Ergodicity and correlation decay in billiards

outline of the PhD thesis

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In the following sections we give an outline of the PhD thesis "Ergodicity and Correlation Decay in Billiards". After a brief introduction we describe the models discussed in the thesis, then present the main results. Finally we have a look at some planned directions of future research in the field.

1 Introduction

The detailed behaviour of dynamical systems is, in general, too complicated to study. Also, it is often more informative to know their asymptotic behaviour instead. Of primary importance is then, to know that the system is ergodic – a basic notion for chaos. Once this is the case, physicists are interested in quantifiable, measurable properties, like the decay rate of correlations (the relaxation of the system) or limit laws of probability.

A billiard is the dynamical system describing the motion of a single point particle which moves freely in a container Q – which can have a complicated shape – and bounces back elastically when reaching the walls. Since billiards – both semi-dispersing and dispersing (Sinai billiards) – which are closely related to hard ball systems (a model for ideal gases) possess a manifest feature of hyperbolicity, which is the key mechanism behind their ergodicity, their study has attracted a lot of interest since at least 1963, when Sinai published his version of Boltzmann's Ergodic Hypothesis [17]. Restricting our attention to the "easier" case of dispersing billiards, our starting point is the remarkable method of Lai-Sang Young [19], which provided exponential decay of correlations for the 2-dimensional case. My final aim has, indeed, been to understand the decay of correlations for multi-dimensional ($d > 2$) dispersing billiards, which is in the focus of experts' attention since Young's work.

The very starting point of the thesis is our work with Bálint, Chernov and Szász [2], which made it clear that all building bricks of Young's method can be extended to the multi-dimensional case except for the growth properties of unstable manifolds distilled from Young's method and formulated for $d > 2$ by Chernov [8].

For the verification of these growth properties for $d > 2$, a prerequisite could have been the boundedness of the curvature of the singularity manifolds. This property has been implicitly assumed to hold in Chernov and Sinai's excellent proof of the local ergodicity theorem, also called the Fundamental Theorem (FTH) [9], and accepted by all experts afterwards. The first result of my thesis is actually a negative one: *In multi-dimensional dispersing billiards, the curvature of the singularity manifolds blows up near a two codimensional "pathology" where the manifold is not differentiable.*

This observation not only made the study of growth properties much more difficult, but also demonstrated an unpleasant gap in all proofs of the Fundamental Theorem. It's important to see that the FTH is essential in the proofs of the Boltmann-Sinai Ergodic Hypothesis for hard ball systems, so the correction of the proof became absolutely necessary for further development. The success has only been partial. This is the second main result of the thesis: *A corrected proof of the Fundamental Theorem is given for the case of algebraic billiards*, that is, when scatterers are defined through algebraic equations. This result, however, covers most of the interesting applications: Hard Ball Systems, Lorentz Processes and Bunimovich-Reháček Stadia.

Having the difficulty of handling pathologies in multi-dimensional billiards, one can try to "go around" the problem by considering another generalization: soft billiards. There is hope that there exist ergodic soft billiards in high dimensions, for which the pathologies do not appear, so a proof of exponential correlation decay can be much easier than for billiards. The first step in that direction is to understand correlation decay in 2-dimensional soft billiards. This is the third main result of the thesis: *Exponential decay of correlations is proven for a class of* 2*-dimensional soft billiards.*

All of this work is based on the material that has appeared in the papers [2], [1], [4] and [3]. Going further in the direction of multi-dimensinal correlation decay, the first step has been made: the hyperbolicity of certain multi-dimensional soft billiards is proven in [5]. However, this result is not part of the thesis. Further work on the topic is in progress.

In Dynamical Systems, there are many notions of "chaotic" or "statistical" behaviour. In this thesis we mainly look at two of these: *ergodicity* and *exponential decay of correlations* (EDC). A third notion, *hyperbolicity* is used excessively as a tool, and a fourth, the *central limit theorem* (CLT) is obtained as a byproduct of EDC.

For a better understanding of the work, we give the definitions we use, of these notions here:

Definition 1. *The dynamical system (M,T,μ) is ergodic if every μ-measurable T-invariant set $A \subset M$ has either $\mu(A)=1$ or $\mu(A)=0$.*

Ergodic components of the dynamical system (M,T,μ) are the atoms of the σ-algebra of T-invariant μ-measurable subsets of M. Obviously, (M,T,μ) is ergodic if and only if it has an ergodic component with measure 1.

