# Undecidability of the Real-Algebraic Structure of Models of Intuitionistic Elementary Analysis

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**Abstract.** We show that true first-order arithmetic is interpretable over the real-algebraic structure of models of intuitionistic analysis built upon a certain class of complete Heyting algebras. From this the undecidability of the structures follows. We also show that Scott's model is equivalent to true second-order arithmetic. In the appendix we argue that undecidability on the language of ordered rings follows from intuitionistically plausible properties of the real numbers.

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## 1 Introduction

Let $L$ denote the language of ordered rings. In [4], using a coding similar to the one used by Cherlin in [1], we showed that the $L$-theory of Scott's topological model for intuitionistic analysis is undecidable by encoding true first-order arithmetic in that structure. (Scott defined the model in question in [8].) In this paper we extend this result by showing that true second-order arithmetic is interpretable in Scott's model and that true first-order arithmetic is interpretable in a class of models which includes the well-known topological models as well as Scowcroft's model (defined in

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[9]) and its generalizations. For the sake of convenience, we restrict ourselves to the set of non-negative reals, so in what follows, “real” will mean “non-negative real”.

## 2 Coding $\mathbb{N}^+$

In this section we interpret the structure of positive integers in the ordered-ring structure of reals, provided that the reals obey certain assumptions true in various models of intuitionistic analysis. In the appendix we shall attempt to give an intuitionistically acceptable justification of these assumptions using a creative subject argument. Here the variables $x$, $y$, $u$, $v$, $w$ etc. will range over reals, $k$, $l$, $n$ will range over natural numbers. In the following formula let $B(y)$ denote $y = 0 \vee y \neq 0$.
Let $\varphi_{\mathbb{N}^+}(x, y, u, v) \equiv$

$$\neg x < 1 \wedge (\neg v = u \vee \neg xv = u \rightarrow B(y)) \wedge \forall w[(\neg wv = u \rightarrow B(y)) \rightarrow$$
$$[(w < 1 \rightarrow B(y)) \wedge (w > 1 \rightarrow \exists w'(w \neq w' \vee \neg w'v = u + v \rightarrow B(y)))]].$$

Then $\varphi_{\mathbb{N}^+}$ is an $L$-formula.

Note that $x \neq y$ is defined as $x < y \vee y < x$ and $\neg x = y$, which is equivalent to $\neg\neg x \neq y$, is weaker intuitionistically than $x \neq y$.
From now on let us assume that the following sentences are true (we shall say that a Heyting algebra $\mathcal{H}$ is *nice* if these sentences are true in the corresponding model of intuitionistic second-order arithmetic, cf. the next section):

1. $\forall y \exists u \exists v (\forall n \in \mathbb{N}^+ (\neg nv = u \rightarrow y = 0 \vee y \neq 0) \wedge \neg v = 0 \wedge \neg\neg \exists n \in \mathbb{N}^+ \, (nv = u))$;
2. $\neg \forall y (y = 0 \vee y \neq 0)$.

Note that if $y = 0 \vee y \neq 0$ is true, then in 1 arbitrary $u = v$ work, so this sentence is true classically.
Note also that 2 is known to be true in any intuitionistic theory of the reals. This is the only classically false ingredient of the proof below.
We have chosen to use these sentences because they seem to be the weakest assumptions making the proofs work.
We shall also use the following intuitionistically valid properties of the reals. The strict ordering $<$ on $\mathbb{R}$ and the operations addition and multiplication extend those of $\mathbb{Q}$ and have the following properties: $\mathbb{R}$ is a commutative ring of characteristic zero, and

$(i)$ $\neg x < x$,

$(ii)$ $x < y \wedge y < z \rightarrow x < z$,

$(iii)$ $x < y \rightarrow x < z \vee z < y$,

$(iv)$ $x < y \rightarrow x + z < y + z$,

$(v)$ $x < y \wedge 0 < z \rightarrow xz < yz$, (*vi*) $x = y \leftrightarrow \neg(x < y \vee y < x)$,

(*vii*) $\neg x < 1 \to \exists n \in \mathbb{N}^+ (n - 1 < x < n + 1)$.

Note that by (*vi*) we may use $=$ between real numbers only as an abreviation. Note also that $x \leq y \not\to (x = y \vee x < y)$. Trichotomy (which, since we do not "exclude the middle" is *not* equivalent to (*vi*)) is replaced by the weaker property (*iii*).

**Theorem 1.** *Let $\psi(x)$ denote the formula $\forall y \exists u \exists v \varphi_{\mathbb{N}^+}(x, y, u, v)$. Then from the axioms and the properties of real numbers mentioned above and from the usual axioms of natural numbers $\exists k \in \mathbb{N}^+ (x = k) \equiv \psi(x)$ follows in two-sorted intuitionistic predicate calculus with equality. (In particular, the proof uses induction in the natural number sort.)*

*Proof.* First we show that $\exists k \in \mathbb{N}^+ (x = k) \to \psi(x)$, so let us assume that $x = k$ is true. Let us fix a real number $y$ and let $B(y)$ denote $y = 0 \vee y \neq 0$ as before. From the first axiom follows that

$$\forall n \in \mathbb{N}^+ (\neg nv = u \to B(y)) \wedge \neg v = 0 \wedge \neg\neg\exists n \in \mathbb{N}^+ (nv = u).$$

for some real numbers $u$ and $v$. Then the first two conjuncts of $\varphi_{\mathbb{N}^+}(x, y, u, v)$ are obviously true.

Next let us assume that $\neg wv = u \to B(y)$ for some real number $w$. First we want to show that in this case $w < 1$ implies $B(y)$. If $w < 1$ and $wv = u$ then, since $v \neq 0$, from the above mentioned properties of the real numbers follows that $\neg\exists n \in \mathbb{N}^+ \; nv = u$. Since $\neg\neg\exists n \in \mathbb{N}^+ \; nv = u$ is true, $w < 1$ implies $\neg wv = u$, which, by our assumption, implies $B(y)$. Next we show that if $w > 1$ then there is some positive natural number $n$ such that $w \neq n \vee \neg(n-1)v = u$ implies $B(y)$. From this $\psi(x)$ follows. If $w > 1$ then $k - 1 < w < k + 1$ for some $k \in \mathbb{N}^+$. Fix such a positive natural number $k$. If $w \neq k$ and $wv = u$ then, since $k - 1 < w < k + 1$ and $v \neq 0$, $\neg\exists n \in \mathbb{N}^+ \; nv = u$ follows, contradicting to one of our assumpions. So if $w \neq k$ then $\neg wv = u$ and $B(y)$ follows as before. If $\neg(k-1)v = u$ and $k$ is at least 2, $B(y)$ follows by our assumption about $u$ and $v$. If $k = 1$ then then $1 < w < 2$ which entails $\neg wv = u$ and $B(y)$ as before.

Finally we show that $\psi(x)$ implies $\exists k \in \mathbb{N}^+ (x = k)$.

As a first step, we show by induction on $k \in \omega$ that if $y$, $u$ and $v$ are arbitrary real numbers then

$$\forall x (\varphi_{\mathbb{N}^+}(x, y, u, v) \wedge k < x < k + 1 \to B(y)).$$

If $k = 0$ then $\varphi_{\mathbb{N}^+}(x, y, u, v) \wedge 0 < x < 1$ is false, so the above implication holds. Let us assume that $\forall x (\varphi_{\mathbb{N}^+}(x, y, u, v) \wedge k < x < k + 1 \to B(y))$ and $\varphi_{\mathbb{N}^+}(w, y, u, v) \wedge k + 1 < w < k + 2$ are true where $w$ is an arbitrary real number. If $\neg wv = u$ then $B(y)$ follows from $\varphi_{\mathbb{N}^+}(w, y, u, v)$, so since $w > 1$ ($k$ is at least 0), $\exists w' ((w \neq w' \vee \neg w'v = u + v) \to B(y))$. Fix such $w'$. Since $k + 1 < w$, $w' < w$ or $k + 1 < w'$ follows. In the first case $w \neq w'$, so $B(y)$ is true. In the second case let $\bar{w} = w' - 1$. If $\neg\bar{w}v = u$ then $\neg w'v = u + v$ and $B(y)$ follows, so $\neg\bar{w}v = u \to B(y)$ is true. But then if $k = 0$,

$0 < \bar{w} < 1$ and $B(y)$ follows from this implication. Otherwise, if $k > 0$, $\neg\bar{w} < 1$, so $\varphi_{\mathbb{N}^+}(\bar{w}, y, u, v) \wedge k < \bar{w} < k+1$ is true and $B(y)$ follows by the inductional hypothesis.

Now let us assume $\psi(x)$ is true. Then $\neg x < 1$, so $k-1 < x < k+1$ for some positive integer $k$. Fix such a $k \in \mathbb{N}^+$. Let $y$ be an arbitrary real number. Then from $\psi(x)$ follows that there are real numbers $u$ and $v$ such that $\varphi_{\mathbb{N}^+}(x, y, u, v)$ is true. If $x \neq k$ then $k-1 < x < k$ or $k < x < k+1$. In both cases $B(y)$ follows. Since $y$ was an arbitrary real number, $\psi(x) \wedge k-1 < x < k+1 \wedge x \neq k$ implies $\forall y B(y)$, which is false by one of our assumptions. From these follows that $\psi(x)$ implies $\exists k \in \mathbb{N}^+\ x = k$. $\square$

Let $\varphi(\vec{x})$ be a formula of the language $L' = \langle 1, +, \times \rangle$. Let the $L$-formula $\tilde{\varphi}$ be defined inductively as follows:

1. If $\varphi$ is atomic, then $\tilde{\varphi} = \varphi$;
2. $\widetilde{\varphi_1 \circ \varphi_2} = \tilde{\varphi}_1 \circ \tilde{\varphi}_2$ where $\circ = \wedge, \vee, \rightarrow$;
3. $\widetilde{\exists x \varphi_1(x)} = \exists x(\psi(x) \wedge \tilde{\varphi}_1(x))$;
4. $\widetilde{\forall x \varphi_1(x)} = \forall x(\psi(x) \rightarrow \tilde{\varphi}_1(x))$.

