Undecidability of the Real-Algebraic Structure of Scott's Model

Miklós Erdélyi-Szabó[1]

Department of Mathematics, Cornell University
White Hall, Ithaca, NY 14853, U.S.A.[2]

Abstract. We show that true first-order arithmetic of the positive integers is interpretable over the real algebraic structure of Scott's topological model for intuitionistic analysis. From this the undecidability of the structure follows.

Mathematics Subject Classification: 03-D35, 03-F55.

Keywords: Undecidability, Intuitionism, Topological model, True first order arithmetic.

1 Introduction

Scott defined his topological model for intuitionistic analysis in [4]. In [5] he showed that the universal fragment of the order structure is decidable. Other decidable fragments of the model were described in the papers of Scowcroft ([6] and [7]) and in [2]. These results led to decidability results in the constructive and intuitionistic theory of reals (cf. [8], [9] and [3]).

In this paper let L denote the language of ordered rings. We shall show, using a coding similar to the one used by Cherlin in [1], that the L-theory of Scott's model is undecidable.

2 Scott's Model

First let us describe the model in question.

[1]I would like to thank Philip Scowcroft for showing me Cherlin's paper. I would also like to thank the referee for helpful suggestions.

[2]e-mail: mszabo@math.cornell.edu

Let T be Baire space (the space of infinite sequences of natural numbers or, equivalently, the space of irrational numbers with the usual metric). The elements of the model $\mathcal{R}$ are the continuous functions from T into the set $\mathbb{R}$ of real numbers. Then $\mathcal{R}$ is a model for the language L with addition and multiplication being the usual operations on continuous functions and $0_{\mathcal{R}}$ and $1_{\mathcal{R}}$ the constant functions with value 1 and 0 respectively.

Without loss of generality we assume that all atomic formulas are of the form $p(\vec{f}) > 0$ where p is a polynomial. We shall use $=$ only as an abreviation: $f = g \leftrightarrow \neg(g - f > 0 \vee f - g > 0)$. This relation has the usual properties of equality in the model.

The truth value $\|\varphi\|$ of a formula $\varphi \in Form_{L\cup\mathcal{R}}$ with parameters from $\mathcal{R}$ (and without free variables) is an open subset of T defined inductively as follows.

1. $\|p(\vec{f}) > 0\| = \{t \in T | \mathbb{R} \models p(f_1(t), \dots, f_n(t)) > 0\}$
   where $p(f_1, \dots, f_n)$ is a term of the language (a polynomial) with parameters among $f_1, \dots, f_n$ and with no free variables,
2. $\|\varphi_1 \vee \varphi_2\| = \|\varphi_1\| \cup \|\varphi_2\|$
3. $\|\varphi_1 \wedge \varphi_2\| = \|\varphi_1\| \cap \|\varphi_2\|$
4. $\|\varphi_1 \to \varphi_2\| = int((T \setminus \|\varphi_1\|) \cup \|\varphi_2\|)$
5. $\|\neg\varphi\| = int(T \setminus \|\varphi\|)$
6. $\|\exists w \varphi(w)\| = \cup_{f\in\mathcal{R}} \|\varphi(f)\|$
7. $\|\forall w \varphi(w)\| = int \cap_{f\in\mathcal{R}} \|\varphi(f)\|$.

A sentence ψ is true in the model (denoted by $\mathcal{R} \models \psi$) just in case $\|\psi\| = T$.

3 Interpretation of $\mathbb{N}^+$ over the Model

For $f \in \mathcal{R}$ let $z(f)$ denote its zero-set. Let $\neq$ denote the apartness relation: $x \neq y \leftrightarrow x < y \vee y < x$. Let $\mathcal{O}(x, y)$ be the L-formula

$$\neg\neg x \neq 0 \to y \neq 0$$

Lemma 1. *For all $f, g \in \mathcal{R}$*

$$z(g) \cap \|\mathcal{O}(f, g)\| \subseteq cl(int(z(f)))$$

where $cl(X)$ denotes the closure of the set X.

Proof.

$$\|\mathcal{O}(f, g)\| = int\Big(cl(int(z(f))) \cup \|g \neq 0\|\Big)$$

□

Let $\zeta(x)$ be the L-formula

$$\forall y((y \neq 0 \to x \neq 0) \vee y \neq 0)$$

Then $\zeta(g)$ states that a function $g \in \mathcal{R}$ has at most one zero locally, i.e.

