DECIDABILITY OF SCOTT'S MODEL AS AN ORDERED $\mathbb{Q}$-VECTORSPACE

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Abstract. Let $L = \langle <, +, h_q, 1\rangle_{q\in\mathbb{Q}}$ where $\mathbb{Q}$ is the set of rational numbers and h_q is a one-place function symbol corresponding to multiplication by q. Then the L-theory of Scott's model for intuitionistic analysis is decidable.

§**1. Introduction.** Scott defined his topological model for intuitionistic analysis in [4]. In [5] he showed that the universal fragment of the order structure is decidable. Later Scowcroft proved that the truth in the model of certain $\forall\exists$-sentences of the language of ordered rings is also decidable (in [6], [7]). In [8] he showed among other facts that if certain sentences are true in Scott's model, then they are provable constructively.

In this paper let $L = \langle <, +, h_q, 1\rangle_{q\in\mathbb{Q}}$, where $\mathbb{Q}$ is the set of rational numbers and h_q is a one-place function symbol corresponding to multiplication by q. We shall show using Rabin's results on the decidability of some monadic second-order theories of several successor functions (cf. [3]), that the set of L-sentences true in the model is decidable.

The proof is classical, mainly because Rabin's results are classical. However, the proof of our main lemma (Proposition 1) which shows that every L-sentence is equivalent to one of special form, uses only results from Bishop's constructive analysis (cf. [1]) and is entirely constructive.

§**2. Scott's model.** Let us sketch briefly those features of Scott's model $\mathcal{R}$ for the intuitionistic continuum which will be relevant in this paper.

Let T be Baire space (the space of infinite sequences of natural numbers or, equivalently, the space of irrational numbers with the usual metric). The elements of the model $\mathcal{R}$ are the continuous functions from T into the set $\mathbb{R}$ of real numbers. Then $\mathcal{R}$ is a model for the language $L = \langle <, +, h_q, 1\rangle_{q\in\mathbb{Q}}$ with $+_{\mathcal{R}}$ and $(h_q)_{\mathcal{R}}$ being

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the usual operations on continuous functions and $1_{\mathcal{R}}$ the constant function with value 1.

The truth value $\|\phi\|$ of a formula $\phi \in Form_{L\cup\mathcal{R}}$ with parameters from $\mathcal{R}$ (and without free variables) is an open subset of T defined inductively as follows:

(i) $\quad \|t_1 < t_2\| = \{t \in T \mid \mathbb{R} \models t_1(f_1(t), \ldots, f_n(t)) < t_2(f_1(t), \ldots, f_n(t))\}$

where $t_1(f_1, \ldots, f_n)$ and $t_2(f_1, \ldots, f_n)$ are terms of the language with parameters among $f_1, \ldots, f_n$ and with no free variables;

(ii) $\quad \|\phi_1 \vee \phi_2\| = \|\phi_1\| \cup \|\phi_2\|$;

(iii) $\quad \|\phi_1 \wedge \phi_2\| = \|\phi_1\| \cap \|\phi_2\|$;

(iv) $\quad \|\phi_1 \to \phi_2\| = int\big((T \setminus \|\phi_1\|) \cup \|\phi_2\|\big)$;

(v) $\quad \|\exists x \phi(x)\| = \bigcup_{f\in\mathcal{R}} \|\phi(f)\|$;

(vi) $\quad \|\forall x \phi(x)\| = int \bigcap_{f\in\mathcal{R}} \|\phi(f)\|$.

Note that we shall use $=$ only as an abbreviation: $f = g \leftrightarrow \neg(f < g \vee f > g)$.

A sentence ψ is true in the model — $\mathcal{R} \models \psi$ — just in case $\|\psi\| = T$.

§3. A constructive "special form" theorem.

Proposition 1. *Let $\phi \in Sent_L$ be a sentence. Then there is a sentence $\psi \in Sent_L$ such that*

(i) $\quad$ *ψ can be obtained from ϕ constructively.*
(ii) $\quad$ *Each variable w of ψ occurs only in atomic subformulas of the form $w > 0$. (ψ is of special form).*
(iii) $\quad$ *$\mathbb{R} \models \phi \leftrightarrow \psi$ where $\mathbb{R}$ denotes the structure of constructive reals from [1].*

The proof is based on the following lemma (whose proof is sketched in the appendix):