Here $\psi(x)$ is the formula defined above.

The next proposition gives the undecidability result.

**Proposition 2.** *Let $\varphi(\vec{x})$ be an $L'$-formula and $\vec{a}$ be a tuple of positive integers. If the structure of reals obey the assumptions above, then $\mathbb{N}^+ \models \varphi[\vec{a}]$ if and only if $\tilde{\varphi}[\vec{a}]$ is true.*

*Proof.* By formula induction. As an illustration, we show the case when the formula starts with an existential quantifier:

Suppose that $\mathbb{N}^+ \models \exists x \varphi(x)$. Then $\mathbb{N}^+ \models \varphi(a)$ for some $a \in \mathbb{N}^+$, so by the induction hypothesis $\tilde{\varphi}(a)$ is true. Also, $\psi(a)$ is true by Theorem 1, so $\widetilde{\exists x \varphi(x)} \equiv \exists x(\psi(x) \wedge \tilde{\varphi}(x))$ is true.

If $\widetilde{\exists x \varphi(x)} \equiv \exists x(\psi(x) \wedge \tilde{\varphi}(x))$ is true then by Theorem 1 $\exists x \exists k \in \mathbb{N}^+(x = k \wedge \tilde{\varphi}(x))$. From this $\exists x \exists k \in \mathbb{N}^+ \tilde{\varphi}(k)$ follows, and by the induction hypothesis $\mathbb{N}^+ \models \varphi(k)$. Thus $\mathbb{N}^+ \models \exists x \varphi(x)$. $\square$

## 3 The Models in Question

Let $\mathcal{H} = (H, \bot, \top, \wedge, \vee, \rightarrow, \leq)$ be a complete Heyting algebra – the *truth value algebra* of the model to be defined. Infinite infimum and supremum will be denoted by $\bigwedge$ and $\bigvee$ respectively. We say that $A \in H$ is *complemented* if there is an element denoted by $A^c$ of $H$ such that $A \wedge A^c = \bot$ and $A \vee A^c = \top$. Note that if A is complemented then $A^c = A \to \bot$. $A \to \bot$ is the *pseudocomplement* of $A$, denoted by $\neg A$. We say that two elements $U_1, U_2 \in H$ are *disjoint* if $U_1 \wedge U_2 = \bot$.

A *choice sequence*, i.e. a sequence of natural numbers is represented in the model by a function $\xi : \omega \times \omega \to H$ such that $\bigvee_m \xi(l, m) = \top$ and $\xi(l, m) \wedge \xi(l, n) = \bot$ for all $l$, $m$, $n$ with $m \neq n$. From this follows that the elements $\xi(l, m)$ are complemented. Here $\xi(l, m)$ is the truth value of the statement that the $l$-th element of the sequence represented by $\xi$ is $m$. Let $\Xi$ denote the set of choice sequences.

The language $L_1$ we shall use is the one used in [9]. It contains two sorts of variables — $x$, $y$, $z$, etc. ranging over the elements of $\omega$, and $\alpha$, $\beta$, etc. ranging over choice sequences — and constant symbols for each $m \in \omega$ and for each choice sequence $\xi$. Since it will not cause any confusion, we shall use the same symbol for the constant and the corresponding element of the model. The language contains symbols for certain primitive recursive functions and relations defined on the elements of $\omega$ - e.g. $|x - y|$, $\leq$ etc. — and we also have the equality symbol $=$. It will be used in atomic formulas of the form $t = t'$ or $\xi(t) = t'$ where $t$ and $t'$ are terms of natural-number sort and $\xi$ is a choice sequence (constant).

Atomic sentences receive truth values as follows:

1. $\|m \leq n\| = \begin{cases} \top & \text{if } m \leq n \\ \bot & \text{if } m \not\leq n \end{cases}$

   and similarly for other primitive-recursive relations.

2. Concerning choice sequences, we have:

   $\|\xi(m) = n\| = \xi(m, n)$.

Arbitrary sentences $\varphi$ receive truth values $\|\varphi\| \in H$ in the usual way:

1. $\|\varphi_1 \vee \varphi_2\| = \|\varphi_1\| \vee \|\varphi_2\|$
2. $\|\varphi_1 \wedge \varphi_2\| = \|\varphi_1\| \wedge \|\varphi_2\|$
3. $\|\varphi_1 \to \varphi_2\| = \|\varphi_1\| \to \|\varphi_2\|$
4. $\|\neg\varphi\| = \|\varphi\| \to \bot$
5. $\|\exists x \varphi(x)\| = \bigvee_{n \in \omega} \|\varphi(n)\|$
6. $\|\forall x \varphi(x)\| = \bigwedge_{n \in \omega} \|\varphi(n)\|$
7. $\|\exists \alpha \varphi(\alpha)\| = \bigvee_{\xi \in \Xi} \|\varphi(\xi)\|$
8. $\|\forall \alpha \varphi(\alpha)\| = \bigwedge_{\xi \in \Xi} \|\varphi(\xi)\|$

A sentence $\psi$ is true in the model just in case $\|\psi\| = \top$.

In [6, pages 134-135] Vesley considers a species $R$ of *real-number generators*: $\xi \in R$ if and only if the sequence $2^{-x}\xi(x)$ $(x \in \omega)$ of diadic fractions is a Cauchy-sequence with

$\forall k \exists x \forall p |2^{-x}\xi(x) - 2^{-x-p}\xi(x+p)| < 2^{-k}$, i.e. if and only if $\forall k \exists x \forall p 2^k |2^p \xi(x) - \xi(x+p)| < 2^{x+p}$.

Equality, ordering, addition and multiplication on $R$ are defined as follows.

1. $\xi = \eta$ if and only if $\forall k \exists x \forall p 2^k |\xi(x+p) - \eta(x+p)| < 2^{x+p}$,
2. $\xi < \eta$ if and only if $\exists k \exists x \forall p 2^k (\eta(x+p) \dot{-} \xi(x+p)) \geq 2^{x+p}$,
3. $(\xi + \eta)(x) := \xi(x) + \eta(x)$ and
4. $(\xi\eta)(x) := \lfloor 2^{-x}\xi(x)\eta(x) \rfloor$.

The following facts are also proved in [6]. If $\xi, \eta \in R$ then $\xi + \eta \in R$, $=$ is a congruence relation with respect to $<$ and $+$. Similar facts are true for multiplication as well (cf. also [5, pages 20-21]).

Let us expand the language $L_1$ with a unary predicate symbol $R$, binary predicate symbols $=$ and $<$, and binary function symbols for addition and multiplication on choice sequences.

The corresponding truth values may be defined as follows:

$$\|R(\xi)\| := \|\forall k \exists x \forall p 2^k |2^p \xi(x) - \xi(x+p)| < 2^{x+p}\|$$

using obvious abbreviations.

Similarly:

$$\|\xi = \eta\| := \|\forall k \exists x \forall p 2^k |\xi(x+p) - \eta(x+p)| < 2^{x+p}\|,$$

and

$$\|\xi < \eta\| := \|\exists k \exists x \forall p 2^k (\eta(x+p) \dot{-} \xi(x+p)) \geq 2^{x+p}\|.$$

Using the facts mentioned above, these definitions can be extended readily to polynomials of choice sequences.

$\xi$ is a *global real-number generator* just in case $\|R(\xi)\| = \top$. Let $\mathcal{G}$ denote their set. $\mathcal{G}$ is closed under addition and multiplication. From now on elements of $\mathcal{G}$ will be denoted by $f$, $g$, $h$ etc.

For each natural number $n$ there is a corresponding global real-number generator $f_n$ defined as follows: $f_n(l,m) = \top$ if $m = n2^l$ and $f_n(l,m) = \bot$ otherwise. Then

$$\|f_n f = \overbrace{f + \cdots + f}^{n}\| = \top.$$

If it does not cause any confusion, we shall denote this particular element of $\mathcal{G}$ by $n$. Then $\mathcal{G}$ is a model for the language of ordered rings with addition and multiplication defined above and the interpretation of 0 and 1 is $f_0$ and $f_1$ respectively. Note that $=$ has the usual properties in the model, e.g. $f = g \leftrightarrow \neg(g < f \vee f < g)$, $f = g \wedge \varphi(f) \rightarrow \varphi(g)$ etc. has truth value $\top$. $f \neq g$ is defined as $f < g \vee g < f$. Finally note that in general $\neg f = g \not\leftrightarrow f \neq g$.

In what follows a *base* of a Heyting algebra is a set of elements with the property that every element of the algebra is the supremum of base elements.

If $\mathcal{H}$ has a base of complemented elements, than it is easy to see that quantification over reals in the corresponding model can be reduced to quantification over global real-number generators. I.e. in this case

$$\|\exists\alpha(R(\alpha) \wedge \varphi(\alpha))\| = \bigvee_{f\in\mathcal{G}} \|\varphi(f)\| \text{ and } \|\forall\alpha(R(\alpha) \to \varphi(\alpha))\| = \bigwedge_{f\in\mathcal{G}} \|\varphi(f)\|.$$

Since $\mathcal{G} \subset \Xi$ and $\|R(f)\| = \top$ for all $f \in \mathcal{G}$,

$$\bigvee_{f\in\mathcal{G}} \|\varphi(f)\| = \bigvee_{f\in\mathcal{G}} \|R(f) \wedge \varphi(f)\| \leq \bigvee_{\xi\in\Xi} \|R(\xi) \wedge \varphi(\xi)\| = \|\exists\alpha(R(\alpha) \wedge \varphi(\alpha))\|.$$

On the other hand, if $U \leq \|R(\xi) \wedge \varphi(\xi)\|$ for some $\xi \in \Xi$ and complemented element $U$, then we can define a real-number generator $f \in \mathcal{G}$ such that $f = \xi$ on $U$ and $f = f_0$ on the complement of $U$. Then $U \leq \|f = \xi \wedge \varphi(\xi)\| \leq \|\varphi(f)\|$, so

$$\|\exists\alpha(R(\alpha) \wedge \varphi(\alpha))\| = \bigvee_{\xi\in\Xi} \|R(\xi) \wedge \varphi(\xi)\| \leq \bigvee_{f\in\mathcal{G}} \|\varphi(f)\|$$

follows.