Lemma 2. *For all $g \in \mathcal{R}$, if $p \in \|\zeta(g)\|$ then there is an open neighborhood U of p such that $|z(g) \cap U| \leq 1$.*

Proof. Fix a function g in $\mathcal{R}$ and an element p in $\|\zeta(g)\|$. If $p \in \|g \neq 0\|$, then $U = \|g \neq 0\|$ fulfills the requirement. So let us suppose that $g(p) = 0$. Let $h \in \mathcal{R}$ be a function whose zero set is $\{p\}$. Since $p \notin \|h \neq 0\|$ and $p \in \|\zeta(g)\|$, $p \in \|h \neq 0 \to g \neq 0\| = int(z(h) \cup \|g \neq 0\|) = int(\{p\} \cup \|g \neq 0\|)$. If U denotes this open set, then obviously $p \in U$ and $z(g) \cap U = \{p\}$. □

The converse of this result also holds:

Lemma 3. *If for some open subset U of T $|z(g) \cap U| \leq 1$, then $U \subseteq \|\zeta(g)\|$.*

Proof. Let U be an open subset of T such that $|z(g) \cap U| \leq 1$ and let $p \in U$. If $p \notin z(g)$, then $p \in \|g \neq 0\| \subseteq \|\zeta(g)\|$. Otherwise $U \setminus \{p\} \subseteq \|g \neq 0\|$. Let $h \in \mathcal{R}$ be an arbitrary function. Then $U \setminus \{p\} \subseteq \|h \neq 0 \to g \neq 0\|$. If $h(p) \neq 0$, then $p \in \|h \neq 0\|$. If $h(p) = 0$, then $U = \{p\} \cup (U \setminus \{p\}) \subseteq z(h) \cup \|g \neq 0\|$, so $U \subseteq int(z(h) \cup \|g \neq 0\|) = \|h \neq 0 \to g \neq 0\|$. In both cases $U \subseteq \|\zeta(g)\|$. □

Let $\varphi_{\mathbb{N}^+}(x, y, u, v) \equiv \neg x < 1 \wedge \mathcal{O}(xv - u, y) \wedge \mathcal{O}(v - u, y) \wedge$
$$\forall w\{[\mathcal{O}(wv - u, y) \wedge (y \neq 0 \vee w > 1 \vee w < 1)] \to$$
$$[(y \neq 0 \vee w > 1) \wedge \exists w'((w \neq w' \to y \neq 0) \wedge \mathcal{O}((w' - 1)v - u, y))]\}$$

Lemma 4. *For all $f, g, h_1, h_2 \in \mathcal{R}$*

1. $\|\neg f < 1\| \cap \|g \neq 0\| \subseteq \|\varphi_{\mathbb{N}^+}(f, g, h_1, h_2)\|$.
2. *If $p \in T$, $g(p) = 0$, and $p \in \|\varphi_{\mathbb{N}^+}(f, g, h_1, h_2)\|$, then $f(p)$ belongs to $\mathbb{N}^+$, the set of positive integers.*

Proof. 1. Obvious. 2. Suppose p has the above properties, but $f(p) \notin \mathbb{N}^+$. We show by induction on $k \in \mathbb{N}^+$ that in this case there are elements $f_k \in \mathcal{R}$ with the following properties:

1. $f_1 = f$
2. $f_{k+1}(p) = f_k(p) - 1$
3. $p \in \|\mathcal{O}(f_k h_2 - h_1, g)\|$
4. $p \in \|f_k > 1\|$

From these follows that $f(p) > k$ for all $k \in \mathbb{N}^+$ and this contradiction proves the lemma.

Let $f_1 = f$, obviously fulfilling the requirements, and suppose that f_l $(l \leq k)$ are defined with the above properties. Then there is some function f'_k in $\mathcal{R}$ such that

$p \in \|f_k \neq f'_k \to g \neq 0\| \cap \|\mathcal{O}((f'_k - 1)h_2 - h_1, g)\|$. So, since $g(p) = 0$, $f'_k(p) = f_k(p)$. Let $f_{k+1} = f'_k - 1$. Then $f_{k+1}(p) = f_k(p) - 1$ and $p \in \|\mathcal{O}(f_{k+1}h_2 - h_1, g)\|$. From the inductional hypothesis follows that $f_k(p) \notin \mathbb{N}^+$, so $p \in \|f_{k+1} > 1 \vee f_{k+1} < 1\|$ is also true. Then $p \in \|f_{k+1} > 1\|$, since $p \in z(g) \cap \|g \neq 0 \vee f_{k+1} > 1\|$. $\square$

Let $\psi(x) \equiv \forall y(\zeta(y) \to \exists u \exists v \varphi_{\mathbb{N}^+}(x, y, u, v))$. The next two lemmas will show that ψ defines the set of positive integers in the model.

For every $f \in \mathcal{R}$ let $\|f \in \mathbb{N}^+\| = \bigcup_{k \in \mathbb{N}^+} \|f = k\|$.

