DECIDABILITY IN THE CONSTRUCTIVE THEORY OF REALS AS AN ORDERED $\mathbb{Q}$-VECTORSPACE

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ABSTRACT. We show that various fragments of the intuitionistic/constructive theory of the reals are decidable.

1. Introduction

In this paper we shall work with the following language of the theory of reals as an ordered $\mathbb{Q}$-vectorspace: $L = \langle <, +, h_q, 1\rangle_{q\in\mathbb{Q}}$ where $\mathbb{Q}$ is the set of rational numbers and h_q is a one-place function symbol corresponding to multiplication by q.

In [5] we gave a constructive proof of a normal-form theorem for formulas of the language L which shows that every L-sentence is equivalent to one whose atomic formulas are of the form $x > 0$, x is a variable, and $h_q(1) > 0$ with $q \in \mathbb{Q}$ (Proposition 1). The proof used only facts true in Bishop's constructive analysis (cf. [1]).

Using this result we shall show constructively that various fragments of the L-theory of the reals are decidable.

In Section 2 we shall prove that the constructive truth of prenex universal sentences of our language can be decided using intuitionistic propositional calculus (which is known to be decidable, cf. [6]). The methods applied in the proof go beyond Bishop-style constructivism: we shall use an intuitionistic theory of lawless sequences in the metatheory. (About lawless sequences see Dummett's book [4, chapter 3] or Troelstra's monograph [7].)

In Section 3 we shall give a simple decision method for sentences not containing universal quantifiers. The proof is syntactic and uses only simple facts accepted by every school of constructive mathematics.

Section 4 combines the previous results giving a decision method for sentences where at most the innermost quantifiers are universal. The metatheory used is of course that of Section 2.

First let us quote the normal-form theorem from [5]:

1991 *Mathematics Subject Classification.* 03-B25, 03-F55, 03-F65.

Key words and phrases. Decidability, constructivism, intuitionism, $\mathbb{Q}$-vectorspace, lawless sequences.

I would like to thank my advisor Professor Philip Scowcroft for his direction and for helpful discussions.

Proposition 1. *Let $\varphi \in Sent_L$ be a sentence. Then there is a sentence $\psi \in Sent_L$ such that*

1. *ψ can be obtained from φ constructively.*
2. *All atomic subformulas of ψ are one of the following forms:*
   (a) *$w > 0$, where w is a variable;*
   (b) *$h_q(1) > 0$, where $q \in \mathbb{Q}$.*
   (Such a ψ is said to be of special form).
3. *$\mathbb{R} \models \varphi \leftrightarrow \psi$ where $\mathbb{R}$ denotes the structure of constructive reals from* [1].

$\square$

2. Prenex Universal Sentences

In this section, using an intuitionistic theory of lawless sequences in the metatheory, we shall show that the constructive truth of prenex universal sentences of the language L can be decided using intuitionistic propositional calculus (which is known to be decidable, cf. [6]). Again $\mathbb{R}$ denotes the structure of reals as defined in [1].

Notation

1. For $n \in \omega$ $n = \{\emptyset, 1, \ldots, n-1\}$, so $l < n$ if and only if $l \in n$.
2. For $n, m \in \omega$ n^m is the set of functions $k : m \to n$ (also called m-tuples or sequences of length m of elements of n). The empty sequence will be denoted by $<>$. Note that $n^\emptyset = \{<>\}$. These tuples can be encoded by natural numbers in the usual way. We shall use the same symbol for the (finite) function as for its code, and assume that the code of $<>$ is 0.
3. If $k \in n^m$ then $lth(k) = m$, $\langle k \rangle_i = k(i)$ for $i < m$, and $k = \langle k(0), \ldots, k(m-1) \rangle$.
4. $n^{<\omega} := \bigcup_{m<\omega} n^m$.
5. n^ω is the set of functions $\alpha : \omega \to n$ (infinite sequences).
6. If $m \in \omega$, $\alpha \in n^\omega$ then $\bar{\alpha}m = \langle \alpha(0), \ldots, \alpha(m-1) \rangle \in n^m$.
7. If $\alpha \in n^\omega$, $k \in n^m$ then $\alpha \in k$ if and only if $\bar{\alpha}m = k$.
8. If $k \in n^m$ and $l \in n$ then $k * \langle l \rangle$ is a sequence of length $m+1$ such that $\langle k * \langle l \rangle \rangle_i = \langle k \rangle_i$ if $i < m$ and $\langle k * \langle l \rangle \rangle_m = l$.
9. In general, if $k \in n^m$, $\alpha \in n^p$, $m < \omega$, and $p \leq \omega$, then $\beta = k * \alpha \in n^{m+p}$ where $\langle \beta \rangle_i = \langle k \rangle_i$ if $i < m$ and $\langle \beta \rangle_i = \langle \alpha \rangle_{i-m}$ if $m \leq i < m+p$.

Let the language L_1 consist of the following symbols:

1. There are two sorts of variables: x, y, i, j, k, μ, ν, etc. ranging over the elements of ω, and α, β, γ, etc. ranging over infinite sequences of elements of ω.
2. We have constant symbols for elements of ω.

3. We have a unary function symbol lth (the length of a finite sequence).
4. We have binary function symbols of various sorts
   (a) $*$: concatenation,
   (b) $\langle x\rangle_k$: the $k+1$-st element of the sequence coded by x,
   (c) $\langle \alpha\rangle_k$: the $k+1$-st element of the sequence α and
   (d) $\bar{\alpha}m$: the finite sequence of the first m elements of α.
5. We have three different "element" relations: $k \in n^m$, $k \in n^{<\omega}$ and $\alpha \in k$. Their meaning is described above.
6. Symbols for certain other primitive-recursive functions and relations ($\dot{-}$, $<$, etc).
7. Equality symbol.

The connectives, quantifiers and bound quantifiers are the usual ones.

We assume the axioms of Heyting arithmetic, and that the usual properties of the introduced functions and relations hold.