A similar reasoning shows that $\|\forall\alpha(R(\alpha) \to \varphi(\alpha))\| = \bigwedge_{f\in\mathcal{G}} \|\varphi(f)\|$.

From now on let us assume that $\mathcal{H}$ has a base of complemented elements.

**Lemma 3.** *$\mathcal{H}$ is nice if there is some $g \in \mathcal{G}$ with $B = \|g = 0 \vee g \neq 0\| \neq \top$, and for each such $g \in \mathcal{G}$ there is a subalgebra $\mathcal{H}_1 \leq \mathcal{H}$ containing the elements $g(l,m)$ with the following properties.*

1. *$\mathcal{H}_1$ has a base $\mathcal{D}$ forming a tree of height $\kappa$ such that elements on different branches are disjoint and every node is less than its predecessor. The level of a node is defined inductively as usual. For $\lambda < \kappa$ let $\mathcal{D}_\lambda$ denote the set of elements of $\mathcal{D}$ of level $\lambda$.*
2. *For every element $C$ of $\mathcal{H}_1$ there is a countably infinite set of pairwise disjoint elements less than $C$.*
3. *If an element of $\mathcal{H}_1$ is the supremum of countably many disjoint elements, then it is the supremum of countably many complemented elements. Without loss of generality these elements are pairwise disjoint.*
4. $\bigwedge\{A \in H_1 : \neg A = \bot\} = \bot$.

*Proof.* Claim 1. First we show that $B$ contains a pairwise disjoint set of elements $\mathcal{B} = \{B_i : i \in I\}$ of $H_1$ with the property that for every $C \in H_1$, if $C \vee B > B$ then there is some $i \in I$ with $B_i \leq C$.
Proof. For $\lambda < \kappa$ let $\mathcal{B}_\lambda = \{D \in \mathcal{D}_\lambda : D \leq B \text{ and } \forall\nu < \lambda,\ \forall E \in \mathcal{B}_\nu,\ E \wedge D = \bot\}$. Let $\mathcal{B} = \bigcup_{\lambda<\kappa} \mathcal{B}_\lambda$. Then $B = \bigvee \mathcal{B}$. Now let $C \in H_1$, $C \vee B > B$. Then there is some $D \in \bigcup_\lambda \mathcal{D}_\lambda$ such that $D \leq C$ but $D \not\leq B$. Then $D \neq \bot$ and since $\neg B = \bot$ ($B$ is *dense*), $D \wedge B \neq \bot$. So there is some $D' \in \mathcal{D}_j$ with $j < \kappa$ minimal such that $D' \leq D \wedge B$. Then $D' \in \mathcal{B}_j$ and $D' \leq C$ proving the claim.

Now let $\mathcal{B} = \{B_i : i \in I\}$ as in Claim 1 and for every $i \in I$ let $\{U_i^n \neq \bot : n \in \mathbb{N}^+\}$ be a set of disjoint elements with $B_i \geq \bigvee_{n\in\mathbb{N}^+} U_i^n$ (here we are using property 2) and $U_i^1 = B_i \wedge \neg \bigvee_{n>1} U_i^n$. Let $U^n = \bigvee_{i \in I} U_i^n$. Then $U^l \wedge U^m = \perp$ if $l \neq m$ and $\neg \bigvee_{n \in \mathbb{N}^+} U^n = \neg B = \perp$.

Claim 2. Let $\{U^n\}$ be a countable sequence of pairwise disjoint elements of $H_1$ as above. Then there are elements $h_1$ and $h_2$ in $\mathcal{G}$ such that

$$\bigvee_{n \in \mathbb{N}^+} U^n = \|h_1 \neq 0\| = \|h_2 \neq 0\| \text{ and } U^n \leq \|nh_2 = h_1\|.$$

Moreover every element of the form $h_i(m,n)$ is in $H_1$, where $i = 1, 2$, and $m$ and $n$ are in $\mathbb{N}$.
Proof. Using property 3. Let $\bigvee_{n \in \mathbb{N}^+} U^n = \bigvee_{i \in \omega} E_i$ where $\{E_i\}$ is a countable sequence of pairwise disjoint complemented elements. Let

$$h_1(l,m) = \begin{cases} (\bigvee_{i \leq l} E_i)^c & \text{if } m = 0 \\ E_i & \text{if } m = 2^{l-i},\ (0 \leq i \leq l) \\ \perp & \text{otherwise,} \end{cases}$$

and let

$$h_2(l,m) = \begin{cases} (\bigvee_{i \leq l} E_i)^c & \text{if } m = 0 \\ \bigvee\{E_i \wedge U^n : m = \lceil \frac{2^{l-i}}{n} \rceil,\ 0 \leq i \leq l\} & \text{if } m = \lceil \frac{2^{l-i}}{n} \rceil \text{ for some } n \text{ and } i \\ \perp & \text{otherwise.} \end{cases}$$

Note that $\|\neg nh_2 = h_1\|$ is in $H_1$ for all $n$ in $\mathbb{N}^+$. Note also that the equation $m = \lceil \frac{k}{n} \rceil$ ($m$ is the least integer greater than or equal to the rational number $k/n$) is definable in our language as $k \leq mn \wedge mn < k + n$. We show that $h_1$ and $h_2$ fulfill the requirements.
(a) $\|R(h_2)\| = \top$. ($\|R(h_1)\| = \top$ is similar.)
Proof. First of all $h_2$ is a choice sequence, since $h_2(l,m)$ and $h_2(l,n)$ are obviously disjoint if $m \neq n$, and

$$\bigvee_{m \in \omega} h_2(l,m) = (\bigvee_{i \leq l} E_i)^c \vee \bigvee_{i \leq l} \bigvee_{n \in \mathbb{N}^+} (E_i \wedge U^n) =$$

$$(\bigvee_{i \leq l} E_i)^c \vee \bigvee_{i \leq l} (E_i \wedge (\bigvee_{n \in \mathbb{N}^+} U^n)) = (\bigvee_{i \leq l} E_i)^c \vee \bigvee_{i \leq l} (E_i \wedge (\bigvee_{j \in \omega} E_j)) = \top.$$

Next we want to show that $\top \leq \|\forall k \exists x \forall p 2^k |2^p h_2(x) - h_2(x+p)| < 2^{x+p}\|$. Fix $k \in \omega$ and let $x > k$. Then for all $i \leq x$, $p \in \omega$ and $n \in \mathbb{N}^+$,

$$E_i \wedge U^n \leq \|h_2(x+p) = \left\lceil \frac{2^{x+p-i}}{n} \right\rceil \|,$$

so

$$E_i \wedge U^n \leq \|2^k |2^p h_2(x) - h_2(x+p)| < 2^k(2^p + 1)\| \wedge \|2^k(2^p+1) \leq 2^{x+p}\|.$$

Also,

$$(\bigvee_{i \leq x} E_i)^c \leq \|h_2(x) = 0\| \wedge \|h_2(x+p) \leq 2^{x+p-(x+1)}\| \wedge \|2^{x+p-(x+1)} = 2^{p-1}\|$$

for all $p$, so

$$(\bigvee_{i\leq x} E_i)^c \leq \bigwedge_{p\in\omega} \|2^k|2^p h_2(x) - h_2(x+p)| \leq 2^{k+p-1}\| \wedge \|2^{k+p-1} < 2^{x+p}\|.$$

From these

$$\top \leq \|\forall k \exists x \forall p 2^k |2^p h_2(x) - h_2(x+p)| < 2^{x+p}\|$$

follows.

$(b)$ For all $n \in \mathbb{N}^+$ and $i \in \omega$, $E_i \wedge U^n \leq \|nh_2 = h_1\|$. (Since $U^n = \bigvee_{i\in\omega}(E_i \wedge U^n)$, from this follows that for all $n \in \mathbb{N}^+$, $U^n \leq \|nh_2 = h_1\|$.)

Proof. If $i \leq l$ then

$$E_i \wedge U^n \leq \|h_2(l) = \left\lceil \frac{2^{l-i}}{n} \right\rceil \| \wedge \|h_1(l) = 2^{l-i}\|,$$

so for all $k \in \omega$, if $x > k$, $x > i$ and $2^x > n2^k$, then for all $p \in \omega$,

$$E_i \wedge U^n \leq \|2^k|nh_2(x+p) - h_1(x+p)| < n2^k\| \wedge \|n2^k < 2^{x+p}\|.$$

From this the statement follows.

$(c)$ $\bigvee_{n\in\mathbb{N}^+} U^n = \|h_1 \neq 0\| = \|h_2 \neq 0\|$.

Proof. We show only that $\bigvee_{n\in\mathbb{N}^+} U^n = \|h_2 \neq 0\|$, $\bigvee_{n\in\mathbb{N}^+} U^n = \|h_1 \neq 0\|$ is similar.

Since $(\bigvee_{i\leq x+p} E_i)^c \leq \|h_2(x+p) = 0\|$, $\|2^k h_2(x+p) \geq 2^{x+p}\| \leq \bigvee_{i\leq x+p} E_i$.