Lemma 5. *For all $f \in \mathcal{R}$, $\|\psi(f)\| \subseteq \|f \in \mathbb{N}^+\|$.*

Proof. Let $p \in \|\psi(f)\|$ and let $g_p \in \mathcal{R}$ be a function with $z(g_p) = \{p\}$. Then by Lemma 3 $p \in \|\zeta(g_p)\|$, so for some h_1, h_2 in $\mathcal{R}$, $p \in \|\varphi_{\mathbb{N}^+}(f, g_p, h_1, h_2)\|$. By Lemma 4.2 $f(p) = k$ for some k in $\mathbb{N}^+$. Since f is continuous and $\|\psi(f)\|$ is open, there is an open neighborhood U of p with $U \subseteq \|\psi(f)\|$ and for all $q \in U$ $f(q) \in (k-1, k+1)$. For every $q \in U$ let $g_q \in \mathcal{R}$ be a function with $z(g_q) = \{q\}$. Then, as before, $q \in \|\zeta(g_q)\| \cap \|\psi(f)\|$, so $q \in \|\varphi_{\mathbb{N}^+}(f, g_q, h'_1, h'_2)\|$ for some $h'_1, h'_2 \in \mathcal{R}$, and $f(q) \in \mathbb{N}^+$ follows. From these $f(q) = k$ for all $q \in U$, so $p \in \|f = k\| \subseteq \|f \in \mathbb{N}^+\|$. $\square$

Lemma 6. *For all $f \in \mathcal{R}$, $\|f \in \mathbb{N}^+\| \subseteq \|\psi(f)\|$.*

Proof. Let $p \in \|f \in \mathbb{N}^+\|$. Then there are an open neighborhood U of p and $k \in \mathbb{N}^+$ such that $f(q) = k$ for all $q \in U$. Then $U \subseteq \|\neg f < 1\|$. The lemma follows from the following claim.

Claim. For all $g \in \mathcal{R}$, $U \subseteq (T \setminus \|\zeta(g)\|) \cup \bigcup_{h_1, h_2 \in \mathcal{R}} \|\varphi(f, g, h_1, h_2)\|$.

Proof. Fix $g \in \mathcal{R}$ and let $q \in U \cap \|\zeta(g)\|$. Then by Lemma 2 there is an open interval $V \subseteq U$ such that $q \in V$ and $|z(g) \cap V| \leq 1$. If $g(q) \neq 0$, then by Lemma 4.1 $q \in \|\varphi_{\mathbb{N}^+}(f, g, h_1, h_2)\|$ for all $h_1, h_2 \in \mathcal{R}$. Suppose that $g(q) = 0$. Let $h_1, h_2 \in \mathcal{R}$ be continuous functions with the following property:

For every $n \in \mathbb{N}^+$ there is an open subset $U_n \subset T$ such that

1. $q \in cl(U_n)$ and $U_n \subseteq z(nh_2 - h_1)$;
2. $\bigcup_n U_n = T \setminus \{q\}$;
3. $z(h_1) = z(h_2) = \{q\}$.

We claim that $V \subseteq \|\varphi_{\mathbb{N}^+}(f, g, h_1, h_2)\|$.

If $q' \in V$ and $q' \neq q$, then $q' \in \|g \neq 0\|$. In this case $q' \in \|\varphi_{\mathbb{N}^+}(f, g, h_1, h_2)\|$ by Lemma 4.1. So it is enough to show that $q \in \|\varphi_{\mathbb{N}^+}(f, g, h_1, h_2)\|$. Since $f|V = k$, $q \in cl(U_k \cap V) \subseteq cl(int(z(fh_2 - h_1)))$. Also, $q \in cl(U_1) \subseteq cl(int(z(h_2 - h_1)))$, so since $V \setminus \{q\} \subseteq \|g \neq 0\|$, $q \in \|\mathcal{O}(fh_2 - h_1, g) \wedge \mathcal{O}(h_2 - h_1, g)\|$. Next let h be an arbitrary element of $\mathcal{R}$, and suppose that $q \in \|\mathcal{O}(hh_2 - h_1, g) \wedge (g \neq 0 \vee h > 1 \vee h < 1)\|$. By Lemma 1 $q \in cl(int(z(hh_2 - h_1)))$, and since $g(q) = 0$, $h(q) > 1$ or $h(q) < 1$. Then there is a sequence $q_n \to q$ such that $q_n \in int(z(hh_2 - h_1))$ and $q_n \neq q$. From this, by the properties of the functions h_1 and h_2 follows that $h(q_n) \in \mathbb{N}^+$ for all n and, using the continuity of h, we can conclude that $h(q) \in \mathbb{N}^+$ and $h(q) > 1$. Then $q \in \|g \neq 0 \vee h > 1\|$. Finally let $f_{h(q)} \in \mathcal{R}$ be the constant function with value $h(q)$. Then $q \in \|h \neq f_{h(q)} \to g \neq 0\|$ and $q \in \|\mathcal{O}((f_{h(q)} - 1)h_2 - h_1, g)\|$ are true, and the lemma follows.