Lemma 4. *Let $\underline{t} = t_1, \ldots, t_k$, $\underline{y} = y_1, \ldots, y_k$, $\underline{y'} = y'_1, \ldots, y'_k$ be disjoint lists of distinct variables and let $\Theta_2(\underline{t}, \underline{y}, \underline{y'})$ be the conjunction of the following formulas (for $1 \le i, j \le k$):*

(α) $\quad \neg(y_i > 0 \wedge y'_i > 0)$
(β) $\quad y'_i > 0 \to t_i < t_j \vee y'_j > 0$
(γ) $\quad y_j > 0 \to t_i < t_j \vee y_i > 0$
(δ) $\quad t_i < t_j \to y_j > 0 \vee y'_i > 0$

Then

$$\mathbb{R} \models \forall\underline{t}\forall\underline{y}\forall\underline{y'}\big(\Theta_2(\underline{t}, \underline{y}, \underline{y'}) \to \exists x \bigwedge_{i=1}^{k} (x < t_i \leftrightarrow y_i > 0) \wedge (x > t_i \leftrightarrow y'_i > 0)\big)$$

Note that the implication from right to left trivially holds.

Proof of Proposition 1.

Let $\phi(\underline{z}, x_1, \ldots, x_n) \in Form_L$ be a formula such that if w is a bound variable occuring in ϕ then it appears only in atomic subformulas of the form $w > 0$. We shall show that there are formulas $\psi_1(\underline{z}, \underline{y})$, $\psi_2(\underline{z}, \underline{y})$ such that $\mathbb{R} \models \forall \underline{z}(\forall \underline{x}\phi(\underline{z}, \underline{x}) \leftrightarrow \forall \underline{y}\psi_1(\underline{z}, \underline{y}))$ and $\mathbb{R} \models \forall \underline{z}(\exists \underline{x}\phi(\underline{z}, \underline{x}) \leftrightarrow \exists \underline{y}\psi_2(\underline{z}, \underline{y}))$. Moreover if v is a bound variable in $\forall \underline{y}\psi_1(\underline{z}, \underline{y}))$ or in $\exists \underline{y}\psi_2(\underline{z}, \underline{y}))$ then v occurs only in atomic subformulas of the form $v > 0$. Repeating this procedure from the innermost quantifiers outwards, we obtain Proposition 1.

Without loss of generality all other atomic subformulas of ϕ — i.e., those not containing bound variables — have the form $t(\underline{z}, x_1, \ldots, x_{l-1}) < x_l$, $t(\underline{z}, x_1, \ldots, x_{l-1}) > x_l$, or do not contain any variable from the list $\underline{x}$. (Notice that every atomic formula is equivalent to one of the above form.) We say that these atomic formulas are of standard form, and let the set of them in ϕ be $\Delta(\underline{z}, \underline{x})$. For $0 \leq l \leq n$ let

$\Delta'_l(\underline{z}, x_1, \ldots, x_l) := \{t(\underline{z}, x_1, \ldots, x_{l-1}) < x_l, t(\underline{z}, x_1, \ldots, x_{l-1}) > x_l \mid t(\underline{z}, \underline{x}) > x_l$ or $t(\underline{z}, \underline{x}) < x_l$ is in $\Delta(\underline{z}, \underline{x})\}$.

Let $\Delta_n(\underline{z}, \underline{x}) := \Delta'_n(\underline{z}, \underline{x})$ and suppose that $\Delta_i(\underline{z}, \underline{x})$ is defined for $i > l$ where $0 \leq l < n$. Then let

$\Delta_l(\underline{z}, \underline{x}) := \Delta'_l(\underline{z}, x_1, \ldots, x_l) \cup \{t(\underline{z}, x_1, \ldots, x_{l-1}) < x_l,\ t(\underline{z}, x_1, \ldots, x_{l-1}) > x_l \mid$ there are $\delta_i(\underline{z}, \underline{x}) \equiv t_i(\underline{z}, \underline{x})\binom{<}{>}x_m$ $(i = 1, 2)$ in $\Delta_m(\underline{z}, \underline{x})$ for some $l < m \leq n$ such that $t(\underline{z}, \underline{x})\binom{<}{>}x_l$ is the standard form of $t_1(\underline{z}, \underline{x}) < t_2(\underline{z}, \underline{x})\}$.

Note that the atomic formulas in $\Delta_0(\underline{z}, \underline{x})$ do not contain variables from $\underline{x}$.

Let $\bar{\Delta}(\underline{z}, \underline{x}) = \bigcup_{i=0}^{n} \Delta_i(\underline{z}, \underline{x})$, a finite set, and for each $\delta_i(\underline{z}, \underline{x}) \in \bar{\Delta}(\underline{z}, \underline{x})$ let y_i be a new variable.