From this follows that

$$\|h_2 \neq 0\| = \bigvee_{k\in\omega} \bigvee_{x\in\omega} \bigwedge_{p\in\omega} \|2^k h_2(x+p) \geq 2^{x+p}\| \leq \bigvee_{k\in\omega} \bigvee_{x\in\omega} \bigwedge_{p\in\omega} \bigvee_{i\leq x+p} E_i =$$

$$\bigvee_{k\in\omega} \bigvee_{x\in\omega} \bigvee_{i\leq x} E_i = \bigvee_{k\in\omega} \bigvee_{x\in\omega} E_x = \bigvee_{x\in\omega} E_x = \bigvee_{n\in\mathbb{N}^+} U^n.$$

On the other hand for $n \in \mathbb{N}^+$ and $i \in \omega$ fixed, if $n < 2^y$ and $x = k = y + i$, then for all $p \in \omega$,

$$E_i \wedge U^n \leq \|2^k h_2(x+p) = 2^k \left\lceil \frac{2^{x+p-i}}{n} \right\rceil \| \wedge \|2^k \left\lceil \frac{2^{x+p-i}}{n} \right\rceil > 2^{k+x+p-i-y}\| \wedge$$
$$\|2^{k+x+p-i-y} = 2^{x+p}\|.$$

From this $\bigvee_{n\in\mathbb{N}^+} U^n \leq \|h_2 \neq 0\|$ follows.

Claim 3. For all $n \in \mathbb{N}^+$ $\|\neg nh_2 = h_1\| \leq B$.

Proof. Otherwise $\|\neg nh_2 = h_1\| \vee B > B$, so by Claim 1 $B_i \leq \|\neg nh_2 = h_1\|$ for some $i \in I$. Thus $U_i^n \leq \|\neg nh_2 = h_1\|$, but $U_i^n \leq \|nh_2 = h_1\|$, a contradiction.

Note that $\neg \bigvee_{n\in\mathbb{N}^+} \|nh_2 = h_1\| = \|h_1 = 0\| = \|h_2 = 0\| = \neg \bigvee_{n\in\mathbb{N}^+} U^n = \perp$ are obviously true.

Finally, let $H_1'$ be the set of elements $A'$ of $H_1$ with the properties $\neg A' = \perp$ and $A'$ is a supremum of a countable sequence of pairwise disjoint complemented elements. From 2 and 3 follows that for each $A$ in $H_1$ with $\neg A = \perp$ there is some $A'$ in $H_1'$ such that $A' \leq A$. So by 4, $\bigwedge H_1' = \perp$. Moreover, the above construction of $h_1$ shows that for each $A'$ in $H_1'$ there is some $g$ in $\mathcal{G}$ with $A' = \|g \neq 0\|$. From these follows that $\bigwedge_{g\in\mathcal{G}} \|g = 0 \vee g \neq 0\| = \perp$.

□

Next we give some examples (three proper classes, to be more exact) of nice Heyting algebras with bases of complemented elements.

*Example 1.* Let $\kappa$ be any cardinal with the discrete topology, and $\mathcal{H}$ be the Heyting algebra of the coperfect open sets, or the Heyting algebra of the open sets of $X = {}^{\omega}\kappa$. Then $\mathcal{H}$ with $\mathcal{H}_1 = \mathcal{H}$ fulfills the requirements of the lemma. So true first-order arithmetic is interpretable in the corresponding real-algebraic structure. Note that the open set algebra gives us Scott's model in the case of $\kappa = \omega$, and the models with the algebra of coperfect open sets generalize the one defined in [9]. This second class of models is investigated by Scowcroft.[4]

*Example 2.* Let again $\kappa$ be an infinite cardinal, but now let $\mathcal{H}$ be the Heyting algebra of the open sets of $X = {}^{\kappa}({}^{\omega}2)$ with the product topology. For each $g \in \mathcal{G}$ there is a subalgebra $\mathcal{H}_1$ isomorphic to the open-set algebra of ${}^{\omega}2$ and containing the elements $g(m, n)$ – the subalgebra of elements with support equal to the support of $g$ (see [10]). Then $\mathcal{H}$ fulfills the requirements of the lemma (this can be shown by using the isomorphism between $\mathcal{H}_1$ and the open-set algebra of ${}^{\omega}2$ and the previous example.) So again, true first-order arithmetic is interpretable in the corresponding real-algebraic structure. Note that if $\kappa > \omega$ than these models are elementarily equivalent to Krol's model defined in [7] (cf. [10]). So the real-algebraic structure of Krol's model is undecidable.

## 4 Coding Second-Order Arithmetic in Scott's Model

In this part let $\mathcal{H}$ be the open-set algebra of the space $T = {}^{\omega}\omega$ with the usual metric. Then $\mathcal{G}$ is isomorphic to Scott's model, i.e. to the structure of continuous functions from $T$ into the set $\mathbb{R}$ of real numbers with the usual operations on continuous functions.

Then for any subset $W$ of $\mathbb{N}^+$, any element $B \in H$ of the form $B = \|g = 0 \vee g \neq 0\|$ where $g \in \mathcal{G}$ and $p \in T \setminus B$ with $\{p\} \cup B \in H$ there are global real-number generators $h_1^W$ and $h_2^W$ such that $W = \{n \in \mathbb{N}^+ : p \notin \|\neg n h_2^W = h_1^W\|\}$. The construction in [4] can easily be modified to show this.

Let $\varphi(\vec{x}, \vec{S})$ be a second-order formula of the language $L' = \langle 1, +, \times\rangle$. Here $\vec{x}$ is a tuple of first-order variables, $\vec{S}$ is a tuple of second-order variables. Without loss of generality, we assume that $\varphi$ does not contain any implications. For each second-order variable $S$ let $v_S$, $u_S$ be a pair of new variables, and let $y$ be a new variable

[4]Private communication.

occuring only in the indicated places. Let $\tilde{\varphi}(\vec{x}, \vec{u}_S, \vec{v}_S, y)$ be a formula of the language of ordered rings defined inductively as follows:

1. If $\varphi$ is first-order atomic, then $\tilde{\varphi} = \varphi$;
2. $\widetilde{x \in S} = \neg x v_S = u_S \to (y = 0 \vee y \neq 0)$;
3. $\widetilde{\varphi_1 \circ \varphi_2} = \tilde{\varphi}_1 \circ \tilde{\varphi}_2$ where $\circ = \wedge, \vee$;
4. $\widetilde{\neg\varphi_1} = \tilde{\varphi}_1 \to (y = 0 \vee y \neq 0)$;
5. $\widetilde{\exists x \varphi_1(x)} = \exists x(\psi(x) \wedge \tilde{\varphi}_1(x))$;
6. $\widetilde{\forall x \varphi_1(x)} = \forall x(\psi(x) \to \tilde{\varphi}_1(x))$;
7. $\widetilde{\exists S \varphi_1(S)} = \exists u_S \exists v_S \tilde{\varphi}_1(u_S, v_S)$;
8. $\widetilde{\forall S \varphi_1(S)} = \forall u_S \forall v_S \tilde{\varphi}_1(u_S, v_S)$.

Here $\psi(x)$ is the formula defined in the previous section.

In the next lemma let $g$ be a fixed element of $\mathcal{G}$, let $B$ denote $\|g = 0 \vee g \neq 0\|$, and let $p$ be an element of $T \setminus B$ such that $\{p\} \cup B \in H$. For $k \in \mathbb{N}^+$ let $f_k \in \mathcal{G}$ be the corresponding real-number generator as before. For $W \subseteq \mathbb{N}^+$ and for the fixed $g \in \mathcal{G}$ and $p \in T$ as above, let $h_1^W$ and $h_2^W$ be arbitrary elements of $\mathcal{G}$ fulfilling the requirement at the beginning of this section.

**Lemma 4.** *Let $\varphi(\vec{x}, \vec{S})$ be a second-order $L'$-formula, $\vec{a}$ be a tuple of positive integers, and $\vec{W}$ be a tuple of subsets of $\mathbb{N}^+$.*

1. *If $\mathbb{N}^+ \models \varphi[\vec{a}, \vec{W}]$ then $\|g = 0 \vee g \neq 0\| \leq \|\tilde{\varphi}[\vec{f}_a, \vec{h}_1^W, \vec{h}_2^W, g]\|$,*
2. *$\mathbb{N}^+ \models \varphi[\vec{a}, \vec{W}]$ if and only if $p \in \|\tilde{\varphi}[\vec{f}_a, \vec{h}_1^W, \vec{h}_2^W, g]\|$*

*where the quantifiers in $\tilde{\varphi}$ range over elements in $\mathcal{G}$.*

*Proof.* 1. By a straightforward formula induction. For the case $\varphi = \forall S \varphi_1(S)$ observe that any pair $h_1, h_2 \in \mathcal{G}$ can be regarded as $h_1^W, h_2^W$ for $W = \{n \in \mathbb{N}^+ : p \notin \|\neg n h_2 = h_1\|\}$.

2. By formula induction.

$(i)$ The case when $\varphi$ is first-order atomic is obvious.

$(ii)$ If $\mathbb{N}^+ \models a \in W$ then by the assumption on $h_1^W$ and $h_2^W$, $p \notin \|\neg f_a h_2^W = h_1^W\|$, so $p \in B \cup \{p\} \leq \|\tilde{\varphi}(f_a, h_1^W, h_2^W, g)\|$.

If $p \in \|\tilde{\varphi}(f_a, h_1^W, h_2^W, g)\| = \|\neg f_a h_2^W = h_1^W\| \to B$ then, since $p \notin B$, $p \notin \|\neg f_a h_2^W = h_1^W\|$, $\mathbb{N}^+ \models a \in W$.

$(iii)$ The cases when $\varphi$ is a disjunction or a conjunction are obvious.