For the sake of completeness we include a simple construction of two functions h_1 and h_2 with the above properties.

Let $V' \subseteq \mathbb{R}$ be an open interval such that $V' \cap T = V$. Let $r_1 < r_2 < \dots$ be a sequence of rational numbers in V' with $\lim_i r_i = q$ and, similarly, let $r'_1 > r'_2 > \dots$ be a sequence of rational numbers in V' again with $\lim_i r'_i = q$. For $i \geq 1$ let $I_i = (r_i, r_{i+1})$, $I'_i = (r'_{i+1}, r'_i) \subseteq V$ be T-clopen intervals. For all i let $r_i < r_i^1 < r_i^2 < \dots$ be a sequence of rational elements of V' with $\lim_j r_i^j = r_{i+1}$, $r'_i > r_i^{1'} > r_i^{2'} > \dots$ be a similar sequence with $\lim_j r_i^{j'} = r'_{i+1}$. Let $I_i^1 = (r_i, r_i^1)$ and $I_i^{1'} = (r_i^{1'}, r_i')$, and for $j > 1$ let $I_i^j = (r_i^{j-1}, r_i^j)$ and $I_i^{j'} = (r_i^{j'}, r_i^{j-1'})$ be T-clopen subintervals of V.

We define the two functions h_1, h_2 from T to $\mathbb{R}$ piecewise.

Let $h_1|I_i = h_1|I'_i = 1/i,\ h_1|(-\infty, r_1) = h_1|(r'_1, +\infty) = 1$ and $h_1(q) = 0$.
Let $h_2|I_i^j = h_2|I_i^{j'} = 1/ij,\ h_2|(-\infty, r_1) = h_2|(r_1{}', +\infty) = 1$ and $h_2(q) = 0$.

These functions are continuous, so $h_1, h_2 \in \mathcal{R}$, and they obviously fulfill the requirements. □

Let $\varphi(\vec{x})$ be a formula of the language $L' = \langle 1, +, \times \rangle$. Let $\tilde{\varphi}$ be defined inductively as follows:

1. If φ is atomic, then $\tilde{\varphi} = \varphi$.
2. $\widetilde{\varphi_1 \circ \varphi_2} = \tilde{\varphi}_1 \circ \tilde{\varphi}_2$ where $\circ = \wedge,\ \vee,\ \to$.
3. $\widetilde{\exists x \varphi_1(x)} = \exists x(\psi(x) \wedge \tilde{\varphi}_1(x))$
4. $\widetilde{\forall x \varphi_1(x)} = \forall x(\psi(x) \to \tilde{\varphi}_1(x))$

Here $\psi(x)$ is the formula defined above.

For $k \in \mathbb{N}^+$ let $f_k \in \mathcal{R}$ be the constant function with value k.

The next proposition gives the undecidability result.

Proposition 7. *Let $\varphi(\vec{x})$ be an L'-formula and $\vec{a}$ be a tuple of positive integers. $\mathbb{N}^+ \models \varphi[\vec{a}]$ if and only if $\mathcal{R} \models \tilde{\varphi}[\vec{f_a}]$*

Proof. By formula induction using the fact that $\|\tilde{\varphi}[\vec{f_a}]\|$ is either T or empty. □

References

[1] Cherlin, G., Rings of continuous functions: Decision problems, in: L. Pacholski, J. Wierzejewski and A. J. Wilkie, eds., Model Theory of Algebra and Arithmetic, Lecture Notes in Mathematics 834 (Springer-Verlag, Berlin, Heidelberg, New York, 1980) 44-91.

[2] Erdélyi-Szabó, M., Decidability of Scott's model as an ordered $\mathbb{Q}$-vectorspace, Journal of Symbolic Logic (to appear).

[3] Erdélyi-Szabó, M., Decidability in the constructive theory of reals as an ordered $\mathbb{Q}$-vectorspace, Mathematical Logic Quarterly (to appear).

[4] Scott, D. S., Extending the topological interpretation to intuitionistic analysis, Compositio Math. 20 (1968) 194-210.

[5] Scott, D. S., Extending the topological interpretation to intuitionistic analysis, II, in: A. Kino, J. Myhill and R.E. Vesley, eds., Intuitionism and Proof Theory (North-Holland, Amsterdam, 1970) 235-255.

[6] Scowcroft, P., The real-algebraic structure of Scott's model of intuitionistic analysis, Ann. Pure Appl. Logic 27 (1984) 275-308.

[7] Scowcroft, P., More on real algebra in Scott's model, Ann. Pure Appl. Logic 30 (1986) 277-291.

[8] Scowcroft, P., A transfer theorem in constructive real algebra, Ann. Pure Appl. Logic 40 (1988) 29-87.

[9] Scowcroft, P., More on Brouwer's refutations, Ann. Pure Appl. Logic 41 (1989) 83-91.