Definition 2. *We say that the dynamical system (M,T,μ) has exponential decay of correlations (EDC) , if for every pair of Hölder-continuous functions $f,g : M \to \mathbb{R}$ there exist constants $C<\infty$ and $a>0$ such that for every $n \in \mathbb{N}$*

$$\left| \int_M f(x)g(T^n x)d\mu(x) - \int_M f(x)d\mu(x) \int_M g(T^n x)d\mu(x) \right| \leq Ce^{-an}.$$

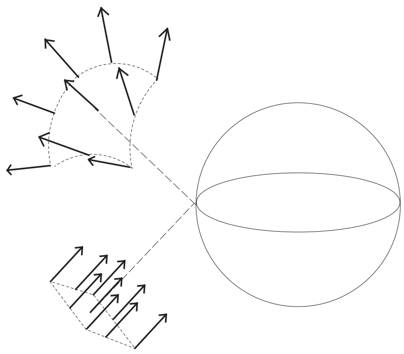
Definition 3. *We say that (M,T,μ) satisfies the central limit theorem (CLT) (for Hölder continuous functions) if for every $\eta>0$ and every Hölder-continuous function $f : M \to \mathbb{R}$*

with $\int f\, d\mu = 0$, there exists a $\sigma_f \geq 0$ such that

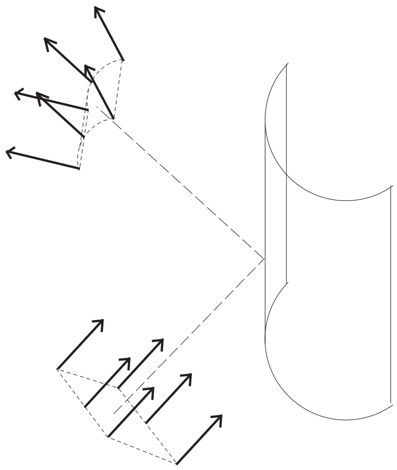
$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \xrightarrow{\text{distr}} \mathcal{N}(0, \sigma_f)$$

where $\mathcal{N}(0, \sigma_f)$ is the Gaussian distribution with variance σ_f^2.

Figure 1 demonstrates the key feature of billiards which is responsible for their chaotic behaviour. This is the "scattering" mechanism which causes (certain) nearby trajectories to divert from each-other. We show two well-distinguished cases. One is a *dispersing* billiard, meaning that scatterers are strictly convex, and so parallel trajectories can be diverted (from each other) in all directions through a single collision. The other is only *semi-dispersing*: scatterers are not strictly convex, resulting in a "less effective" scattering.



(a) dispersing



(b) semi-dispersing

Figure 1: scattering in dispersing and semi-dispersing billiards

1.1 History

Mathematical studies of billiards have begun over thirty years ago. Ya. Sinai in his seminal paper of 1970 [18] described the first large class of billiards with truly chaotic behaviour – with nonzero Lyapunov exponents, positive entropy, enjoying ergodicity, mixing, and (as was later discovered by G. Gallavotti and D. Ornstein [12]) the Bernoulli property. Sinai billiards are defined in two dimensions ($d = 2$), i.e. the container $Q \subset \mathbb{R}^2$ or $Q \subset \mathbb{T}^2$, and the boundary of Q must be concave (i.e., convex outward Q), similarly to the Lorentz process (where the scatterers are convex). Due to the geometric concavity, the boundary ∂Q scatters (disperses) bundles of geodesic lines falling upon it, see Figure 1. For this reason, Sinai billiards are said to be *dispersing*.

Lorentz processes in two dimension have been studied very thoroughly since 1970. Many fine ergodic and statistical properties have been established by various researchers, including P. Bleher, L. Bunimovich, N. Chernov, J. Conze, C. Dettmann, G. Gallavotti,

A. Krámli, J. Lebowitz, D. Ornstein, K. Schmidt, N. Simányi, Ya. Sinai, D. Szász, and others. The latest major result for this model (the exponential decay of correlations) was obtained by L.-S. Young [19]. The success in these studies had significant impact on modern statistical mechanics. The methods and ideas originally developed for the planar Lorentz process were applied to many other classes of physical models – see recent reviews by Cohen, Gallavotti, Ruelle and Young [11, 15, 20].

On the other hand, the progress in the study of the multi-dimensional Lorentz process (where $d > 2$) has been much slower and somewhat controversial. Relatively few papers were published covering specifically the case $d > 2$, especially in contrast to the big number of works on the 2-D case. Furthermore, the arguments in the published articles were usually rather sketchy, as in Chernov's paper [7]. It was commonly assumed that the geometric properties of the multi-dimensional Lorentz process were essentially similar to those of the 2-D system, and so the basic methods of study should be extended from 2-D to any dimension at little cost. Thus, the authors rarely elaborated on details.