$(iv)$ If $\mathbb{N}^+ \models \neg\varphi$, then $\mathbb{N}^+ \not\models \varphi$, so by the induction hypothesis $p \notin \|\tilde{\varphi}\|$, so $p \in \|\tilde{\varphi}\| \to B = \|\widetilde{\neg\varphi}\|$.

If $p \in \|\widetilde{\neg\varphi}\| = \|\tilde{\varphi}\| \to B$ then, since $p \notin B$, $p \notin \|\tilde{\varphi}\|$. By the induction hypothesis $\mathbb{N}^+ \not\models \varphi$, so $\mathbb{N}^+ \models \neg\varphi$,

$(v)$ Suppose that $\mathbb{N}^+ \models \exists x\varphi(x)$. Then $\mathbb{N}^+ \models \varphi(a)$ for some $a \in \mathbb{N}^+$, so by the induction hypothesis $p \in \|\tilde{\varphi}(f_a)\|$. Also, $p \in \|\psi(f_a)\| = \top$ by Theorem 1, so $p \in \|\exists x(\psi(x) \wedge \tilde{\varphi}(x))\| = \|\widetilde{\exists x\varphi(x)}\|$.

On the other hand, since for all $f \in \mathcal{G}$ $\|\psi(f)\| = \bigvee_{a\in\mathbb{N}^+} \|f = f_a\|$ by Theorem 1, $\|\widetilde{\exists x\varphi(x)}\| = \|\exists x(\psi(x) \wedge \tilde{\varphi}(x))\| = \bigvee_{f\in\mathcal{G}} \bigvee_{a\in\mathbb{N}^+} \|f = f_a \wedge \varphi(f)\| \leq \bigvee_{f\in\mathcal{G}} \bigvee_{a\in\mathbb{N}^+} \|\varphi(f_a)\|$ by the infinite distributive law. So if $p \in \|\widetilde{\exists x\varphi(x)}\| = \bigvee_{a\in\mathbb{N}^+} \|\varphi(f_a)\|$ then $p \in \|\tilde{\varphi}(f_a)\|$ for some $a \in \mathbb{N}^+$. By the induction hypothesis $\mathbb{N}^+ \models \varphi(a)$ and $\mathbb{N}^+ \models \exists x\varphi(x)$ follows.

$(vi)$ If $\mathbb{N}^+ \models \forall x\varphi(x)$, then $\mathbb{N}^+ \models \varphi(a)$ for all $a \in \mathbb{N}^+$, so by the induction hypothesis and 1. $B \cup \{p\} \leq \|\tilde{\varphi}(f_a)\|$ for all $a \in \mathbb{N}^+$. Also, $\|\widetilde{\forall x\varphi(x)}\| = \|\forall x(\psi(x) \to \tilde{\varphi}(x))\| = \bigwedge_{f\in\mathcal{G}} \bigwedge_{a\in\mathbb{N}^+} \|f = f_a \to \tilde{\varphi}(f)\|$. Since for all $f \in \mathcal{G}$ and $a \in \mathbb{N}^+$ $\|f = f_a\| \wedge (B \cup \{p\}) \leq \|f = f_a\| \wedge \|\tilde{\varphi}(f_a)\| \leq \|\tilde{\varphi}(f)\|$, $p \in \|\widetilde{\forall x\varphi(x)}\|$ follows.

If $p \in \|\widetilde{\forall x\varphi(x)}\| = \|\forall x(\psi(x) \to \tilde{\varphi}(x))\|$ then, since $\|\psi(f_a)\| = \top$, $p \in \|\tilde{\varphi}(f_a)\|$ for all $a \in \mathbb{N}^+$. Again, by the induction hypothesis $\mathbb{N}^+ \models \varphi(a)$ for all $a \in \mathbb{N}^+$ and $\mathbb{N}^+ \models \forall x\varphi(x)$ follows.

$(vii)$ If $\mathbb{N}^+ \models \exists S\varphi(S)$ then $\mathbb{N}^+ \models \varphi(W)$ for some $W \subseteq \mathbb{N}^+$, so by the induction hypothesis $p \in \|\tilde{\varphi}(h_1^W, h_2^W)\|$ for any $h_1^W$ and $h_2^W$ with $W = \{n \in \mathbb{N}^+ : p \notin \neg\|nh_2^W = h_1^W\|\}$. Since by the second paragraph of this section such $h_1^W$ and $h_2^W$ exist, from this follows that $p \in \|\exists u_S \exists v_S \tilde{\varphi}(u_S, v_S)\|$.

If $p \in \|\exists u_S \exists v_S \tilde{\varphi}(u_S, v_S)\|$ $p \in \|\tilde{\varphi}(h_1, h_2)\|$ for some $h_1$ and $h_2$ in $\mathcal{G}$. Let $W = \{n \in \mathbb{N}^+ : p \notin \neg\|nh_2 = h_1\|\}$. Then $h_i$ $(i = 1, 2)$ can be regarded as $h_i^W$, so by the induction hypothesis $\mathbb{N}^+ \models \varphi(W)$ and $\mathbb{N}^+ \models \exists S\varphi(S)$.

$(viii)$ If $\mathbb{N}^+ \models \forall S\varphi(S)$, then $\mathbb{N}^+ \models \varphi(W)$ for all $W \subseteq \mathbb{N}^+$, so by the induction hypothesis and 1. $B \cup \{p\} \leq \|\tilde{\varphi}(h_1, h_2)\|$ for any $h_1$ and $h_2$ in $\mathcal{G}$, so $p \in \|\forall u_S \forall v_S \tilde{\varphi}(u_S, v_S)\|$.

If $p \in \|\forall u_S \forall v_S \tilde{\varphi}(u_S, v_S)\|$ then $p \in \|\tilde{\varphi}(h_1^W, h_2^W)\|$ for any $W \subseteq \mathbb{N}^+$. By the induction hypothesis $\mathbb{N}^+ \models \varphi(W)$ for all $W \subseteq \mathbb{N}^+$, so $\mathbb{N}^+ \models \forall S\varphi(S)$. □

The following lemma is from [4].

Let $\zeta(x)$ be the $L$-formula

$$\forall y((y \neq 0 \to x \neq 0) \vee y \neq 0)$$

Then $\zeta(g)$ states that a real number generator $g \in \mathcal{G}$ has at most one zero locally, i.e.

**Lemma 5.** *For all $g \in \mathcal{G}$, if $p \in \|\zeta(g)\|$ then there is an open neighborhood $U$ of $p$ such that $|z(g) \cap U| \leq 1$ where $z(g)$, the zero-set of $g$, is the complement of $\|g \neq 0\|$. Conversely, if for some open subset $U$ of $T$ $|z(g) \cap U| \leq 1$, then $U \subseteq \|\zeta(g)\|$.*

*Proof.* In [4]. □

Now for an $L'$-sentence $\varphi$ let $\hat{\varphi}$ be the sentence $\forall y(\zeta(y) \to \tilde{\varphi}(y))$. Then from the previous lemma follows the following theorem.

**Theorem 6.** *If* $\mathbb{N}^+ \models \varphi$ *then* $\|\hat{\varphi}\| = \top$, *and if* $\mathbb{N}^+ \not\models \varphi$ *then* $\|\hat{\varphi}\| = \bot$.

*Proof.* First suppose that $\mathbb{N}^+ \models \varphi$ and let $g \in \mathcal{G}$. We want to show that $\|\zeta(g)\| \leq \|\tilde{\varphi}(g)\|$. Let $p \in \|\zeta(g)\|$. Then there is a neighborhood $U$ of $p$ such that $|\zeta(g) \cap U| \leq 1$. If $p \in \|g \neq 0\|$ then $p \in B$, so by Lemma 4.1 $p \in \|\tilde{\varphi}(g)\|$. If $p \notin \|g \neq 0\|$ then $z(g) \cap U = \{p\}$ and $U \setminus \{p\} \leq B$, so $p \in U \leq \{p\} \cup B \in H$ and Lemma 4.2 applies. So $p \in \|\tilde{\varphi}(g)\|$.

If $\mathbb{N}^+ \not\models \varphi$ let $p \in T$ be an arbitrary point and let $g \in \mathcal{G}$ be a real-number generator with $z(g) = \{p\}$. Again, Lemma 4.2 applies since $\{p\} \cup B = \top \in H$, so $p \notin \|\tilde{\varphi}(g)\|$. But by Lemma 5 $\|\zeta(g)\| = \top$, so $p \notin \|\zeta(g) \to \tilde{\varphi}(g)\|$, and $p \notin \|\hat{\varphi}\|$ follows. Since $p$ was an arbitrary element, $\|\hat{\varphi}\| = \bot$. □

## 5 Coding Scott's Model in True Second-Order Arithmetic

In this section we encode the topological model for natural numbers and choice sequences based on the Heyting algebra of open subsets of $T$ in true second-order arithmetic. We regard $T$ as a subspace of $\mathbb{R}$, i.e. as the metric space of positive irrational numbers. We shall code the language $L_1$, from Section 3, in the language of second-order arithmetic. Since Scott's model (as described earlier) is definable in $L_1$, the equivalence of the real-algebraic structure of the model to true second-order arithmetic follows.

Since intervals with rational endpoints form a (clopen) base for the topology of $T$, any element $U$ of the open-set algebra of $T$ can be encoded as the set of rational intervals contained in $U$, or using the usual representation of rationals as pairs of natural numbers and a definable bijection from $\omega^4$ to $\omega$, as a set of natural numbers.

We shall need some auxiliary definitions and results (in what follows the rational intervals are closed and of positive length):

Let $\mathbb{N} \models Int(m)$ if and only if $m$ codes a rational interval (denoted by $I_m$) of $\mathbb{R}$.