Let $\hat{\phi}(y) = \phi[y_i > 0/\delta_i(\underline{z}, \underline{x})]$, the formula obtained by substituting the atomic formula $y_i > 0$ for the subformula $\delta_i(\underline{z}, \underline{x})$ in ϕ. (So $\phi(\underline{z}, \underline{x}) = \hat{\phi}[\delta_i(\underline{z}, \underline{x})/y_i > 0)]$.) Let $Ax(\underline{z}, \underline{y})$ be the conjunction of the following formulas:

(*i*) If y_i corresponds to $t_i(\underline{z}, \underline{x}) \underset{(>)}{<} x_m \in \bar{\Delta}(\underline{z}, \underline{x})$ then $t_i(\underline{z}, \underline{x}) \underset{(<)}{>} x_m \in \bar{\Delta}(\underline{z}, \underline{x})$. Let y'_i be the corresponding variable. Then $\neg(y_i > 0 \wedge y'_i > 0)$ is a conjunct.

(*ii*) If y_i (y'_i) corresponds to $t_i(\underline{z}, \underline{x}) > x_m$ $(t_i(\underline{z}, \underline{x}) < x_m$ resp.) and y_j (y'_j) corresponds to $t_j(\underline{z}, \underline{x}) > x_m$ $(t_j(\underline{z}, \underline{x}) < x_m$ resp.) then the standard form of $t_i(\underline{z}, \underline{x}) < t_j(\underline{z}, \underline{x})$ is in $\bar{\Delta}(\underline{z}, \underline{x})$. Let y_{ij} be the corresponding variable. Then $(y_j > 0 \rightarrow y_{ij} > 0 \vee y_i > 0) \wedge (y'_i > 0 \rightarrow y_{ij} > 0 \vee y'_j > 0) \wedge (y_{ij} > 0 \rightarrow y_j > 0 \vee y'_i > 0)$ is a conjunct.

(*iii*) If y corresponds to some $\delta(\underline{z}) \in \Delta_0(\underline{z})$ then $y > 0 \leftrightarrow \delta(\underline{z})$ is a conjunct.

Then

$$(1) \qquad \mathbb{R} \models \forall \underline{z}\forall \underline{x} Ax[\delta_i(\underline{z}, \underline{x})/y_i > 0]$$

Claim 1. Let $\psi_1(\underline{z}, \underline{y}) \equiv Ax(\underline{z}, \underline{y}) \rightarrow \hat{\phi}(\underline{y})$ Then $\mathbb{R} \models \forall \underline{z}(\forall \underline{x}\phi(\underline{z}, \underline{x}) \leftrightarrow \forall \underline{y}\psi_1(\underline{z}, \underline{y}))$

Proof. Let $\bar{\underline{z}}$ be a tuple of elements of $\mathbb{R}$ of the same length as $\underline{z}$.

(i) Suppose $\mathbb{R} \models \forall \underline{y}\psi_1(\bar{\underline{z}}, \underline{y})$ and let $\bar{\underline{x}}$ be a tuple in $\mathbb{R}$ of length n. Let $\hat{t}_i(\underline{z}, \underline{x})$ be terms with $\mathbb{R} \models \forall \underline{z} \forall \underline{x}(\delta_i(\underline{z}, \underline{x}) \leftrightarrow \hat{t}_i(\underline{z}, \underline{x}) > 0)$. Then $\mathbb{R} \models Ax(\bar{\underline{z}}, \hat{t}_i(\bar{\underline{z}}, \bar{\underline{x}})/y_i)$ by (1). So $\mathbb{R} \models \hat{\phi}[\hat{t}_i(\bar{\underline{z}}, \bar{\underline{x}})/y_i]$ or equivalently $\mathbb{R} \models \phi(\bar{\underline{z}}, \bar{\underline{x}})$.

(ii) Suppose $\mathbb{R} \models \forall \underline{x}\phi(\underline{z}, \underline{x})$ and let $\bar{\underline{y}}$ be a tuple of elements of $\mathbb{R}$ of the same length as $\underline{y}$ such that $\mathbb{R} \models Ax(\bar{\underline{z}}, \bar{\underline{y}})$. We shall show the following:

(2) There is $\bar{\underline{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that $\mathbb{R} \models \delta_i(\bar{\underline{z}}, \bar{\underline{x}}) \leftrightarrow \bar{y}_i > 0$ for all $\delta_i \in \bar{\Delta}(\underline{z}, \underline{x})$.