Let S be a new relation symbol on infinite sequences. Its meaning is described by the following axioms on the language $L_1 \cup \{S\}$ (we use the notation $\alpha \in S$ for $S(\alpha)$, n is a fixed constant symbol denoting an element of ω):

1. $\forall \alpha \in S\, \forall i\, \langle\alpha\rangle_i < n$ (i.e. $S \subset n^\omega$);
2. $\forall k \in n^{<\omega} \exists \alpha \in S(\alpha \in k)$;
3. if the formula $\varphi(\alpha)$ does not contain any (free or bound) sequence variables beside α, then the principle of open data is true for φ:
$$\varphi(\alpha) \to \exists x \forall \beta \in S(\bar{\beta}x = \bar{\alpha}x \to \varphi(\beta));$$
4. if $Q(x)$ does not contain any (free or bound) sequence variables, then monotone bar induction holds for Q:
$$[\forall \alpha \in S \exists x\, Q(\bar{\alpha}x) \wedge \forall x < n \forall y \in n^{<\omega}(Q(y) \to Q(y * \langle x\rangle))$$
$$\wedge \forall y \in n^{<\omega}(\forall x < n Q(y * \langle x\rangle) \to Q(y))] \to Q(0).$$

By [2] this axiom system has a model. In what follows we shall work inside such a model, so let us fix a subclass S of infinite sequences obeying 1-4. Note that the result follows from Proposition 1, the properties of finite and infinite sequences mentioned above, the axioms for Heyting arithmetic, and the definition of constructive real numbers (cf. [1]); the reasoning occurs in three-sorted intuitionistic predicate calculus, with sorts for natural numbers, elements of S, and real numbers.

Definition 2. *$S' \subset S$ is $\emptyset$-definable if there is a formula $\varphi(\alpha)$, with no (free or bound) sequence variables besides α, such that $\alpha \in S'$ if and only if $\varphi(\alpha)$ is true.*

Notation: $S' = \lceil \varphi \rceil = \{\alpha \in S \mid \varphi(\alpha)\}$

Let $Def(S)$ be the set of $\emptyset$-definable subsets of S.

Lemma 3. *Let $\lceil \varphi \rceil \Rightarrow \lceil \psi \rceil = \lceil \varphi \to \psi \rceil$. Then $(Def(S), \emptyset, S, \cup, \cap, \Rightarrow, \subseteq)$ is a Heyting algebra: i.e. $(Def(S), \subseteq)$ is a partially ordered set, $\cup$ is the least upper bound, $\cap$ is the greatest lower bound, $\Rightarrow$ is an implication, $\emptyset$ is the least element and S is the greatest element of $Def(S)$ (cf.* [3, page 60]*). Moreover $\lceil \varphi \rceil \cup \lceil \psi \rceil = \lceil \varphi \vee \psi \rceil$ and $\lceil \varphi \rceil \cap \lceil \psi \rceil = \lceil \varphi \wedge \psi \rceil$.*

Proof. We have to show that $\lceil \varphi \rceil \cup \lceil \psi \rceil = \lceil \varphi \vee \psi \rceil$, $\lceil \varphi \rceil \cap \lceil \psi \rceil = \lceil \varphi \wedge \psi \rceil$, $\Rightarrow$ is an implication, $\emptyset$ is the least element and S is the greatest element of $Def(S)$. From these the statement follows.

(*i*) The proof that $\lceil \varphi \rceil \cup \lceil \psi \rceil = \lceil \varphi \vee \psi \rceil$ and $\lceil \varphi \rceil \cap \lceil \psi \rceil = \lceil \varphi \wedge \psi \rceil$ is straightforward.

(*ii*) $\Rightarrow$ *is an implication:*

$$\begin{aligned} \lceil \varphi \rceil \cap (\lceil \varphi \rceil \Rightarrow \lceil \psi \rceil) &= \lceil \varphi \wedge (\varphi \to \psi) \rceil = \{\alpha \in S \mid \varphi(\alpha) \wedge (\varphi(\alpha) \to \psi(\alpha))\} \\ &\subseteq \{\alpha \in S \mid \psi(\alpha)\} = \lceil \psi \rceil \end{aligned}$$

Now suppose that $\lceil \theta \rceil \cap \lceil \varphi \rceil \subseteq \lceil \psi \rceil$ for some θ. Then $\{\alpha \in S \mid \theta(\alpha) \wedge \varphi(\alpha)\} \subseteq \{\alpha \in S \mid \psi(\alpha)\}$. Suppose that $\theta(\alpha)$ is true. Then if $\varphi(\alpha)$ is true, $\theta(\alpha) \wedge \varphi(\alpha)$ and consequently $\psi(\alpha)$ follows. So $\{\alpha \in S \mid \theta(\alpha)\} \subseteq \{\alpha \in S \mid \varphi(\alpha) \to \psi(\alpha)\}$ i.e. $\lceil \theta \rceil \subseteq \lceil \varphi \rceil \Rightarrow \lceil \psi \rceil$.

(*iii*) $\emptyset = \{\alpha \in S \mid 0 \neq 0\} \subseteq \lceil \varphi \rceil$ for all φ, so $\emptyset$ is really the least (zero) element in $Def(S)$.

(*iv*) Finally $\lceil \varphi \rceil \subseteq S = \{\alpha \in S \mid 0 = 0\}$ for all φ, so S is the greatest element in $Def(S)$.

□

Definition 4. *Let L' be a language. An L'-structure T is decidable if for all atomic sentence A with parameters from T, $T \models A$ or $T \models \neg A$ has a constructive proof.*

Let $\mathcal{T}$ be a finite decidable tree with nodes $\{0, 1, \dots, n-1\}$ where 0 is the root. For nodes τ and σ, $\tau \leq \sigma$ if and only if τ is a descendant (successor) of σ. A subset U of the nodes is open if it is downward closed. U is decidable, if the relation $\tau \in U$ is decidable. Let $\mathcal{O}(\mathcal{T})$ be the Heyting algebra of the decidable open sets of $\mathcal{T}$.

Proposition 5. *There is an Ha-homomorphism*

$$e : \mathcal{O}(\mathcal{T}) \to Def(S)$$

such that e is weakly injective: i.e.,

$$\exists\tau(\tau \in U_1 \wedge \neg\tau \in U_2) \rightarrow \neg\forall\alpha(\alpha \in e(U_1) \rightarrow \alpha \in e(U_2))$$

where each $U_i \in \mathcal{O}(\mathcal{T})$.

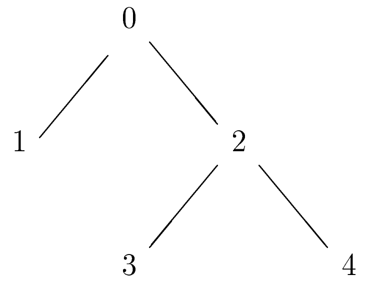
Note that the construction of e corresponds to the conversion of Kripke models to Beth models (cf. [4, Sections 5.1-5.4]).