Recent discoveries proved that spatial dispersing billiards are *very much* different from planar ones. Bunimovich and Reháček studies of *astigmatism* [6], in the somewhat different context of focusing billiards, emphasized the known fact that billiard trajectories that are initially parallel, may focus very rapidly in one plane and very slowly in the orthogonal planes. Astigmatism is unique to 3-D (and higher dimensional) billiards, it cannot occur on a plane.

The history of soft billiards is as old as that of the hard ones. After many works of others, Donnay and Liverani in [10] have proven ergodicity of 2-dimensional systems under nearly necessary conditions. Exponential decay is discussed for some of these systems in this thesis. For the higher dimensional case, to my knowledge, the first result is the proof of hyperbolicity in [5], not yet appeared.

2 Models of the thesis

2.1 The hard billiard dynamical dystem

A billiard is a dynamical system describing the motion of a point particle in a connected, compact domain $Q \subset \mathbb{T}^d$. In general, the boundary of the domain, ∂Q is assumed to be piecewise C^3-smooth. Inside Q the motion is uniform while the reflection at the boundary ∂Q is elastic. As the absolute value of the velocity is a first integral of motion, we fix this value to be 1. So the phase space of the billiard flow is the set Q of possible configuration points, each point equipped with possible unit velocity vectors. The set of possible velocity vectors is typically $\mathbf{S}^{d-1}$, however, at ∂Q, where momentary collisions take place, it is only a hemisphere – pre-collision (incoming) and post-collision (outgoing) velocities have to be identified, so that for every moment in time, the state of the particle can be uniquely defined. So, the phase space is $\mathcal{M} = Q \times \mathbf{S}^{d-1}/\sim$, where $\sim$ is the equivalence of appropriate incoming and outgoing velocities. Still we will denote a phase point x as $x = (q, v)$ with $q \in Q$ and $v \in \mathbb{R}^d$, $|v| = 1$. The dynamics S^t is the function from $\mathcal{M}$ to $\mathcal{M}$ that tells in which point $S^t(x)$ of the phase space the system will be at time t, if it is in x at time zero. The usual invariant measure for this system is the Liouville probability measure μ_0 on $\mathcal{M}$, which is essentially the product of the Lebesgue measures, i.e. $d\mu_0 = \text{const.} d\mathcal{L}_0 d\mathcal{H}$ where $\mathcal{L}_0$ is the Lebesgue measure on Q and $\mathcal{H}$ is the Riemannian measure on $\mathbf{S}^{d-1}$.

Definition 4. *The resulting dynamical system* $(\mathcal{M}, \{S^t, t \in \mathbb{R}\}, \mu_0)$ *is the billiard flow.*

Instead of the flow, one often considers a discrete time version of the system by looking only at collision moments. In other words, the boundary ∂Q defines a natural cross-section for the billiard flow. Consider namely

$$M = \{(q, v) \in \mathcal{M} \mid q \in \partial Q\}.$$

$n(q)$ will denote the unit normal vector of ∂Q at q, pointing inward Q. φ will denote the collision angle, which is the angle of $n(q)$ and the outgoing velocity v, such that $\cos\varphi = \langle v, n(q)\rangle$. When $d = 2$, it will be convenient to consider φ as a signed angle, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

We will always use outgoing velocities, such that

$$M = \{(q, v) \in \mathcal{M} \mid q \in \partial Q,\ \langle n(q), v\rangle \geq 0\} = \partial Q \times \mathbf{S}_+^{d-1},$$

where $+$ means that we only take into account the hemisphere of the outgoing velocities.

The discrete time dynamics T will denote the first return map to M: For any $x \in \mathcal{M}$ we set $t^+(x) := \inf\{t > 0 |\ S^t x \in M\}$, and $T^+x := S^{t^+(x)}x$ (of course, $T^+ : \mathcal{M} \to M$). Then the Poincaré section map $T : M \to M$ is defined as $Tx := T^+x$ for $x \in M$.

The natural invariant measure for this map, which is the 'projection' of the flow-invariant measure μ_0 onto M by the dynamics, is again absolutely continuous with respect to the product of Lebesgue measures:

$$d\mu = \text{const.}\langle n(q), v\rangle d\mathcal{L}d\mathcal{H} = \text{const.}\cos\varphi d\mathcal{L}d\mathcal{H},$$

where $\mathcal{L}$ is the Riemannian measure on ∂Q and $\mathcal{H}$ is still the Riemannian measure on $\mathbf{S}^{d-1}$.