Since $\mathbb{R} \models \phi(\bar{\underline{z}}, \bar{\underline{x}})$, from (2) it follows that $\mathbb{R} \models \hat{\phi}[\delta_i(\bar{\underline{z}}, \bar{\underline{x}})/y_i > 0]$ and $\mathbb{R} \models \hat{\phi}(\bar{\underline{y}})$. Thus $\mathbb{R} \models \forall \underline{y}\psi_1(\bar{\underline{z}}, \underline{y})$ and Claim 1 is true.

We show (2) by proving that for all $l \leq n$ the following is true:

(3) There are $\bar{x}_1, \ldots, \bar{x}_l$ such that $\mathbb{R} \models \delta_i(\bar{\underline{z}}, \bar{x}_1, \ldots, \bar{x}_m) \leftrightarrow \bar{y}_i > 0$ for all $\delta_i \in \Delta_m(\underline{z}, \underline{x})$ when $m \leq l$.

The proof of (3) goes by induction on l and the statement follows when $l = n$.

If $l = 0$, $\delta_i(\underline{z}) \in \Delta_0(\underline{z})$ then $\delta_i(\underline{z}) \leftrightarrow y_i > 0$ is a conjunct in $Ax(\underline{z}, \underline{y})$, so since $\mathbb{R} \models Ax(\bar{\underline{z}}, \bar{\underline{y}})$, $\mathbb{R} \models \delta_i(\bar{\underline{z}}) \leftrightarrow \bar{y}_i > 0$.

Next suppose $l \geq 0$ and (3) is true. For each element of $\Delta_{l+1}(\underline{z}, \underline{x})$ let y_i (y'_i) correspond to $t_i(\underline{z}, \underline{x}_1, \ldots, \underline{x}_l) < x_{l+1}$ $(t_i(\underline{z}, \underline{x}) > x_{l+1}$ resp.) and let y_{ij} correspond to the standard form of $t_i(\underline{z}, \underline{x}) < t_j(\underline{z}, \underline{x})$. Let $\bar{y}_i, \bar{y}'_i$ and $\bar{y}_{ij}$ denote the elements of $\mathbb{R}$ corresponding to the variables y_i, y'_i and y_{ij} respectively. Then by the induction hypothesis $\mathbb{R} \models \bar{y}_{ij} > 0 \leftrightarrow t_i(\bar{\underline{z}}, \bar{\underline{x}}) < t_j(\bar{\underline{z}}, \bar{\underline{x}})$. So by the definition of Ax and Θ_2, $\mathbb{R} \models \Theta_2(\underline{t}(\bar{\underline{z}}, \bar{x}_1, \ldots, \bar{x}_l), \bar{\underline{y}}, \bar{\underline{y}}')$, and by the Lemma 4. there is $\bar{x}_{l+1} \in \mathbb{R}$ such that $\mathbb{R} \models \bigwedge_i (t_i(\bar{\underline{z}}, \bar{x}_1, \ldots, \bar{x}_l) < \bar{x}_{l+1} \leftrightarrow \bar{y}_i > 0) \wedge (t_i(\bar{\underline{z}}, \bar{x}_1, \ldots, \bar{x}_l) > \bar{x}_{l+1} \leftrightarrow \bar{y}'_i > 0)$. Thus (3) is true for $l+1$ and the statement follows.

Claim 2. Let $\psi_2(\underline{z}, \underline{y}) \equiv Ax(\underline{z}, \underline{y}) \wedge \hat{\phi}(\underline{y})$ Then $\mathbb{R} \models \forall \underline{z}(\exists \underline{x}\phi(\underline{z}, \underline{x}) \leftrightarrow \exists \underline{y}\psi_2(\underline{z}, \underline{y}))$

Proof. Let $\bar{\underline{z}}$ be a tuple of elements of $\mathbb{R}$ as in Claim 1.