Proof. For each node μ that has $l \geq 0$ immediate successors $\{\sigma_1, \ldots, \sigma_l\}$ in $\mathcal{T}$, let $P_\mu = \{p_0^\mu, \ldots, p_l^\mu\}$ be a partition of the set $\{0, \ldots, n-1\}$ such that $0 \in p_0^\mu$. Then there is a formula $E(y, \mu, \nu)$, determined by the structure of the tree $\mathcal{T}$, which is true if and only if (using the above notation)

$$(\mu = \nu \wedge y \in p_0^\mu) \vee \bigvee_{i=1}^{l} (\nu = \sigma_i \wedge y \in p_i^\mu).$$

Example.

Let $\mathcal{T}$ be the following tree:



Let $p_0^0 = \{0,1\}$, $p_1^0 = \{2\}$, $p_2^0 = \{3,4\}$, $p_0^1 = p_0^3 = p_0^4 = \{0,1,2,3,4\}$, $p_0^2 = \{0,4\}$, $p_1^2 = \{2\}$, and $p_2^2 = \{1,3\}$. (The indices go from left to right, so e.g. p_1^0 corresponds to the node 1 and p_2^2 corresponds to the node 4).

Then

$$\begin{aligned}
E(y,\mu,\nu) \quad \equiv \quad & (\mu = 0 \wedge \nu = 0 \wedge (y = 0 \vee y = 1)) \\
\vee \quad & (\mu = 0 \wedge \nu = 1 \wedge y = 2) \\
\vee \quad & (\mu = 0 \wedge \nu = 2 \wedge (y = 3 \vee y = 4)) \\
\vee \quad & (\mu = 2 \wedge \nu = 4 \wedge (y = 1 \vee y = 3)) \\
\vee \quad & (\mu = 2 \wedge \nu = 2 \wedge (y = 0 \vee y = 4)) \\
\vee \quad & (\mu = 2 \wedge \nu = 3 \wedge y = 2) \\
\vee \quad & \left(\bigvee_{j=1,3,4} \mu = j \wedge \nu = j \wedge \bigvee_{0 \leq i \leq 4} y = i \right)
\end{aligned}$$

Let

$$e^*(k,t) \equiv \exists m(lth(k) = m \wedge \left(\vee_{i=0}^{n-1}(\langle k\rangle_0 = i \wedge \langle t\rangle_0 = 0)\right)$$
$$\wedge \forall l(0 < l < m \rightarrow E(\langle k\rangle_l, \langle t\rangle_{l-1}, \langle t\rangle_l))),$$

let

$$e(\alpha,\tau) \equiv \exists m \exists t\,(e^*(\bar{\alpha}m, t) \wedge \langle t\rangle_{m-1} = \tau)$$

and for $U \in \mathcal{O}(\mathcal{T})$ let

$$e_U(\alpha) \equiv \bigvee_{\tau \in U} e(\alpha,\tau).$$

Then $e(U) := \{\alpha \in S \mid e_U(\alpha)\} \in Def(S)$ since U is decidable.

Statements (i)-(vi) below show that e fulfills the requirements.

(i) *e is weakly injective:*

$$\exists\tau(\tau \in U_1 \wedge \neg\tau \in U_2) \rightarrow \neg\forall\alpha(e_{U_1}(\alpha) \rightarrow e_{U_2}(\alpha)) \quad (U_i \in \mathcal{O}(\mathcal{T}))$$

Proof. (Using bar induction.) Fix $\tau \in U_1$, $\tau \notin U_2$ and suppose that $\forall\alpha(e_{U_1}(\alpha) \rightarrow e_{U_2}(\alpha))$. There are k, t (both can be constructed from τ) such that $e^*(k,t)$ and $\langle t\rangle_{lth(t)-1} = \tau$. Fix such a k, and let $m = lth(k)$.

Claim 1. If $e^*(k,t)$ and $e^*(k,t')$ then $t = t'$.

Proof. By induction on $lth(k)$.

Now let $Q(l) \equiv \bigvee_{\mu \in U_2} \exists t' \left(e^*(k * l, t') \wedge \langle t'\rangle_{lth(t')-1} = \mu\right)$.

Claim 2. $\forall\alpha \in S \exists x Q(\bar{\alpha}x)$

Proof. Fix $\alpha \in S$ and let $\beta = k * \alpha$. Then, since $lth(k) = lth(t) = m$, $\bar{\beta}m = k$, $\langle t\rangle_{m-1} = \tau$, so $e^*(\bar{\beta}m, t)$, $e(\beta,\tau)$ and $e_{U_1}(\beta)$ are true. So $e_{U_2}(\beta)$ is true by the hypothesis, and from this follows that there are $\mu \in U_2$, $n' > 0$ and $t' \in n^{n'}$ such that $e^*(\bar{\beta}n', t') \wedge \langle t'\rangle_{n'-1} = \mu$. If $n' \leq m$ then $\tau \in U_2$ and $Q(\bar{\alpha}0)$. If $n' > m$ let $x = n' - m$. Then $Q(\bar{\alpha}x)$ is true.

Claim 3. $\forall x < n \forall y \in n^{<\omega}(Q(y) \rightarrow Q(y * \langle x\rangle))$.

Proof. Fix x, y such that $Q(y)$ is true, i.e. $e^*(k * y, t') \wedge \langle t'\rangle_{lth(t')-1} = \mu$ for some t' and $\mu \in U_2$. Let $\{\mu_1, \ldots, \mu_l\}$ be the set of the immediate successors of μ and let $P_\mu = \{p_1, \ldots, p_l\}$ be the corresponding partition. Since U_2 is open, $\mu_i \in U_2$ for all $1 \leq i \leq l$. If $x \in p_0$ let $t'' = t' * \mu$. If $x \in p_i$ $(1 \leq i \leq l)$ let $t'' = t' * \mu_i$. Then $\bigvee_{\mu \in U_2} \left(e^*(k * y * \langle x\rangle, t'') \wedge \langle t''\rangle_{lth(t'')-1} = \mu\right)$ is true.

Claim 4. $\forall y \in n^{<\omega}(\forall x < n Q(y * \langle x\rangle) \rightarrow Q(y))$.

Proof. Fix y such that $\forall x < n Q(y * \langle x\rangle)$: i.e.,

$$\forall x < n \bigvee_{\mu \in U_2} \exists t' \left(e^*(k * y * \langle x\rangle, t') \wedge \langle t'\rangle_{lth(t')-1} = \mu\right).$$

Then $e^*(k * y * \langle 0\rangle, t') \wedge \langle t'\rangle_{lth(t')-1} = \mu$ for some t' and $\mu \in U_2$. $t' = t'' * \mu$ and by the definition of e^*, $\langle t''\rangle_{lth(t'')-1} = \mu$. So $e^*(k * y, t'') \wedge \langle t''\rangle_{lth(t'')-1} = \mu$ is true and $Q(y)$ follows.