Definition 5. *The resulting discrete time dynamical system* (M, T, μ) *is the billiard map.*

We will work with semi-dispersing billiards, which means that the smooth components of ∂Q are convex (as seen from outside Q). If these components are strictly convex, the billiard is dispersing. In other words:

Definition 6. *A billiard system is semi-dispersing if for any point q of a smooth component of ∂Q, the curvature operator (or second fundamental form) of ∂Q – which is actually the derivative operator of the function $n(q)$ – is positive semi-definite. If this operator is positive definite for every $q \in \partial Q$, the billiard is dispersing.*

The name 'dispersing' comes from the effect that particles reaching ∂Q at nearby points with parallel velocities will be diverted from each-other – a planar front will be 'scattered' (see Figure 1 and Section 2.1.1). In a semi-dispersing billiard, planar fronts are scattered in some directions. In these cases, components of the complement of Q are often called 'scatterers'.

2.1.1 Fronts

Our most important tools in describing hyperbolicity are local orthogonal manifolds or simply *fronts.* A front $\mathcal{W}$ is defined in the flow phase space rather than in the Poincaré section.

Definition 7. *Take a smooth one codimensional submanifold E of the whole configuration space Q, and add the unit normal vector $v(q)$ of this submanifold at every point q as a velocity, continuously. Consequently, at every point the velocity points to the same side of the submanifold E. The set*

$$\Sigma = \{(q, v(q)) \mid q \in E\} \subset \mathcal{M}, \tag{1}$$

where $v : E \to \mathbf{S}^{d-1}$ is continuous (smooth) and $v \perp E$ at every point of E, is called a front.

The derivative of this function v, called B plays a crucial role: $dv = Bdq$ for tangent vectors (dq, dv) *of the front.* B acts on the tangent plane $\mathcal{T}_q E$ of E, and takes its values from the tangent plane $\mathcal{J} = \mathcal{T}_{v(q)}\mathbf{S}^{d-1}$ of the velocity sphere. These are both naturally embedded in the configuration space Q, and can be identified through this embedding. So we just write $B : \mathcal{J} \to \mathcal{J}$. B is nothing else than the curvature operator of the submanifold E. Yet we will prefer to call it second fundamental form (SFF), in order to avoid confusion with other curvatures that are coming up. Obviously, B is symmetric.

Definition 8. *A front is called convex if its SFF B is positive semi-definite. A front is called strictly convex if its SFF B is positive definite.*

In billiards, fronts remain fronts during time evolution – at least locally, and apart from some singularity lines.

2.1.2 Hyperbolicity

Hyperbolicity is a key property of most billiard systems, used heavily in the proof of other properties representing strong statistical behaviour. In general, hyperbolicity means that all Lyapunov exponents are non-zero. [1] In Hamiltonian systems (including billiards) this means that at every point of the phase space the tangent space can be decomposed into two subspaces of equal dimension, on one of which (the *unstable* subspace) the dynamics is expanding, and on the other (the *stable* subspace) contracting. However, in billiards the notion of hyperbolicity is always connected to the geometrical picture of instability: the unstable subspace corresponds to some strictly convex front. See the outgoing front of Figure 1 (dispersing case) for an example. Phase points forming such a front will depart from each other, corresponding to expansion in the unstable direction. In semi-dispersing billiards, convex fronts remain convex during their time evolution, so this expansion is preserved. In other examples, including focusing billiards (e.g. the famous Bunimovich stadium) and soft billiards, the mechanism is more complicated, but the essence is the same: convex fronts remain convex. We will need the following notion related to hyperbolicity (which, for simplicity, we define for the flow here):

Definition 9. *The trajectory segment from time* 0 *to b of a point x is said to be sufficient if a planar front at x (that is, at time zero) evolves into a strictly convex front at $S^b(x)$ (that is, at time b). A phase point x is said to be sufficient if its entire trajectory contains a finite sufficient trajectory segment.*

[1] In the case of a billiard flow, one Lyapunov exponent – corresponding to the flow direction – is necessarily zero. Hyperbolicity means that all *others* are nonzero.