Let $\mathcal{O}(S) \equiv \forall m(m \in S \to Int(m))$. If $\mathbb{N} \models \mathcal{O}(S)$ then $S$ is a code of the union $U_S$ of the intervals coded by elements of $S$. Vice versa, if $U$ is an open subset of $T$, let $S_U$ be the set of codes rational intervals $I$ with $I \cap T$ contained in $U$. Then $\mathbb{N} \models \mathcal{O}(S_U)$. Moreover, if $U$ is as above, then $U_{S_U} \cap T = U$.

We shall need a formula expressing that an interval $I_p \cap T$ is contained in an open set $U_S \cap T$.

$I_p \cap T \subseteq U_s \cap T$ if and only if for every subinterval $I_q$ of $I_p$ there is an element $r$ in $S$ with $I_q \cap I_r$ non-empty, and whenever we divide $S$ into two disjoint pieces $S_1$ and $S_2$ with $U_{S_1} \cap T$ lying on the left of $U_{S_2} \cap T$ in the natural order of $\mathbb{R}$, if $I_p$ intersects both $U_{S_1}$ and $U_{S_2}$, then they are separated by a rational point. This can be expressed using the following formulas.

$\mathbb{N} \models Meet(m,n)$ if and only if $m$ and $n$ code rational intervals with non-empty intersection in $\mathbb{R}$;

$\mathbb{N} \models Mt(m,n,p)$ if and only if $m$, $n$ and $p$ code rational intervals with $I_p$ the intersection of $I_m$ and $I_n$;

$\mathbb{N} \models Left(m,a,b)$ if and only if $\mathbb{N} \models Int(m)$ and the right endpoint of $I_m$ is less then or equal to $a/b$;

$\mathbb{N} \models Right(m,a,b)$ if and only if $\mathbb{N} \models Int(m)$ and the left endpoint of $I_m$ is greater then or equal to $a/b$;

$\mathbb{N} \models Sub(m,n)$ if and only if $m$ and $n$ code rational intervals, and $I_n$ is a subinterval of $I_m$. Finally let

$S_1 \prec S_2 \equiv \mathcal{O}(S_1) \wedge \mathcal{O}(S_1) \wedge \forall p \forall q(p \in S_1 \wedge q \in S_2 \to \exists a \exists b(Left(p,a,b) \wedge Right(q,a,b)))$, and

$In(p,S) \equiv Int(p) \wedge \forall q(Sub(p,q) \to \exists r(r \in S \wedge Meet(q,r))) \wedge \forall S_1 \forall S_2[\forall q(q \in S \leftrightarrow q \in S_1 \vee q \in S_2) \wedge S_1 \prec S_2 \wedge \exists q_1 \exists q_2(q_1 \in S_1 \wedge q_2 \in S_2 \wedge Meet(q_1,p) \wedge Meet(q_2,p)) \to \exists a \exists b \forall q_1 \forall q_2(q_1 \in S_1 \wedge q_2 \in S_2 \to Left(q_1,a,b) \wedge Right(q_2,a,b))$

**Lemma 7.** *Suppose that $\mathbb{N} \models Int(p) \wedge \mathcal{O}(S)$. Then $I_p \cap T \subseteq U_S \cap T$ if and only if $\mathbb{N} \models In(p,S)$.*

*Proof.* The implication from left to right is obvious. Suppose that $I_p \cap T \not\subseteq U_S \cap T$ and $\mathbb{N} \models In(p,S)$. Then there is some $t \in T \cap I_p$ with $t \notin U_S$. Let $I_{q_1}$ and $I_{q_2}$ be subintervals of $I_p$ on the left and on the right of $t$ respectively. Since $\mathbb{N} \models In(p,S)$, from the second conjunct follows that there are elements $r_1$ and $r_2$ of $S$ with $I_{q_i} \cap I_{r_i} \neq \emptyset$ $(i = 1,2)$. Then $I_{r_1}$ lies on the left, $I_{r_2}$ lies on the right of $t$. Let $S_1 = \{q \in S : I_q \text{ is on the left of } t\}$, $S_2 = \{q \in S : I_q \text{ is on the right of } t\}$. Then $S = S_1 \cup S_2$, $S_1 \prec S_2$, $r_i \in S_i$ $(i = 1,2)$, $Meet(r_1,p)$ and $Meet(r_2,p)$. So there is a rational point $a/b$ separating $U_{S_1}$ and $U_{S_2}$. But then there is a subinterval of $I_p$ in the interval $[a/b,t)$ which is disjoint from $U_S$, contradicting the second conjunct of $In(p,S)$. $\square$

We also need a formula expressing that for two sets $S_1$ and $S_2$ with $\mathbb{N} \models \mathcal{O}(S_i)$ $(i = 1,2)$, the intersection of the two corresponding open subsets of $T$ is empty. So let $\mathbb{N} \models Disjoint(p,q)$ if and only if $p$ and $q$ code rational intervals, and the intersection of $I_p$ and $I_q$ does not contain any open subinterval. For example $Disjoint(p,q) \equiv \exists a \exists b[(Left(p,a,b) \wedge Right(q,a,b)) \vee (Left(q,a,b) \wedge Right(p,a,b))]$ works.

Let $S_1|S_2 \equiv \mathcal{O}(S_1) \wedge \mathcal{O}(S_2) \wedge \forall p \forall q(p \in S_1 \wedge q \in S_2 \to Disjoint(p,q))$. Then, if $\mathbb{N} \models S_1|S_2$ then $U_{S_1} \cap U_{S_2} \cap T = \emptyset$. Vice versa, if $U_1 \cap U_2 \cap T = \emptyset$ then $\mathbb{N} \models S_{U_1}|S_{U_2}$.

Next we encode choice sequences as follows. We shall define a formula $\Xi(S)$ with the property that $\mathbb{N} \models \Xi(S)$ if and only if $S$ consists of codes of triples of natural numbers denoted by $\langle m, n, p\rangle$ such that there is some choice sequence $\xi$ with $\{p : \langle m, n, p\rangle \in S\}$ being a code for $\xi(m, n)$. We have to express the fact that if $n_1 \neq n_2$ then $\{p : \langle m, n_1, p\rangle \in S\}$ and $\{q : \langle m, n_2, q\rangle \in S\}$ code disjoint open sets for all $m$ (i.e. $\mathbb{N} \models \{p : \langle m, n_1, p\rangle \in S\} | \{q : \langle m, n_2, q\rangle \in S\}$), and that the set $\{p : \exists n\langle m, n, p\rangle \in S\}$ codes $T$ for every $m$. So let

$\Xi(S) \equiv \forall r[r \in S \to \exists m\exists n\exists p(r = \langle m, n, p\rangle \wedge Int(p))] \wedge \forall m\forall n_1\forall n_2[n_1 \neq n_2 \to \forall p\forall q(\langle m, n_1, p\rangle \in S \wedge \langle m, n_2, q\rangle \in S \to Disjoint(p, q))] \wedge \forall m\exists S'[\forall p(p \in S' \leftrightarrow \exists n\langle m, n, p\rangle \in S) \wedge \forall q(Int(q) \to In(q, S'))]$

For $S \subseteq \mathbb{N}$ if $\mathbb{N} \models \Xi(S)$, let $\xi_S : \omega\times\omega \to H$ ($H$ is the set of open-sets of $T$) be defined as follows. $\xi_S(m, n)$ is the intersection of the open set coded by $\{p : \langle m, n, p\rangle \in S\}$ with $T$. Then $\xi_S$ is a choice sequence, and we say that $S$ codes $\xi_S$. Vice versa, if $\xi$ is a choice sequence, let $S_\xi$ consist of the codes of triples $\langle m, n, p\rangle$ where $I_p \cap T$ is a clopen subinterval of $\xi(m, n)$. Then $\mathbb{N} \models \Xi(S_\xi)$ and $\xi_{S_\xi} = \xi$.

Let $\varphi(\vec{\xi}, \vec{m})$ be an $L_1$-sentence with choice sequence and natural number parameters shown. We define inductively a second-order arithmetic formula $\bar{\varphi}(\vec{S_\xi}, \vec{m}, S)$ containing the parameters shown, where $S_\xi$ is any subset of $\mathbb{N}$ coding the choice sequence $\xi$, $m \in \mathbb{N}$ and $S$ codes the open set $U_S$ such that $U_S \cap T = \|\varphi(\vec{\xi}, \vec{m})\|$.

1. If $\varphi(\vec{m})$ is atomic and does not contain any choice sequence parameters then let $\bar{\varphi}(\vec{m}, S) \equiv (\varphi(\vec{m}) \wedge \forall p(p \in S \leftrightarrow Int(p))) \vee (\neg\varphi(\vec{m}) \wedge \forall p(p \notin S))$.
2. If $\varphi \equiv \xi(m) = n$ then let $\bar{\varphi}(S_\xi, m, n, S) \equiv \forall p(p \in S \leftrightarrow \langle m, n, p\rangle \in S_\xi)$.
3. If $\varphi \equiv \varphi_1 \vee \varphi_2$ then let $\bar{\varphi}(S) \equiv \exists S_1\exists S_2(\bar{\varphi}_1(S_1) \wedge \bar{\varphi}_2(S_2) \wedge \forall p(p \in S \leftrightarrow p \in S_1 \vee p \in S_2))$.
4. If $\varphi \equiv \varphi_1 \wedge \varphi_2$ then let $\bar{\varphi}(S) \equiv \exists S_1\exists S_2[\bar{\varphi}_1(S_1) \wedge \bar{\varphi}_2(S_2) \wedge \forall p(p \in S \leftrightarrow \exists q_1\exists q_2(q_1 \in S_1 \wedge q_2 \in S_2 \wedge Sub(q_1, p) \wedge Sub(q_2, p)))]$.
5. If $\varphi \equiv \varphi_1 \to \varphi_2$ then let $\bar{\varphi}(S) \equiv \exists S_1\exists S_2[\bar{\varphi}_1(S_1) \wedge \bar{\varphi}_2(S_2) \wedge \forall p(p \in S \leftrightarrow \forall q(q \in S_1 \to \exists r(Mt(p, q, r) \wedge In(r, S_2))))]$.
6. If $\varphi \equiv \exists x\varphi_1(x)$ then let $\bar{\varphi}(S) \equiv \forall p(p \in S \leftrightarrow \exists m\exists S'(\bar{\varphi}_1(m, S') \wedge p \in S'))$.
7. If $\varphi \equiv \forall x\varphi_1(x)$ then let $\bar{\varphi}(S) \equiv \forall p(p \in S \leftrightarrow \forall m\forall S'(\bar{\varphi}_1(m, S') \to In(p, S')))$.
8. If $\varphi \equiv \exists\alpha\varphi_1(\alpha)$ then let $\bar{\varphi}(S) \equiv \forall p(p \in S \leftrightarrow \exists S^\alpha\exists S'(\Xi(S^\alpha) \wedge \bar{\varphi}_1(S^\alpha, S') \wedge p \in S'))$.
9. If $\varphi \equiv \forall\alpha\varphi_1(\alpha)$ then let $\bar{\varphi}(S) \equiv \forall p(p \in S \leftrightarrow \forall S^\alpha\forall S'(\Xi(S^\alpha) \wedge \bar{\varphi}_1(S^\alpha, S') \to In(p, S')))$.