Finally, bar induction shows that $Q(0)$ is true, so $\tau \in U_2$ by Claim 1. This contradiction proves statement (i).

(ii) $e(U_1 \cup U_2) = e(U_1) \cup e(U_2)$.

Proof.

$$e(U_1 \cup U_2) = \{\alpha \in S \mid e_{U_1 \cup U_2}(\alpha)\} = \{\alpha \in S \mid \bigvee_{\tau \in U_1 \cup U_2} e(\alpha, \tau)\}$$

$$= \{\alpha \in S \mid \bigvee_{\tau \in U_1} e(\alpha, \tau)\} \cup \{\alpha \in S \mid \bigvee_{\tau \in U_2} e(\alpha, \tau)\} = e(U_1) \cup e(U_2).$$

(iii) $e(U_1 \cap U_2) = e(U_1) \cap e(U_2)$.

Proof.

$$e(U_1 \cap U_2) = \{\alpha \in S \mid e_{U_1 \cap U_2}(\alpha)\} = \{\alpha \in S \mid \bigvee_{\tau \in U_1 \cap U_2} e(\alpha, \tau)\}$$

$$\subseteq \{\alpha \in S \mid \bigvee_{\tau \in U_1} e(\alpha, \tau)\} \cap \{\alpha \in S \mid \bigvee_{\tau \in U_2} e(\alpha, \tau)\} = e(U_1) \cap e(U_2)$$

On the other hand:

If $\alpha \in e(U_1) \cap e(U_2)$ then $e(\alpha, \tau_1) \cap e(\alpha, \tau_2)$ for some $\tau_i \in U_i$. Then from the definition of e follows that $\tau_1 \leq \tau_2$ (τ_1 is a descendant of τ_2), or $\tau_2 \leq \tau_1$. From this follows that $\tau_1 \in U_1 \cap U_2$ or $\tau_2 \in U_1 \cap U_2$. In both cases $\alpha \in e(U_1 \cap U_2)$, so $e(U_1 \cap U_2) = e(U_1) \cap e(U_2)$.

(iv) *e is monotone.*

Proof. Let $U_1 \subseteq U_2$. Then $U_1 \cap U_2 = U_1$, $e(U_1) = e(U_1 \cap U_2) = e(U_1) \cap e(U_2) \subseteq e(U_2)$, so e is monotone.

(v) *$e(\emptyset) = \emptyset$ and $e(\mathcal{T}) = S$ are trivial.*

(vi) $e(U_1 \Rightarrow U_2) = e(U_1) \Rightarrow e(U_2)$

Proof.

(a) $e(U_1 \Rightarrow U_2) \subseteq e(U_1) \Rightarrow e(U_2)$ if and only if $e(U_1 \Rightarrow U_2) \cap e(U_1) \subseteq e(U_2)$. But $e(U_1 \Rightarrow U_2) \cap e(U_1) = e((U_1 \Rightarrow U_2) \cap U_1) \subseteq e(U_2)$ by (iii) and (iv).

(b) $e(U_1) \Rightarrow e(U_2) \subseteq e(U_1 \Rightarrow U_2)$:
Let $\alpha \in e(U_1) \Rightarrow e(U_2)$: i.e., $e_{U_1}(\alpha) \to e_{U_2}(\alpha)$. Then by the principle of open data

$$\exists y > 0 \forall \beta \in \bar{\alpha} y (e_{U_1}(\beta) \to e_{U_2}(\beta)) \qquad (**)$$

Fix such a y.

Claim 1. $\forall k \in n^{<\omega} \exists t (e^*(k, t))$.

Proof. By induction on $lth(k)$ the statement follows from the definition of e^*.

So $e^*(\bar{\alpha}y, t)$ is true for some t. Let $\tau = \langle t \rangle_{y-1}$.

Claim 2. $\tau \in U_1 \Rightarrow U_2$.

Proof. It is enough to show that $\{\sigma \leq \tau \mid \sigma \in U_1\} \subseteq U_2$. Let $\sigma \in U_1$, $\sigma \leq \tau$. Then $\exists \beta \in \bar{\alpha}y \exists x > 0 \exists t' \in n^{<\omega} \left(y \leq x \wedge e^*(\bar{\beta}x, t') \wedge \langle t' \rangle_{lth(t')-1} = \sigma \right)$. Fix such β, x and t'. So $e(\beta, \sigma)$, $e_{U_1}(\beta)$ and by $(**)$ $e_{U_2}(\beta)$ follows. Now let us apply bar induction to the following formula — Claim 2 will follow from $Q(0)$:

$$Q(l) \equiv \bigvee_{\mu \in U_2} \exists t_1 \left(e^*(\bar{\beta}x * l, t_1) \wedge \langle t_1 \rangle_{lth(t_1)-1} = \mu \right)$$

.

Claim 3. $\forall \gamma \in S \exists z\, Q(\bar{\gamma}z)$

Proof. Let us fix $\gamma \in S$ and let $\delta = \bar{\beta}x * \gamma$. Then $e^*(\bar{\delta}x, t')$ and $\langle t' \rangle_{lth(t')-1} = \sigma$ are true. So $e_{U_1}(\delta)$ is true, and since $\delta \in \bar{\alpha}y$, $e_{U_2}(\delta)$ is true by $(**)$. So again there are $\mu \in U_2$, $w > 0$ and $t_1 \in n^w$ such that $e^*(\bar{\delta}w, t_1)$ and $\langle t_1 \rangle_{w-1} = \mu$ are true. If $w \leq x$ then $\sigma \in U_2$ and $Q(\bar{\gamma}0)$ is true. If $w > x$ then let $z = w - x$. Then $Q(\bar{\gamma}z)$ is true.

The rest of the proof that $Q(0)$ is true is the same as the proof of Claim 3 and Claim 4 in (i).

So $Q(0)$ is true and $\sigma \in U_2$ follows. Then $\{\sigma \mid \sigma \leq \tau\} \cap U_1 \subseteq U_2$, $\{\sigma \mid \sigma \leq \tau\} \subseteq U_1 \Rightarrow U_2$ and $\tau \in U_1 \Rightarrow U_2$ which proves Claim 2.