(i) Suppose $\mathbb{R} \models \exists \underline{x}\phi(\bar{\underline{z}}, \underline{x})$ i.e. $\mathbb{R} \models \phi(\bar{\underline{z}}, \bar{\underline{x}})$ for some tuple $\bar{\underline{x}}$ in $\mathbb{R}$ of length n. Then $\mathbb{R} \models \hat{\phi}[\delta_i(\bar{\underline{z}}, \bar{\underline{x}})/y_i > 0)$. Again let $\hat{t}_i(\underline{z}, \underline{x})$ be terms with $\mathbb{R} \models \delta_i(\underline{z}, \underline{x}) \leftrightarrow \hat{t}_i(\underline{z}, \underline{x}) > 0$. Then $\mathbb{R} \models \hat{\phi}(\hat{t}_i(\bar{\underline{z}}, \bar{\underline{x}})/y_i)$ and $\mathbb{R} \models Ax(\bar{\underline{z}}, \hat{t}_i(\bar{\underline{z}}, \bar{\underline{x}})/y_i)$ by (1). So $\mathbb{R} \models \exists \underline{y}\psi_2(\bar{\underline{z}}, \underline{y}))$

(ii) Suppose $\mathbb{R} \models \exists \underline{y}\psi_2(\bar{\underline{z}}, \underline{y})$ and let $\bar{\underline{y}}$ be a tuple in $\mathbb{R}$, of the same length as $\underline{y}$, such that $\mathbb{R} \models \psi_2(\bar{\underline{z}}, \bar{\underline{y}})$. Then $\mathbb{R} \models Ax(\bar{\underline{z}}, \bar{\underline{y}})$. As in the proof of Claim 1 there are $\bar{x}_1, \ldots, \bar{x}_n$ such that $\mathbb{R} \models \delta_i(\bar{\underline{z}}, \bar{\underline{x}}) \leftrightarrow \bar{y}_i > 0$ for all $\delta_i \in \bar{\Delta}(\underline{z}, \underline{x})$. Thus $\mathbb{R} \models \hat{\phi}[\delta_i(\bar{\underline{z}}, \bar{\underline{x}})/y_i > 0]$, so $\mathbb{R} \models \phi(\bar{\underline{z}}, \bar{\underline{x}})$ and $\mathbb{R} \models \exists \underline{x}\phi(\bar{\underline{z}}, \underline{x})$ follows. $\square$

Note that if ϕ is an L-sentence and ψ is its standard form obtained as in the proof then ϕ and ψ have the same quantifier structure in the following sense: there is a bijection F from the set of blocks of quantifiers of ϕ onto the set of blocks of quantifiers of ψ such that the image of a universal block is universal, and the image of an existential block is existential. Moreover, if block $\underline{Q}$ is within the scope of block $\underline{Q'}$ in ϕ, then block $F(\underline{Q})$ is within the scope of block $F(\underline{Q'})$ in ψ.

Note also that the proof goes through in Bishop's constructive analysis, so the proposition is true in the various models of intuitionistic analysis.

§4. The decidability theorem.

Proposition 2. *The L-theory of Scott's model is decidable.*

Proof. By [3] and by Proposition 1 it is enough to show that if $\psi \in Sent_L$ is a sentence of special form then there is a sentence $\lceil\psi\rceil \in Sent_{L'}$ with $\mathcal{R} \models \psi$ if and only if $T \models \lceil\psi\rceil$, where L' is a monadic second order language whose atomic formulas are of the form $true$, $false$, or $t \in U$ where t ranges over T and U ranges over the open subsets of T.

Recall that $\mathcal{R} \models \psi$ if and only if $T \models \forall t(t \in \|\psi\|)$. For a variable x_i of the language L let U_i be a corresponding set variable of the language L'. For a formula $\phi(x_1, \ldots, x_l) \in Form_L$ that is a subformula of a sentence in special form, there is a corresponding formula $\lceil\phi\rceil(U_1, \ldots, U_l, t) \in Form_{L'}$ defined as follows (we identify $h_q 1$ with q):

(*i*) $\lceil q > 0\rceil = true$ if $\mathbb{R} \models q > 0$, $false$ if $\mathbb{R} \models q \leq 0$ $(q \in \mathbb{Q})$

(*ii*) $\lceil x_i > 0\rceil = t \in U_i$

(*iii*) $\lceil\phi_1(\underline{x}) \overset{\wedge}{\vee} \phi_2(\underline{x})\rceil = \lceil\phi_1\rceil(\underline{U}, t) \overset{\wedge}{\vee} \lceil\phi_2\rceil(\underline{U}, t)$

(*iv*) $\lceil\phi_1(\underline{x}) \to \phi_2(\underline{x})\rceil = \exists V\big(t \in V \wedge \forall t'(t' \in V \wedge \lceil\phi_1\rceil(\underline{U}, t') \to \lceil\phi_2\rceil(\underline{U}, t'))\big)$