**Lemma 8.** *Let $\varphi(\vec{\xi}, \vec{m})$ be an $L_1$-sentence with choice sequence and natural number parameters shown and let $S_\xi$ be any subset of $\mathbb{N}$ coding the choice sequence $\xi$. Then*

1. *there is some $S \subseteq \mathbb{N}$ such that $\mathbb{N} \models \bar{\varphi}(\vec{S_\xi}, \vec{m}, S)$, and*
2. *for $S \subseteq \mathbb{N}$, if $\mathbb{N} \models \bar{\varphi}(\vec{S_\xi}, \vec{m}, S)$ then $S$ codes the open set $U_S$ such that $U_S \cap T = \|\varphi(\vec{\xi}, \vec{m})\|$.*

*Proof.* By formula induction. 1. is obvious.

As an illustration, we show only 2. for conjunction, implication and existential sequence variable. The other cases are similar.

$(a)$ $\varphi \equiv \varphi_1 \wedge \varphi_2$ and $\mathbb{N} \models \bar{\varphi}(S)$. Our claim is that $U_S \cap T = \|\varphi(\vec{\xi}, \vec{m})\|$.

Let $t \in U_S \cap T$. Then $t \in I_p \cap T$ for some $p \in S$. Since $\mathbb{N} \models \bar{\varphi}(S)$, there are $S_1$, $S_2 \subseteq \mathbb{N}$ such that $\mathbb{N} \models \bar{\varphi}_i(S_i)$ $(i = 1, 2)$ and there are elements $q_i \in S_i$ $(i = 1, 2)$ such that $Sub(q_i, p)$ $(i = 1, 2)$, i.e. $I_p \subseteq I_{q_1} \cap I_{q_2}$. Then $t \in I_p \cap T \subseteq I_{q_1} \cap I_{q_2} \cap T \subseteq U_{S_1} \cap U_{S_2} \cap T = \|\varphi_1\| \cap \|\varphi_2\|$, where the last equation follows by the induction hypothesis. So $T \cap U_S \subseteq \|\varphi\|$.

Now suppose that $t \in \|\varphi_1\| \cap \|\varphi_2\|$. By 1. and the induction hypothesis there are subsets $S_i \subseteq \mathbb{N}$ $(i = 1, 2)$ with $\mathbb{N} \models \bar{\varphi}_i(S_i)$ and $U_{S_i} \cap T = \|\varphi_i\|$. From this follows that there are $q_i \in S_i$ $(i = 1, 2)$ with $t \in I_{q_1} \cap I_{q_2} \cap T$. Then $I_{q_1} \cap I_{q_2}$ is a rational interval of positive length coded by some number $p$, so $t \in I_p \subseteq I_{q_1} \cap I_{q_2}$. But then $p \in S$ follows, so $t \in U_S \cap T$.

$(b)$ $\varphi \equiv \varphi_1 \to \varphi_2$ and $\mathbb{N} \models \bar{\varphi}(S)$. Again, first assume that $t \in U_S \cap T$, i.e. that $t \in I_p \cap T$ for some $p \in S$. Then there are $S_i \subseteq \mathbb{N}$ such that $\mathbb{N} \models \bar{\varphi}_i(S_i)$ $(i = 1, 2)$ and by Lemma 7, for all $q$ in $S_1$, $I_q \cap I_p \cap T \subseteq U_{S_2} \cap T$. Then $I_p \cap U_{S_1} \cap T \subseteq U_{S_2} \cap T$. By the induction hypothesis $U_{S_i} \cap T = \|\varphi_i\|$, so $I_p \cap T \cap \|\varphi_1\| \subseteq \|\varphi_2\|$. Then $I_p \cap T \subseteq \|\varphi_1 \to \varphi_2\| = \|\varphi\|$ and $t \in \|\varphi\|$ follows.

Next suppose that $t \in \|\varphi\|$. Then there is some interval $I_p$ with $t \in I_p \cap T \subseteq \|\varphi\|$. Again, by the inductive hypothesis there are subsets $S_i \in \mathbb{N}$ such that $\mathbb{N} \models \bar{\varphi}_i(S_i)$ and $U_{S_i} \cap T = \|\varphi_i\|$. It is enough to show that for all $q$ in $S_1$, $I_q \cap I_p \cap T \subseteq U_{S_2} \cap T$, so let $t' \in I_q \cap I_p \cap T$. Then $t' \in U_{S_1} \cap T$, so $t' \notin (U_{S_1} \cap T)^c$. But $t' \in I_p \cap T \subseteq (U_{S_1} \cap T)^c \cup (U_{S_2} \cap T)$, so $t' \in (U_{S_2} \cap T)$.

$(c)$ Let us assume that $\varphi \equiv \exists\alpha\varphi_1(\alpha)$ and $\mathbb{N} \models \bar{\varphi}(S)$.

Let $t \in I_p \cap T$ for some $p \in S$. Then there are $S^\alpha$, $S' \subseteq \mathbb{N}$ such that $\mathbb{N} \models \Xi(S^\alpha) \wedge \bar{\varphi}_1(S^\alpha, S') \wedge p \in S'$. Then $S^\alpha$ codes the choice sequence $\xi_{S^\alpha}$, so by the induction hypothesis $U_{S'} \cap T = \|\varphi_1(\xi_{S^\alpha})\|$, and since $p \in S'$, $t \in I_p \cap T \subseteq \|\varphi_1(\xi_{S^\alpha})\| \subseteq \|\varphi\|$, i.e. $U_S \cap T \subseteq \|\varphi\|$.

Finally let $t \in \|\varphi\| = \bigcup_\xi \|\varphi_1(\xi)\|$, i.e. $t \in \|\varphi_1(\xi)\|$ for some choice sequence $\xi$. Let $S_\xi \subseteq \mathbb{N}$ code $\xi$. Then $\mathbb{N} \models \Xi(S_\xi)$. By the induction hypothesis there is some $S' \subseteq \mathbb{N}$ with $\mathbb{N} \models \bar{\varphi}_1(S_\xi, S')$ and $U_{S'} \cap T = \|\varphi_1(\xi)\|$. Then $t \in U_{S'} \cap T$, so $t \in I_p \cap T$ for some $p \in S'$. Then $p \in S$ and $t \in U_S \cap T$ follows. $\square$

The following theorem is an immediate corollary.

**Theorem 9.** *Let $\varphi$ be a parameter-free $L_1$ sentence. Then $\|\varphi\| = T$ in the model if and only if* $\mathbb{N} \models \exists S(\bar{\varphi}(S) \wedge \forall p(Int(p) \to In(p, S)))$.

So the real-algebraic structure of Scott's model is equivalent to true second-order arithmetic.

## 6 Appendix

Here we give a randomized version of the creative subject argument leading to a strengthening of Kripke's schema.[5]

As we shall see, from this version the sentence

$$\forall y \exists u \exists v (\forall n \in \mathbb{N}^+ (\neg nv = u \to y = 0 \vee y \neq 0) \wedge \neg v = 0 \wedge \neg\neg\exists n \in \mathbb{N}^+ (nv = u))$$

used in Section 2 follows. We have put this exposition into an appendix, because even the original argument behind Kripke's schema is regarded as "controversial" in the literature (cf. e.g. the chapter 'Controversial Subjects' in [5], or pages 323 – 325 in [2]), and the fact that we use that argument with a "randomly chosen number" certainly will not facilitate its acceptance.

It is also conceivable that the above sentence has other, maybe more palatable justifications. However, we would like to point out that the above mentioned strengthening of Kripke's schema is true in the models described in examples 1 and 2 above.

So we propose the following axiom schema of second-order arithmetic:

$$S(\varphi) \equiv \exists \beta [(\exists n(\beta(n) > 0) \to \varphi) \wedge (\neg \exists n(\beta(n) > 0) \to \neg\varphi) \wedge$$

$$\forall k > 0 (\neg \exists n(\beta(n) = k) \to \varphi \vee \neg\varphi) \wedge \forall k > 0 \forall n((\beta(n) = k) \to \forall m \geq n(\beta(m) = k))]$$

Note that from $S$ the schema $KS$ in [3], page 132, obviously follows.

The following argument in favor of the schema $S$ (adapted from [3]) is a modification of the usual exposition justifying various versions of Kripke's schema.