Finally, since $e(\alpha, \tau)$ is true, $\alpha \in e(U_1 \Rightarrow U_2)$, thus $e(U_1) \Rightarrow e(U_2) = e(U_1 \Rightarrow U_2)$ which proves (vi) and the Proposition. □

Lemma 6. *Let $U \in \mathcal{O}(\mathcal{T})$. Then there is a function $f : S \to \mathbb{R}$ such that*

$$e(U) = \{\alpha \in S \mid \mathbb{R} \models f(\alpha) > 0\}$$

$\{\alpha \in S \mid \mathbb{R} \models f(\alpha) > 0\}$ will be denoted by $\|f > 0\|$.

Proof. For each $k \in n^{<\omega}$ let τ_k be the unique node of $\mathcal{T}$ such that

$$\exists t \left(e^*(k, t) \wedge \langle t \rangle_{lth(t)-1} = \tau_k \right)$$

For $\alpha \in S$ let $f(\alpha)(0) = 0$ and for $m > 0$ let

$$f(\alpha)(m) := \begin{cases} 0 & \text{if } \tau_{\bar{\alpha}m} \notin U \\ \max(m^{-1}, f(\alpha)(m-1)) & \text{if } \tau_{\bar{\alpha}m} \in U \end{cases}$$

Since $\tau_{\bar{\alpha}m}$ is determined by α and m, and U is decidable, $f(\alpha)$ is well defined for all $\alpha \in S$.

Claim 1. $f(\alpha) \in \mathbb{R}$, i.e. $|f(\alpha)(n) - f(\alpha)(m)| \leq n^{-1} + m^{-1}$.

Proof. Fix $n > m$. Then from the definition of e^* follows that $\tau_{\bar{\alpha}n}$ is a descendant of $\tau_{\bar{\alpha}m}$. So since U is open, if $\tau_{\bar{\alpha}n} \notin U$ then $\tau_{\bar{\alpha}m} \notin U$ and $|f(\alpha)(n) - f(\alpha)(m)| = 0 \leq n^{-1} + m^{-1}$. If $\tau_{\bar{\alpha}n} \in U$, let $k = \min\{l \mid \tau_{\bar{\alpha}l} \in U\}$. Then $k \leq n$. If $k \leq m$ then $f(\alpha)(n) = f(\alpha)(m) = k^{-1}$, so $f(\alpha)(n) - f(\alpha)(m) = 0 < m^{-1} + n^{-1}$. If $m < k \leq n$ then $f(\alpha)(m) = 0$, $f(\alpha)(n) = k^{-1}$, so $|f(\alpha)(n) - f(\alpha)(m)| = k^{-1} < m^{-1} < m^{-1} + n^{-1}$. From these $f(\alpha) \in \mathbb{R}$ follows.

Claim 2. $\|f > 0\| = e(U) \in Def(S)$.

Proof. First let $\alpha \in \|f > 0\|$: i.e., $\mathbb{R} \models f(\alpha) > 0$. Then there is a natural number m such that $f(\alpha)(m) > m^{-1}$. Then $\tau_{\bar{\alpha}m} \in U$, so $\alpha \in e(U)$.

Next suppose that $\alpha \in e(U)$. Then there are some $\tau \in U$ and $m, t > 0$ such that $e^*(\bar{\alpha}m, t)$ and $\langle t \rangle_{m-1} = \tau$ are true — i.e. $\tau = \tau_{\bar{\alpha}m} \in U$. Then $f(\alpha)(m) \geq m^{-1}$ and $f(\alpha)(m+1) \geq m^{-1} > (m+1)^{-1}$. From this $f(\alpha) > 0$ follows. □

Note also that f is continuous.

Lemma 7. *Let P be a propositional sentence built from the letters $p_1, \ldots, p_k$ and from the propositional constants $\top$ denoting true and $\bot$ denoting false. Let $\|p_i\| \in \mathcal{O}(\mathcal{T})$ $(i = 1, \ldots, k)$ and let $\|P\| \in \mathcal{O}(\mathcal{T})$ be defined inductively as follows.*

1. $\|\top\| = \mathcal{T}$
2. $\|\bot\| = \emptyset$
3. $\|P_1 \vee P_2\| = \|P_1\| \cup \|P_2\|$
4. $\|P_1 \wedge P_2\| = \|P_1\| \cap \|P_2\|$
5. $\|P_1 \rightarrow P_2\| = \|P_1\| \Rightarrow \|P_2\|$
6. $\|\neg P'\| = \|P' \rightarrow \bot\|$

By the previous lemma there are functions $f_i : S \rightarrow \mathbb{R}$ such that

$$e\|p_i\| = \{\alpha \in S \mid \mathbb{R} \models f_i(\alpha) > 0)\} \qquad (i = 1, \ldots, k).$$

Let $\psi(\alpha)$ be a formula obtained from P by substituting $f_i(\alpha) > 0$ for p_i, arbitrary true sentences for the occurrences of $\top$, and false sentences for $\bot$. Then

$$e\|P\| = \{\alpha \in S \mid \mathbb{R} \models \psi(\alpha)\} \in Def(S).$$

Proof. By formula induction using the previous results.

First note that $e\|P\| \in Def(S)$ by the definition of e.

(*i*) $P \equiv p_i$.

Then $\psi(\alpha) \equiv f_i(\alpha) > 0$ and $e\|P\| = e\|p_i\| = \{\alpha \in S \mid \mathbb{R} \models f_i(\alpha) > 0\}$.

(*ii*) The cases when $P \equiv \top$ or $P \equiv \bot$ are obvious.

(iii) $P \equiv P_1 \vee P_2$.

Then $\psi(\alpha) \equiv \psi_1(\alpha) \vee \psi_2(\alpha)$ where $\psi_1(\alpha)$ and $\psi_2(\alpha)$ correspond to P_1 and P_2 respectively, and

$$\begin{aligned} e\|P\| &= e(\|P_1 \vee P_2\|) = e(\|P_1\| \cup \|P_2\|) \\ &= e\|P_1\| \cup e\|P_2\| \quad \text{(by Lemma 3)} \\ &= \{\alpha \in S \,|\, \mathbb{R} \models \psi_1(\alpha)\} \cup \{\alpha \in S \,|\, \mathbb{R} \models \psi_2(\alpha)\} \\ &= \{\alpha \in S \,|\, \mathbb{R} \models \psi_1(\alpha) \vee \psi_2(\alpha)\} \\ &= \{\alpha \in S \,|\, \mathbb{R} \models \psi(\alpha)\} \end{aligned}$$

(iv) The case when $P \equiv P_1 \wedge P_2$ is similar.

(v) $P \equiv P_1 \to P_2$.