(*v*) $\lceil\forall x_i \phi'(x_i, \underline{x})\rceil = \exists V\big(t \in V \wedge \forall t'(t' \in V \to \forall U_i \lceil\phi'\rceil(U_i, \underline{U}, t'))\big)$

(*vi*) $\lceil\exists x_i \phi'(x_i, \underline{x})\rceil = \exists U_i \lceil\phi'\rceil(U_i, \underline{U}, t)$

If $\bar{x} \in \mathcal{R}$ then $\|\bar{x} > 0\|$ is an open subset of T; and if $\bar{U}$ is an open subset of T, $\bar{U} = \|\bar{x} > 0\|$ for some $\bar{x} \in \mathcal{R}$. With the help of these one can easily prove that for $\bar{x}_i \in \mathcal{R}$, if $\bar{U}_i = \|\bar{x}_i > 0\|$ then

$$T \models \forall t\big(t \in \|\phi(\bar{x}_1, \ldots, \bar{x}_l)\| \leftrightarrow \lceil\phi\rceil(\bar{U}_1, \ldots, \bar{U}_l, t)\big)$$

and from this the proposition follows. □

§5. Other corollaries of Proposition 1.

First let $\forall \underline{y}\psi(y_1, \ldots, y_n) \in Sent_L$ be a prenex universal sentence in special form. Let $P_1, \ldots, P_n$ be propositional letters. From the fact that the open subsets of T are complete with respect to the intuitionistic propositional calculus it follows that

$$\mathcal{R} \models \forall \underline{y}\psi(\underline{y}) \qquad \text{if and only if} \qquad \vdash_{int} \psi[P_i/y_i > 0]$$

where $\vdash_{int}$ denotes derivability in intuitionistic propositional calculus.

We know that the intuitionistic propositional logic is decidable. This fact together with Proposition 1 — i.e. that there is an algorithm which generates for every L-sentence an equivalent one in special form and with the same quantifier structure (cf. the first note after the proof of Proposition 1) — gives another decision method in Scott's model for prenex universal L-sentences.

Second, using Proposition 1 in Scowcroft's model for the continuum (defined in [9]) together with some trivial extensions of Rabin's work, one can prove similar decidability results in that model too. The main difference between Scott's model and Scowcroft's is that in the latter the structure of the truth values is the Heyting algebra of the coperfect open subsets of T and the elements of the model are functions from T into $\mathbb{R}$ with at most countably many points of discontinuity. Note that the two models have different L-theories (cf. [9, p. 146]).

Finally note that using the theory of lawless sequences we can give an intuitionistically acceptable proof for the decidability of prenex universal L-sentences in the L-theory of the reals. Also, a constructive proof exists for the decidability of L-sentences not containing universal quantifiers.

§**6. Appendix: proof of Lemma 4.** For proving Lemma 4 we need some technical results.

Lemma 1. *Let $\delta(x, u, z', z)$ be the following formula:*

$$[x \leq u \wedge (z' > 0 \wedge z > 0 \rightarrow x < u) \wedge (x < u \rightarrow z' > 0 \vee z > 0)] \rightarrow$$

$$\rightarrow \exists x'[x \leq x' \wedge x' \leq u \wedge (x < x' \rightarrow z' > 0) \wedge (x' < u \rightarrow z > 0) \wedge$$

$$\wedge (z' > 0 \wedge z > 0 \rightarrow x < x' \wedge x' < u)]$$

Then $\mathbb{R} \models \forall x \forall u \forall z' \forall z \delta(x, u, z', z)$.

Proof. Let us use the notation $\delta \equiv \delta_1 \rightarrow \exists x' \delta_2$.

Let $x = \{x_n\}$, $u = \{u_n\}$, $z' = \{z'_n\}$ and $z = \{z_n\}$ real numbers (cf. [1, p.15]) such that $\delta_1(x, u, z', z)$ is true in $\mathbb{R}$.

Let $\{w_n\}$ be the following sequence:

(i) $w_n := \frac{u_n + x_n}{2}$ if for all $k \leq n$ $|u_k - x_k| \leq \frac{2}{k}$.

If $|u_m - x_m| > \frac{2}{m}$ and m is minimal with respect to this property , then $x < u$ and $z' > 0$ or $z > 0$ follows. In this case we have two possibilities:

(ii) There is an m' such that for all $k \leq m'$ $|z_k| \leq \frac{1}{k}$, for all $k < m'$ $|z'_k| \leq \frac{1}{k}$ and $|z'_{m'}| > \frac{1}{m'}$.