Let us suppose that the formula $\varphi$ is given. The values $\beta(n)$ are obtained in the following way: We fix some sequence of moments of time 0, 1, 2, etc. If at moment $n$ $\varphi$ has not been proved yet, we set $\beta(n) = 0$. Otherwise we set $\beta(n) = k$ where $k$ is a random number chosen at the moment $\varphi$ is proved. The point here is that there is no information about $k$ before $\varphi$ has been proved. So if $\beta(n) = k$ for some $n$ and $k > 0$, then $\beta(m) = k$ for every $m \geq n$. So the last conjunct in the matrix of the schema $S$ is true. If at some moment $\neg\varphi$ is proved, we set $\beta(n) = 0$ for every $n$. As usual, we assume that at each moment either $\varphi$ is proven, or it is not, so we can use the above case analysis, and $\beta(n) = 0 \vee \neg\beta(n) = 0$ is true for every $n$.

Now if $\exists n(\beta(n) > 0)$, then this means that $\varphi$ was proved at some moment, so $\varphi$ is true.

If $\varphi$ is true, then it is plausible to assume that it is impossible that $\varphi$ will never be proved. I.e. $\varphi \to \neg\neg\exists n(\beta(n) > 0)$ which is equivalent to $\neg\exists n(\beta(n) > 0) \to \neg\varphi$.

Finally let us assume that $\neg\exists n(\beta(n) = k)$ for some $k > 0$, i.e. $\forall n(\neg\beta(n) = k)$. By the way $\beta$ is constructed it is plausible to assume that there are only two ways to obtain this result. The first one is that $\varphi$ was proved at some moment $m$ (so it is true), and the random number chosen at that moment is not $k$. The second possibility is that $\neg\varphi$ was proved (so it is true), in which case we know that $\forall n(\beta(n) = 0 \neq k)$. So $\forall k > 0(\neg\exists n(\beta(n) = k) \to \varphi \vee \neg\varphi)$.

[5]This is a joint work with Philip Scowcroft.

Next we show how

$$\forall y \exists u \exists v (\forall n \in \mathbb{N}^+ (\neg nv = u \to y = 0 \vee y \neq 0) \wedge \neg v = 0 \wedge \neg\neg \exists n \in \mathbb{N}^+ (nv = u))$$

follows from the schema $S$.

Fix a real number $y$ and let $\varphi \equiv y = 0 \vee y \neq 0$. Note that in this case $\neg\varphi$ is false. Let $\beta$ be the choice sequence given by $S(\varphi)$.

We define two Cauchy sequences $(u_n)$ and $(v_n)$ as follows.

If $\forall l \leq n(\beta(l) = 0)$, let $u_n = v_n = 0$.

If $m = \mu n(\beta(n) > 0)$, then for all $n \geq m$ let $u_n = 1/m$ and $v_n = 1/m\beta(m)$.

We claim that $u$ and $v$ defined by these sequences have the required properties.

Suppose first that $v = 0$ and $\exists n(\beta(n) > 0)$. Then $m = \mu n(\beta(n) > 0)$ exists. But then $v = 1/m\beta(m) \neq 0$, a contradiction. So if $v = 0$, then $\neg\exists n(\beta(n) > 0)$. But then $\neg(y = 0 \vee y \neq 0)$, a contradiction again, since $\neg\neg(y = 0 \vee y \neq 0)$ is true for every real number $y$. So $\neg v = 0$ follows.

Next suppose that $\neg\exists k \in \mathbb{N}^+(kv = u)$ and $\exists n(\beta(n) > 0)$. Then again we get a contradiction, since if $m = \mu n(\beta(n) > 0)$ and $p = \beta(m)$, then $pv = u$. So, as before, $\neg\neg\exists k \in \mathbb{N}^+(kv = u)$ follows.

Finally, let us assume that $\neg kv = u$ for some $k \in \mathbb{N}^+$. Then $\neg\exists n(\beta(n) = k)$ and $\varphi \vee \neg\varphi$ follows. Since $\neg\varphi$ is false, $\varphi$, i.e. $y = 0 \vee y \neq 0$ follows.

In the last part of the paper we show that the examples of Section 3 are models of the schema $S$. The proof is a modification of Lemma 3.

Let us suppose that $\mathcal{H}$ is nice and the strong Kripke's schema holds in the corresponding model, i.e. for every $\varphi$ not containing $g$ free $\exists g(\varphi \equiv g \neq 0)$. Here $g \in \mathcal{G}$. This schema is valid in every structure mentioned in the examples.

Fix the formula $\varphi$ and let $\mathcal{H}_1$ be a subalgebra of $\mathcal{H}$ described in Lemma 3 containing the elements $g(m, n)$, where $g$ is the global real-number generator given by the schema above. Then $\|\varphi\|$ and $\|\neg\varphi\|$ are elements of $\mathcal{H}_1$.

First, $\|\varphi\|$ is the supremum of pairwise disjoint elements $\mathcal{B} = \{B_i \mid i \in I\}$ of $\mathcal{H}_1$ with the property that for every $C \in \mathcal{H}_1$, if $C \vee \|\varphi \vee \neg\varphi\| > \|\varphi \vee \neg\varphi\|$ then $B_i < C$ for some $i \in I$.

The construction of $\mathcal{B}$ is the same as before (we use the notation of Lemma 3). For $\lambda < \kappa$ let $\mathcal{B}_\lambda = \{D \in \mathcal{D}_\lambda : D \leq \|\varphi\| \text{ and } \forall\nu < \lambda,\ \forall E \in \mathcal{B}_\nu,\ E \wedge D = \perp\}$. Let $\mathcal{B} = \bigcup_{\lambda<\kappa} \mathcal{B}_\lambda$. Then $\|\varphi\| = \bigvee \mathcal{B}$.

We can show that $\mathcal{B} = \{B_i \mid i \in I\}$ has the required property as before, using the fact that if $D \wedge \|\varphi\| = \perp$, then $D \leq \|\neg\varphi\| \leq \|\varphi \vee \neg\varphi\|$. So if $C \in H_1$ and $C \vee \|\varphi \vee \neg\varphi\| > \|\varphi \vee \neg\varphi\|$, then there is some $D \in \bigcup_\lambda \mathcal{D}_\lambda$ such that $D \leq C$ but $D \not\leq \|\varphi \vee \neg\varphi\|$. Then $D \wedge \|\varphi\| \neq \perp$. So there is some $D' \in \mathcal{D}_j$ with $j < \kappa$ minimal such that $D' \leq D \wedge \|\varphi\|$. Then $D' \in \mathcal{B}_j$ follows.

Again, for every $i \in I$ let $\{U_i^n \neq \perp : n \in \mathbb{N}^+\}$ be a set of disjoint elements with $B_i \geq \bigvee_{n \in \mathbb{N}^+} U_i^n$ and let $U^n = \bigvee_{i \in I} U_i^n$. Then $U^l \wedge U^m = \perp$ if $l \neq m$. Without loss of generality we assume that $\neg \bigvee_{n \in \mathbb{N}^+} U^n \wedge \|\varphi\| = \perp$, i.e. that $\neg \bigvee_{n \in \mathbb{N}^+} U^n \leq \|\neg\varphi\|$.

Finally, as before, let $\bigvee_n U^n = \bigvee_i E_i$ where $\{E_i\}$ is a countable sequence of pairwise disjoint complemented elements.

Now we are ready to define the required choice sequence $\beta$. Let

$\beta(l,m) = \begin{cases} (\bigvee_{i\le l} E_i)^c & \text{if } m = 0 \\ \bigvee_{i\le l}(E_i \wedge U^m) & \text{if } m > 0 \end{cases}$

It is easy to check that $\beta$ is a choice sequence (cf. the proof of the same fact about $h_2$).

Next we check that $\beta$ has the required properties.

1. $\|\exists n(\beta(n) > 0)\| \le \bigvee_n \bigvee_m U^m \le \|\varphi\|$

2. Since by assumption $\neg \bigvee_{m\in\mathbb{N}^+} U^m \le \|\neg\varphi\|$, to show that $\|\neg\exists n(\beta(n) > 0)\| \le \|\neg\varphi\|$, it is enough to check that $\|\neg\exists n(\beta(n) > 0)\| \le \neg \bigvee_{m\in\mathbb{N}^+} U^m$, i.e. that

$$\|\neg\exists n(\beta(n) > 0)\| \wedge \bigvee_{m\in\mathbb{N}^+} U^m = \bot.$$

But $\|\neg\exists n(\beta(n) > 0)\| \wedge \bigvee_{m\in\mathbb{N}^+} U^m = \|\neg\exists n(\beta(n) > 0)\| \wedge \bigvee_{m\in\mathbb{N}^+} \bigvee_i (U^m \wedge E_i) \le \|\neg\exists n(\beta(n) > 0)\| \wedge \bigvee_{m\in\mathbb{N}^+} \bigvee_i \|\beta(i) = m\| \le \|\neg\exists n(\beta(n) > 0)\| \wedge \|\exists n(\beta(n) > 0)\| = \bot$.

3. Let $l > 0$. We want to show that $\|\neg\exists n(\beta(n) = l)\| \le \|\varphi \vee \neg\varphi\|$. If not, then $\|\neg\exists n(\beta(n) = l)\| \vee \|\varphi\vee\neg\varphi\| > \|\varphi\vee\neg\varphi\|$, so by the assumptions, $B_i \le \|\neg\exists n(\beta(n) = l)\|$ for some $i \in I$. Thus $U_i^l \le \|\neg\exists n(\beta(n) = l)\|$, but $U_i^l \le U^l \wedge \bigvee_j E_j = \exists n(\beta(n) = l)$, a contradiction.

4. Since $\beta(l,m) \le \beta(l+1,m)$ for all $l$ and $m > 0$, $\forall k > 0(\forall n(\beta(n) = k) \to \forall m \ge n(\beta(m) = k))$ is obvious. □

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