Then $\psi(\alpha) \equiv \psi_1(\alpha) \to \psi_2(\alpha)$ where $\psi_1(\alpha)$ and $\psi_2(\alpha)$ correspond to P_1 and P_2 respectively.

$$\begin{aligned} e\|P\| &= e(\|P_1 \to P_2\|) = e(\|P_1\| \Rightarrow \|P_2\|) \\ &= e\|P_1\| \Rightarrow e\|P_2\| \quad \text{(by Lemma 3)} \\ &= \{\alpha \in S \,|\, \mathbb{R} \models \psi_1(\alpha)\} \Rightarrow \{\alpha \in S \,|\, \mathbb{R} \models \psi_2(\alpha)\} \\ &= \{\alpha \in S \,|\, \mathbb{R} \models \psi_1(\alpha) \to \psi_2(\alpha)\} \\ &= \{\alpha \in S \,|\, \mathbb{R} \models \psi(\alpha)\} \end{aligned}$$

(vi) The case when $P \equiv \neg P_1 \equiv P_1 \to \bot$ follows from (ii) and (v).

$\square$

Proposition 8. *Let P be as in the previous lemma and suppose that $\nvdash_{int} P$: i.e., P is not a theorem of intuitionistic propositional logic. Then there are functions $f_i : S \to \mathbb{R}$ $(i = 1, \ldots, k)$ such that*

$$\neg\forall\alpha(\mathbb{R} \models \psi(\alpha))$$

where $\psi(\alpha)$ is defined as in the previous lemma.

Proof. By [4, Sections 5.1-5.3] there is a finite decidable tree $\mathcal{T}$ whose root is denoted by 0 and whose structure of decidable open sets is denoted by $\mathcal{O}(\mathcal{T})$, and a valuation $\|p_i\| \in \mathcal{O}(\mathcal{T})$ $(i = 1, \ldots, k)$ of the propositional letters such that $0 \notin \|P\|$. Since e is weakly injective, $\neg\forall\alpha(\alpha \in e\|P\|)$ follows. By the previous lemma there are functions $f_i : S \to \mathbb{R}$ such that $e\|P\| = \{\alpha \in S \,|\, \mathbb{R} \models \psi(\alpha)\}$. Thus $\neg\forall\alpha(\mathbb{R} \models \psi(\alpha))$. $\square$

These results imply the following theorem:

Theorem 9. *Let $\varphi(\vec{x})$ be a quantifier-free formula. Then by Proposition 1*

$$\mathbb{R} \models \forall \vec{x} \varphi(\vec{x}) \leftrightarrow \forall \vec{y} \psi(\vec{y})$$

where the atomic subformulas of $\psi(\vec{y})$ are of the form $y_i > 0$ and $h_q(1) > 0$ with $q \in \mathbb{Q}$. Let $P = \psi(p_i/y_i > 0)$ be the propositional sentence obtained by substituting the propositional letters p_i for the atomic subformulas $y_i > 0$, $\top$ for $h_q(1) > 0$ if $q > 0$ and $\bot$ for $h_q(1) > 0$ if $q \leq 0$. Then

1. *if $\vdash_{int} P$ then $\mathbb{R} \models \forall \vec{x} \varphi(\vec{x})$,*
2. *if $\nvdash_{int} P$ then $\mathbb{R} \models \neg \forall \vec{x} \varphi(\vec{x})$.*

Proof. The first statement is trivial. If $\nvdash_{int} P$ then by the previous proposition there are functions $f_i : S \to \mathbb{R}$ such that

$$\neg \forall \alpha (\mathbb{R} \models \psi(f_i(\alpha)/y_i))$$

So $\mathbb{R} \models \neg \forall \vec{y} \psi(\vec{y})$ and the statement follows. □

3. ∀-free Sentences

In this section every sentence is an L-sentence in special form. We shall also use the symbols $\top$ and $\bot$ denoting a true and a false atomic formula respectively.

Definition 10. *A sentence φ is ∀-free if it does not contain universal quantifiers.*

We shall give a constructive proof of the following theorem:

Theorem 11. *If φ is an ∀-free sentence, then φ is decidable over $\mathbb{R}$.*

The first step in the proof is a simple elimination result.

Proposition 12. *Let $\varphi(x, \vec{y})$ be a quantifier-free formula in disjunctive normal form. Then there is a quantifier-free formula $\psi(\vec{y})$ in disjunctive normal form such that $\{\neg(0 > 0), 1 > 0\} \vdash_{int} \exists x \varphi(x, \vec{y}) \leftrightarrow \psi(\vec{y})$.*

Proof. Let $\varphi(x, \vec{y}) \equiv \bigvee_i \bigwedge_{j<n_i} \varphi_{ij}(x, \vec{y})$. Then

$$\{\neg(0 > 0), 1 > 0\} \vdash_{int} \exists x \varphi(x, \vec{y}) \leftrightarrow \bigvee_i \exists x \bigwedge_{j<n_i} \varphi_{ij}(x, \vec{y}) \leftrightarrow \bigvee_i \hat{\varphi}_i(\vec{y})$$

where $\hat{\varphi}_i(\vec{y})$ is $\bot$ if $x > 0 \wedge \neg(x > 0)$ is a subformula of $\bigwedge_j \varphi_{ij}$, and otherwise $\hat{\varphi}_i(\vec{y})$ is the conjunction of the subformulas $\varphi_{ij}(\vec{y})$ not containing x. The empty conjunction is $\top$. It is easy to see that this formula works. □

The following facts about the inuitionistic predicate calculus will help to reduce the general ∀-free case to this special one.

Lemma 13. *Let φ be a quantifier-free formula, $\hat{\varphi}$ be one of its disjunctive normal forms. Then $\vdash_{int} \neg\neg\varphi \leftrightarrow \neg\neg\hat{\varphi}$.* □

Lemma 14. $\vdash_{int} (\forall x(\neg\neg\varphi_1(x) \leftrightarrow \neg\neg\varphi_2(x))) \to (\neg\neg\exists x\varphi_1(x) \leftrightarrow \neg\neg\exists x\varphi_2(x))$. *Here the formulas might contain other free variables beside x.* □

Lemma 15. *Let $\exists x\varphi$ be an $\forall$-free sentence and suppose that $\varphi(0)$ and $\varphi(1)$ are decidable in the intuitionistic predicate calculus from the axioms $\neg(0 > 0)$ and $1 > 0$: i.e., $\{\neg(0 > 0), 1 > 0\} \vdash_{int} (\varphi(0) \vee \neg\varphi(0)) \wedge (\varphi(1) \vee \neg\varphi(1))$. Then*

$$\{\neg(0 > 0), 1 > 0\} \vdash_{int} \exists x\varphi(x) \to \varphi(0) \vee \varphi(1).$$