Let $n \geq m$.

(α) If for all $l \leq n$ $|z_l| \leq \frac{1}{l}$, let $w_n := u_n$.

(β) If $l \le n$, $|z_l| > \frac{1}{l}$ and l is minimal with respect to these properties, let $l' = \max(l, m)$ and let $w_n := \max(u_n - \frac{1}{l'}, \frac{u_n + x_n}{2})$ for all $n \ge l'$.

(iii) There is an m' such that for all $k < m'$ $|z'_k| \le \frac{1}{k}$, $|z_k| \le \frac{1}{k}$ and $|z_{m'}| > \frac{1}{m'}$.

Again let $n \ge m$.

(α) If for all $l \le n$ $|z'_l| \le \frac{1}{l}$, let $w_n := x_n$.

(β) If $l \le n$, $|z'_l| > \frac{1}{l}$ and l is minimal with respect to these properties, let $l' = \max(l, m)$ and let $w_n := \min(x_n + \frac{1}{l'}, \frac{u_n + x_n}{2})$ for all $n \ge l'$.

Finally let $x'_n := w_{3n}$.

It is easy to check that $x' = \{x'_n\} \in \mathbb{R}$ and $\mathbb{R} \models \delta_2(x, u, z', z, x')$. $\square$

Lemma 2. *Let $\underline{z} = z_1, \ldots, z_k$, $\underline{z'} = z'_1, \ldots, z'_k$, $\underline{u} = u_1, \ldots, u_k$ be disjoint lists of distinct variables and let $\Theta_1(\underline{z}, \underline{z'}, \underline{u})$ be the conjunction of the following formulas:*

(i) $\neg(z_l > 0 \wedge z'_l > 0) \qquad (1 \le l \le k)$

and for $1 \le l \le k-1$

(ii) $(z_l > 0 \rightarrow z_{l+1} > 0) \wedge (z'_{l+1} > 0 \rightarrow z'_l > 0) \wedge u_l \le u_{l+1}$
(iii) $z'_l > 0 \rightarrow z'_{l+1} > 0 \vee u_l < u_{l+1}$
(iv) $z_{l+1} > 0 \rightarrow z_l > 0 \vee u_l < u_{l+1}$
(v) $u_l < u_{l+1} \rightarrow z_{l+1} > 0 \vee z'_l > 0$

Then
$\mathbb{R} \models \forall \underline{z}, \underline{z'}, \underline{u}\big(\Theta_1(\underline{z}, \underline{z'}, \underline{u}) \rightarrow \exists x \bigwedge_{l=1}^{k}[(x < u_l \leftrightarrow z_l > 0) \wedge (x > u_l \leftrightarrow z'_l > 0)]\big)$

Proof. Suppose $\Theta_1(\underline{z}, \underline{z'}, \underline{u})$ is true and let $x_1 := u_1 - max(0, z_1)$. Then

$$\mathbb{R} \models x_1 \le u_1 \wedge (x_1 < u_1 \leftrightarrow z_1 > 0)$$

Suppose x_l is defined such that for all $j \le l$

(vi) $x_{j-1} \le x_j \le u_j$
(vii) $x_{j-1} < x_j \rightarrow z'_{j-1} > 0$
($viii$) $z'_{j-1} > 0 \wedge z_j > 0 \rightarrow x_{j-1} < x_j < u_j$
(ix) $x_j < u_j \leftrightarrow z_j > 0$

are true in $\mathbb{R}$.

Then $\mathbb{R} \models \delta_1(x_l, u_{l+1}, z'_l, z_{l+1})$ and by Lemma 1 there is an element $x_{l+1} \in \mathbb{R}$ such that $\mathbb{R} \models \delta_2(x_l, u_{l+1}, z'_l, z_{l+1}, x_{l+1})$. Then $\mathbb{R} \models z_{l+1} > 0 \rightarrow x_{l+1} < u_{l+1}$ and $(vi) - (ix)$ follow for $j = l+1$.

So by induction there are $x_1, \ldots, x_k$ obeying $(vi) - (ix)$.

Finally let $x := x_k + max(0, z'_k)$. Then $\mathbb{R} \models x_k \le x \wedge (x_k < x \leftrightarrow z'_k > 0)$.

