Sample exercises from the 1st part of the course

1. Let \( U := \{ x \in \mathbb{C}^3 \mid x_1 + ix_2 = 0 \} \) and \( W := \text{Span}\{w\} \subset \mathbb{C}^3 \) where \( w_1 = w_2 = w_3 = 1 \). Verify that \( U \) and \( W \) are complementary and calculate the matrix (in the std. basis) of the projection onto \( U \) along \( W \).

2. Let \( P \) and \( Q \) be two projections of the vector space \( V \). Prove that \( P + Q \) is again a projection iff \( \text{Im}(P) \subset \text{Ker}(Q) \) and \( \text{Ker}(P) \supset \text{Im}(Q) \).

3. Let \( x_1, x_2, \ldots, x_{n+1} \in V \) be \( n+1 \) vectors in the scalar product space \( V \) and assume that \( \| x_k \| = 1 \) and \( |\langle x_k, x_j \rangle| < 1/n \) for all \( k, j = 1, \ldots, n + 1 \). Show that in this case \( x_1, \ldots, x_{n+1} \) must be linearly independent.

4. Let \( V \) be a finite dimensional vector space, and suppose that both \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) is a scalar product on \( V \). Prove that there exists a bases \( \mathcal{B} = (b_1, \ldots, b_n) \) such that the vectors of \( \mathcal{B} \) are pairwise orthogonal with respect to both scalar products.

5. Let \( A \in \mathcal{B}(\mathcal{H}) \) be such \( A^2 = 0 \) and \( AA^* + A^*A = 1 \). Show that \( \text{Sp}(A + A^*) = \{1, -1\} \) and that the dimensions of the two eigenspaces of \( A + A^* \) are equal.

6. Show that for an operator \( U \) on a finite dimensional Hilbert space \( \mathcal{H} \) the following three properties are equivalent:
   - \( U \) is invertable and \( \|U\| = \|U^{-1}\| = 1 \)
   - \( \|Ux\| = \|x\| \) for all \( x \in \mathcal{H} \)
   - \( U \) is a unitary operator.

7. Let \( A, B \) be two self-adjoint operators. Show that \( \text{Tr}((AB)^2) \in \mathbb{R} \) and \( \text{Tr}(A^2B^2) \in \mathbb{R}^+ \cup \{0\} \) and that
   \[
   \text{Tr}((AB)^2) \leq \text{Tr}(A^2B^2)
   \]
with equality holding if and only if \( A \) and \( B \) commute.

8. Let \( A, B \) be two positive operators. Prove that \( \text{Ker}(A + B) = \text{Ker}(A) \cap \text{Ker}(B) \).

9. Show, by example, that if \( A, B \) are positive operators such that \( A \geq B \), then it does not follow that \( A^2 \geq B^2 \). Decide if on the other hand the (weaker) inequality \( \text{Tr}(A^2) \geq \text{Tr}(B^2) \) follows.