Proof. Let φ be an $\forall$-free formula. Let ψ_φ be a quantifier-free formula defined inductively as follows:

1. If φ is quantifier-free then ψ_φ is one of its disjunctive normal forms.
2. If $\varphi \equiv \varphi_1 \circ \varphi_2$ then ψ_φ is a disjunctive normal form of $\psi_{\varphi_1} \circ \psi_{\varphi_2}$ where $\circ = \wedge, \vee$ or $\to$.
3. If $\varphi \equiv \neg\varphi_1$ then ψ_φ is a disjunctive normal form of $\neg\psi_{\varphi_1}$.
4. If $\varphi \equiv \exists x\varphi_1$, let $\hat{\varphi} \equiv \exists x\psi_{\varphi_1}$. By Proposition 12 there is a quantifier-free formula $\psi_{\hat{\varphi}}$ in disjunctive normal form such that $\{\neg(0 > 0), 1 > 0\} \vdash_{int} \exists x\psi_{\varphi_1} \leftrightarrow \psi_{\hat{\varphi}}$. Let $\psi_\varphi \equiv \psi_{\hat{\varphi}}$.

Claim 1. $\{\neg(0 > 0), 1 > 0\} \vdash_{int} \neg\neg\varphi \leftrightarrow \neg\neg\psi_\varphi$

Proof. The proof goes inductively according to the definition of ψ_φ.

1. By Lemma 13.
2. Easy.
3. Easy.
4. Using the induction hypothesis and Lemma 14 $\{\neg(0 > 0), 1 > 0\} \vdash_{int} \neg\neg\varphi \leftrightarrow \neg\neg\exists x\varphi_1 \leftrightarrow \neg\neg\exists x\psi_{\varphi_1} \leftrightarrow \neg\neg\psi_{\hat{\varphi}} \leftrightarrow \neg\neg\psi_\varphi$.

Claim 2. If ψ is a quantifier-free formula in disjunctive normal form, then

$$\{\neg(0 > 0), 1 > 0\} \vdash_{int} \exists x\psi \leftrightarrow \psi(0) \vee \psi(1)$$

Proof. From right to left the statement is obvious. Proving from left to right without loss of generality we may suppose that $\psi \equiv \wedge_i\psi_i$ where each ψ_i is atomic or negated atomic. We have the following possibilities:

- If $x > 0$ is among the ψ_i's then $\{\neg(0 > 0), 1 > 0\} \vdash_{int} \exists x\psi \to \psi(1)$
- If $\neg(x > 0)$ is among the ψ_i's then $\{\neg(0 > 0), 1 > 0\} \vdash_{int} \exists x\psi \to \psi(0)$
- Otherwise ψ does not depend on x and the statement is obvious.

Now we can conclude the proof of our lemma.

$$\begin{aligned}
\{\neg(0>0), 1>0\} \ \vdash_{int} \ & \exists x\varphi \to \neg\neg\exists x\varphi \\
\leftrightarrow \ & \neg\neg\exists x\psi_\varphi \quad \text{(by Claim 1 and Lemma 14)} \\
\leftrightarrow \ & \neg\neg(\psi_\varphi(0) \vee \psi_\varphi(1)) \quad \text{(by Claim 2)} \\
\leftrightarrow \ & \neg\neg\psi_\varphi(0) \vee \neg\neg\psi_\varphi(1) \quad \text{(since } \psi_\varphi(0) \text{ and } \psi_\varphi(1) \\
& \text{are quantifier-free sentences)} \\
\leftrightarrow \ & \neg\neg\varphi(0) \vee \neg\neg\varphi(1) \quad \text{(by Claim 1.)} \\
\leftrightarrow \ & \varphi(0) \vee \varphi(1) \quad \text{(by the hypothesis)}
\end{aligned}$$

□

We shall need one more definition and then (at last) we are ready to prove Theorem 11.

Definition 16. *Let φ be an $\forall$-free sentence. The subsentences of φ are defined inductively as follows:*

1. $Sub(0 > 0) := \{0 > 0\}$.
2. $Sub(1 > 0) := \{1 > 0\}$.
3. $Sub(\varphi_1 \circ \varphi_2) := Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi_1 \circ \varphi_2\}$ *where* $\circ = \wedge, \vee, \to$.
4. $Sub(\neg\varphi_1) := Sub(\varphi_1) \cup \{\neg\varphi_1\}$.
5. $Sub(\exists x\varphi_1(x)) := Sub(\varphi_1(0)) \cup Sub(\varphi_1(1)) \cup \{\exists x\varphi_1\}$.

Proof of Theorem 11. Since $0 > 0$ and $1 > 0$ are decidable, it is enough to show that the following claim is true.

Claim. If each $\psi \in Sub(\varphi)$, $\psi \not\equiv \varphi$ is decidable from the axioms $\neg(0 > 0)$ and $1 > 0$, then φ is decidable from these axioms.

Proof. If φ is atomic, $\varphi \equiv \varphi_1 \circ \varphi_2$ ($\circ = \wedge, \vee, \to$) or $\varphi \equiv \neg\varphi_1$ then the statement is obvious. Let $\varphi \equiv \exists x\varphi_1(x)$ and suppose that $\varphi_1(0)$ and $\varphi_1(1)$ are decidable. Then $\varphi_1(0) \vee \varphi_1(1)$ is decidable, and since $\varphi_1(0) \vee \varphi_1(1) \to \exists x\varphi_1(x)$ the statement (and thus the theorem) follows from Lemma 15. □

Note that the above proof cannot be "dualized" in order to deal with $\exists$-free sentences. Although the dual of Lemma 13 and (with the help of Brouwer's principle) the dual of Proposition 12 can be shown to be true — both of them with conjunctive instead of disjunctive normal form — the dual of Lemma 14 is false, and Lemma 15 resists to any attempt at dualization.

The (intuitive) reason is roughly that existential quantification is able to eliminate the differences in truth values caused by double negation, while universal quantification tends to enlarge these differences to the extreme.

E.g. let $\psi(x) \equiv x > 0 \vee \neg(x > 0)$. Then $\exists x\psi(x)$, $\exists x\neg\neg\psi(x)$ and $\forall x\neg\neg\psi(x)$ are true, but $\forall x\psi(x)$ is false.

The best result we were able to obtain in this direction is described in the next section.