It is easy to see that $\mathbb{R} \models x < u_l \leftrightarrow z_l > 0$ for $1 \le l \le k$, and a downward induction on l shows that $\mathbb{R} \models u_l < x \leftrightarrow z'_l > 0$ also true for $1 \le l \le k$. $\square$

Lemma 3. *Let $\underline{t} = t_1, \ldots, t_k$, $\underline{y} = y_1, \ldots, y_k$, $\underline{y'} = y'_1, \ldots, y'_k$ be disjoint lists of distinct variables and let $\Theta_2(\underline{t}, \underline{y}, \underline{y'})$ be the conjunction of the following formulas (for $1 \leq i, j \leq k$):*

(α) $\neg(y_i > 0 \wedge y'_i > 0)$
(β) $y'_i > 0 \to t_i < t_j \vee y'_j > 0$
(γ) $y_j > 0 \to t_i < t_j \vee y_i > 0$
(δ) $t_i < t_j \to y_j > 0 \vee y'_i > 0$

Then

$$\mathbb{R} \models \forall \underline{t} \forall \underline{y} \forall \underline{y'} \big(\Theta_2(\underline{t}, \underline{y}, \underline{y'}) \to \Theta_1(\underline{z}, \underline{z'}, \underline{u})\big)$$

where

$$z_1 = \min_j y_j, \ldots, z_l = \max_{|I|=l-1} \min_{j \notin I} y_j, \ldots, z_k = \max_j y_j$$

$$z'_1 = \max_j y'_j, \ldots, z'_l = \min_{|I|=l-1} \max_{j \notin I} y'_j, \ldots, z'_k = \min_j y'_j$$

$$u_1 = \min_j t_j, \ldots, u_l = \max_{|I|=l-1} \min_{j \notin I} t_j, \ldots, u_k = \max_j t_j$$

Proof. A constructive proof of this statement uses the fact that the following (classically obvious) claims about reals are true constructively (the somewhat lengthy details are omitted):

(i) $z'_l > 0$ if and only if there is an index set I with l elements such that $y'_i > 0$ for $i \in I$.

(ii) $z_l > 0$ if and only if there is an index set I with $l-1$ elements such that $y_j > 0$ for $j \notin I$.

(iii) If $u_l < u_{l+1}$ then there is an index set I with $|I| = l$ such that $\bigwedge_{i \in I} \bigwedge_{j \notin I} t_i \leq u_l < u_{l+1} \leq t_j$. If $\bigwedge_{i \in I} \bigwedge_{j \notin I} t_i < t_j$ for a set I with l elements, then again $\bigwedge_{i \in I} \bigwedge_{j \notin I} t_i \leq u_l < u_{l+1} \leq t_j$. $\square$

Now we can sketch the proof of Lemma 4:

Lemma 4.

$$\mathbb{R} \models \forall \underline{t} \forall \underline{y} \forall \underline{y'} \big(\Theta_2(\underline{t}, \underline{y}, \underline{y'}) \to \exists x \bigwedge_{i=1}^{k} (x < t_i \leftrightarrow y_i > 0) \wedge (x > t_i \leftrightarrow y'_i > 0)\big)$$

Proof. Suppose $\mathbb{R} \models \Theta_2(\underline{t}, \underline{y}, \underline{y'})$. Then $\mathbb{R} \models \Theta_1(\underline{z}, \underline{z'}, \underline{u})$ by the previous lemma. So by Lemma 2 there is an $x \in \mathbb{R}$ such that

$$\mathbb{R} \models \bigwedge_{i=1}^{k} (x < u_i \leftrightarrow z_i > 0 \wedge x > u_i \leftrightarrow z'_i > 0)$$

Claim 1. $y_m > 0 \to x < t_m$

Proof. If $y_m > 0$ then $z_k = \max_j y_j > 0$ and an induction on l shows that if $z_l > 0$ then $x < t_m$.

Claim 2. $x < t_m \to y_m > 0$

Proof. Suppose that $x < t_m$. Then $x < u_k = \max_j t_j$, so $z_k > 0$ from above. Again, induction on l shows that if $z_l > 0$ then $y_m > 0$.

Claim 3. $y'_m > 0 \to x > t_m$

Proof. If $y'_m > 0$ then $z'_1 > 0$. A downward induction shows that if $z'_l > 0$ then $x > t_m$.

Claim 4. $x > t_m \to y'_m > 0$

Proof. If $x > t_m$ then $x > u_1$ and $z'_1 > 0$. A downward induction shows that if $z'_l > 0$ then $y'_m > 0$.

Note that in the induction steps one uses the facts $(i) - (iii)$ from the previous lemma. □

A more detailed version of the proofs is found in [2].

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