Ergodicity and correlation decay in billiards

summary of the PhD thesis

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My final aim during my PhD work has been to understand the decay of correlations for multi-dimensional ($d > 2$) dispersing billiards.

The starting point of the thesis is our work [2], which made it clear that all building bricks of Young's method to prove exponential correlation decay can be extended to the multi-dimensional case except for the growth properties of unstable manifolds distilled from Young's method and formulated for $d > 2$ by Chernov.

For the verification of these growth properties for $d > 2$, a prerequisite could have been the boundedness of the curvature of the singularity manifolds. This property has been implicitly assumed to hold in Chernov and Sinai's excellent proof of the local ergodicity theorem, also called the Fundamental Theorem (FTH), and accepted by all experts afterwards. The first result of my thesis is actually a negative one: *In multi-dimensional dispersing billiards, the curvature of the singularity manifolds blows up near a two codimensional "pathology" where the manifold is not differentiable.*

This observation not only made the study of growth properties much more difficult, but also demonstrated an unpleasant gap in all proofs of the Fundamental Theorem. It's important to see that the FTH is essential in the proofs of the Boltmann-Sinai Ergodic Hypothesis for hard ball systems, so the correction of the proof became absolutely necessary for further development. The success has only been partial. This is the second main result of the thesis: *A corrected proof of the Fundamental Theorem is given for the case of algebraic billiards*, that is, when scatterers are defined through algebraic equations. This result, however, covers most of the interesting applications: Hard Ball Systems, Lorentz Processes and Bunimovich-Reháček Stadia.

Having the difficulty of handling pathologies in multi-dimensional billiards, one can try to "go around" the problem by considering another generalization: soft billiards. There is hope that there exist ergodic soft billiards in high dimensions, for which the pathologies do not appear, so a proof of exponential correlation decay can be much easier than for billiards. The first step in that direction is to understand correlation decay in 2-dimensional soft billiards. This is the third main result of the thesis: *Exponential decay of correlations is proven for a class of 2-dimensional soft billiards.*

All of this work is based on the material that has appeared in the papers [2], [1], [4] and [3]. Going further in the direction of multi-dimensinal correlation decay, the first step has been made: the hyperbolicity of certain multi-dimensional soft billiards is proven in [5]. However, this result is not part of the thesis. Further work on the topic is in progress.

[1] P. Bálint, N. I. Chernov, D. Szász and I. P. Tóth, *Multi-dimensional Semi-dispersing Billiards: Singularities and the Fundamental Theorem*, Annales Henri Poincaré **3** (2002) 451–482.

[2] P. Bálint, N. I. Chernov, D. Szász and I. P. Tóth, *Geometry of Multi-dimensional Dispersing Billiards*, Astérisque **286** (2003) 119–150.

[3] P. Bálint and I. P. Tóth, *Correlation decay in certain soft billiards*, Commun. Math. Phys. **243** (2003) 55–91.

[4] P. Bálint and I. P. Tóth, *Mixing and its rate in 'soft' and 'hard' billiards motivated by the Lorentz process*, Physica D **187** (2004) 128–135.

[5] P. Bálint and I. P. Tóth, *Hyperbolicity of multi-dimensional soft billiards*, submitted to Discrete and Cont. Dyn. Syst.