4. Sentences with Innermost Universal Quantifiers

Using the results about prenex universal and $\forall$-free sentences and the same metatheory as in section 2, we may prove the following result (again each formula is in special form):

Theorem 17. *Let φ be an L-sentence such that if $\forall\vec{x}\psi(\vec{x},\vec{y})$ is a subformula of φ, then ψ is quantifier-free (i.e., at most the innermost quantifiers of φ may be universal). Then φ is decidable over $\mathbb{R}$.*

The proof is based on the following lemma:

Lemma 18. *Let $\psi(\vec{x},\vec{y})$ be a quantifier-free L-formula. Then we can find constructively (using the intuitionistic metatheory described in section 2) a quantifier-free formula $\hat{\psi}(\vec{y})$ such that*

$$\mathbb{R} \models \forall\vec{y}(\neg\neg\forall\vec{x}\psi(\vec{x},\vec{y}) \leftrightarrow \neg\neg\hat{\psi}(\vec{y})).$$

Proof. Let $\vec{y} = y_0, \ldots, y_{n-1}$. For I_1, $I_2 \subseteq n$, $I_1 \cap I_2 = \emptyset$, let

$$\varphi_{I_1,I_2}(\vec{y}) = \bigwedge_{i\in I_1} y_i > 0 \wedge \bigwedge_{i\in I_2} \neg(y_i > 0)$$

Let $J = \{(I_1, I_2) \mid \mathbb{R} \models \forall\vec{y}\forall\vec{x}(\varphi_{I_1,I_2}(\vec{y}) \to \varphi(\vec{x},\vec{y}))\}$. By the results in section 2, the set J is decidable. Let

$$\hat{\psi}(\vec{y}) \equiv \bigvee_{(I_1,I_2)\in J} \varphi_{I_1,I_2}(\vec{y}).$$

Then

$$\mathbb{R} \models \forall\vec{y}(\hat{\psi}(\vec{y}) \to \forall\vec{x}\varphi(\vec{x},\vec{y})),$$

so

$$\mathbb{R} \models \forall\vec{y}(\neg\neg\hat{\psi}(\vec{y}) \to \neg\neg\forall\vec{x}\varphi(\vec{x},\vec{y})).$$

Now fix $\underline{y} \in \mathbb{R}^n$ and suppose that

$$\mathbb{R} \models \bigwedge_{i\in n}[\underline{y}_i > 0 \vee \neg(\underline{y}_i > 0)] \wedge \forall\vec{x}\varphi(\vec{x},\underline{y}).$$

Then

$$\mathbb{R} \models \bigwedge_{i\in I_1} \underline{y}_i > 0 \wedge \bigwedge_{i\in I_2} \neg(\underline{y}_i > 0)$$

for some I_1, $I_2 \subseteq n$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = n$.

Claim: $(I_1, I_2) \in J$

Proof. Let $\vec{\tilde{y}} \in \mathbb{R}^n$ such that $\mathbb{R} \models \varphi_{I_1,I_2}(\vec{\tilde{y}})$ i.e.

$$\mathbb{R} \models \bigwedge_{i \in I_1} \tilde{y}_i > 0 \wedge \bigwedge_{i \in I_2} \neg(\tilde{y}_i > 0)$$

then

$$\mathbb{R} \models \bigwedge_i (\tilde{y}_i > 0 \leftrightarrow \underline{y}_i > 0).$$

Since φ is in special form, one concludes that

$$\mathbb{R} \models \forall \vec{x}(\varphi(\vec{x}, \underline{\vec{y}}) \leftrightarrow \varphi(\vec{x}, \vec{\tilde{y}})).$$

Thus

$$\mathbb{R} \models \forall \vec{x}\varphi(\vec{x}, \underline{\vec{y}}) \leftrightarrow \forall \vec{x}\varphi(\vec{x}, \vec{\tilde{y}}),$$

and since $\mathbb{R} \models \forall \vec{x}\varphi(\vec{x}, \underline{\vec{y}})$, $\mathbb{R} \models \forall \vec{x}\varphi(\vec{x}, \vec{\tilde{y}})$ is true. So

$$\mathbb{R} \models \forall \vec{y}(\varphi_{I_1,I_2}(\vec{y}) \rightarrow \forall \vec{x}\varphi(\vec{x}, \vec{y})).$$

But then

$$\mathbb{R} \models \forall \vec{y} \forall \vec{x}(\varphi_{I_1,I_2}(\vec{y}) \rightarrow \varphi(\vec{x}, \vec{y}))$$

which proves the claim.

Now since $\mathbb{R} \models \varphi_{I_1,I_2}(\underline{\vec{y}})$,

$$\mathbb{R} \models \forall \vec{y} \left[\left(\bigwedge_{i \in n} (y_i > 0 \vee \neg(y_i > 0)) \wedge \forall \vec{x}\varphi(\vec{x}, \vec{y}) \right) \rightarrow \hat{\psi}(\vec{y}) \right].$$

From this

$$\mathbb{R} \models \forall \vec{y} \left[\neg\neg \left(\bigwedge_{i \in n} (y_i > 0 \vee \neg(y_i > 0)) \wedge \forall \vec{x}\varphi(\vec{x}, \vec{y}) \right) \rightarrow \neg\neg\hat{\psi}(\vec{y}) \right]$$

and since $\neg\neg(y_i > 0 \vee \neg(y_i > 0))$ is true,

$$\mathbb{R} \models \forall \vec{y}(\neg\neg\forall \vec{x}\psi(\vec{x}, \vec{y}) \rightarrow \neg\neg\hat{\psi}(\vec{y})) \text{ follows.}$$

□

Using this result we can modify Lemma 15 and Definition 16 in such a way that we are able to handle the sentence φ of Theorem 17:

Lemma 19. *Let $\exists x\varphi$ be a sentence such that at most the innermost quantifiers of φ are universal, and suppose that $\varphi(0)$ and $\varphi(1)$ are decidable over $\mathbb{R}$. Then $\mathbb{R} \models \exists x\varphi(x) \rightarrow \varphi(0) \vee \varphi(1)$.*

Proof. The same as the proof of Lemma 15 except we have to add the case when $\varphi \equiv \forall \vec{x}\varphi_1$. Lemma 18 implies that Claim 1 remains true in the form of $\mathbb{R} \models \neg\neg\varphi \leftrightarrow \neg\neg\psi_\varphi$, and the rest of the proof is parallel to the one given for Lemma 15. □

Definition 16 has to be extended with the following case: if $\varphi_1(\vec{x})$ is quantifier-free,

$$Sub(\forall\vec{x}\varphi_1(\vec{x})) := \{\forall\vec{x}\varphi_1(\vec{x})\}.$$

Note that $\forall\vec{x}\varphi_1(\vec{x})$ is decidable. The rest of the proof is the same as in the $\forall$-free case.

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