F}^c)| \le \binom{n}{k}$. However, the lower estimate of $|\sigma_{n-2k}(\mathcal{F}^c)|$ obtainable by Theorem 2.8 or the iterated application of Theorem 2.7 leads to a contradiction. ($\mathcal{F}^c$ is $(n-2k+1)$-intersecting.)

2.8. Let A be an ℓ-element subset of X, $|X|=n$. Take $\mathcal{F} = \{F: A \subset F, |F|=k\}$.

2.9. Observe that condition (2.38) implies that both $\mathcal{F}$ and $\mathcal{F}^c$ are intersecting inclusion-free families. Prove that the maximum number of members of $\mathcal{F}$ consecutive along a given cyclic permutation is at most $\lfloor \frac{n}{2} \rfloor$. This follows by Exercise 2.4 if one of the sets if of size $\le \lfloor \frac{n}{2} \rfloor$. Otherwise $\mathcal{F}^c$ should be considered. Count the number of pairs $(\mathcal{C}, F)$, where $\mathcal{C}$ is a cyclic permutation, $F \in \mathcal{F}$ is consecutive along $\mathcal{C}$ and use $|F|!\,(n-|F|)! \ge \lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !$.

2.10. Prove first the analogous statement for consecutive sets along a cyclic permutation. Let $\mathcal{C}$ be such a family and $A\in\mathcal{C}$. Prove that $|\mathcal{C}|$ plus the number of k-element subsets not belonging to $\mathcal{C}\cup\tilde{\mathcal{C}}$ and "being left" from A is at most k. The same is true for "right". This implies $2|\mathcal{C}|+n-|\mathcal{C}\cup\tilde{\mathcal{C}}| \le 2k$ which leads to the desired inequality. To prove the final inequality rewrite it into the form $\frac{n-k}{k}|F| \le |\tilde{F}|$ and compare the number of pairs $(\mathcal{C},A)$ and $(\mathcal{C},\tilde{A})$ where $A\in F$, $\tilde{A}\in\tilde{F}$ are consecutive sets in $\mathcal{C}$.

2.11. Observe that the connected components contain intersecting subfamilies of F. Then, by Theorem 2.1, $|F| \le \sum \binom{|P|}{k} + \sum \binom{|S|-1}{k-1}$ where P and S are the ground sets of the components of size $\le 2k-1$ and $\ge 2k$, resp. This is maximum for one component of size n.