2.1.3 Singularities

Consider the set of tangential reflections, i.e.

$$\mathcal{S}_0 := \{(q,v) \in M \mid \langle v, n(q)\rangle = 0\} = \partial M.$$

It is easy to see that the map T is not continuous at the set $T^{-1}\mathcal{S}_0$. Further pre-images of $\mathcal{S}_0$ are

$$\mathcal{S}_k := T^{-k}\mathcal{S}_0 \quad (k > 0),$$

the so-called 'higher order' singularities. The singularity set (i.e. where it is not continuous) for a higher iterate T^n is

$$\mathcal{S}^{(n)} = \cup_{i=1}^{n} \mathcal{S}_i. \tag{2}$$

What is even worse, the derivative of the dynamics blows up near these singularities. In the multi-dimensional case ($d > 2$) this is crowned by a vast anisotropy: expansion is blowing up in one direction, while disappearing in the other (both within the unstable subspace). This is the heart of the phenomenon we discuss in Section 3.1 of this outline.

Generally it was assumed in the literature that the set $\mathcal{S}^{(n)}$ is a finite collection of smooth and compact submanifolds of the Poincaré phase space M. However, for multi-dimensional semi-dispersing billiards these manifolds can be treated as submanifolds of M only in a topological sense. So we view all these sets as (finite unions of) topologically embedded one codimensional compact submanifolds with boundary. They have smooth manifold structure in the interiour, however, the behaviour at the boundary is irregular (the curvature diverges).

Above the singularities have been introduced for the Poincaré section map T. Singularities for the flow can be defined analogously.

2.2 The soft billiard dynamical system

Consider the motion of a point particle in a planar periodic array of circular scatterers. There are several physically relevant dynamical systems related to such a geometrical configuration. For all of these, motion outside the scatterers is uniform, i.e. the point particle proceeds along a straight (geodesic) line with constant (unit) velocity. If the particle bounces off the circles according to the laws of elastic collision (angle of incidence equals to the angle of reflection), we talk about the *hard Lorentz process*.

Here we consider the following natural modification. Consider finitely many disjoint circles of radius R on the unit two-dimensional flat torus $\mathbb{T}^2$. (Thinking of a periodic array of circular disks on the Euclidean plane $\mathbb{R}^2$ would not be very much different.) We require that the configuration has finite horizon: there is a certain constant $\tau_{\max}$ such that any straight segment longer than $\tau_{\max}$ on $\mathbb{R}^2$ intersects at least one of the scatterers.

Let the Hamiltonian motion of our point particle be described by a potential which is identically zero outside and is some rotation symmetric function $V(r)$ inside the circular scatterers (here r is the distance from the center of the scatterer). We fix the mass and the full energy of our point particle as

$$m = 1, \qquad E = \frac{1}{2}.$$

This way the free flight velocity has unit length, $|v| = 1$.

We know that as a result of the assumptions we will make, the Hamiltonian dynamical system restricted to this surface of constant full energy is ergodic with respect to Liouville measure (fc. Remark 23.) Our aim is *to study the rate of mixing.* We work with the Poincaré section M of outgoing velocities.

s denotes the arclength parameter along the scatterer (starting from a point arbitrarily fixed), describing position of the outgoing particle.

φ denotes the collision angle, the angle that the outgoing velocity makes with the normal vector of the scatterer in the point s. Clearly $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The position can be equivalently described by another angle parameter $\Theta \in [0, 2\pi]$, for which $s = R\Theta$ (here R is the radius of the scatterer).

The dynamics T is the first return map onto M. Notation for the Lebesgue measure on M is m.

Denote by μ the natural invariant probability measure on M. μ is absolutely continuous w.r.t. Lebesgue, and the density is of the form

$$d\mu = \text{ const. } \cos(\varphi)\, dm = \text{ const. } \cos(\varphi)\, ds\, d\varphi. \tag{3}$$

Definition 10. *The dynamical system (M, T, μ) is the soft billiard map.*

To describe the first return map T we decompose the motion into two parts: free flight among the scatterers and flight in the potential of the scatterers. Free flight can be treated completely analogously to the billiard case. The particle leaves one of the scatterers in the point s_0 with velocity φ_0 and reaches some other scatterer in point s (or equivalently, Θ) with unit incoming velocity that makes an angle φ with the (opposite) normal vector $n(s)$ at the point of income. After some inter-potential motion the particle leaves the circle in some point $s_1 (= R\Theta_1)$ with outgoing velocity specified by φ_1. Out of symmetry reasons $\varphi_1 = \varphi$, thus the only nontrivial quantity is the angle difference $\Delta\Theta = \Theta_1 - \Theta$. Again out of symmetry reasons $\Delta\Theta$ depends only on the angle φ.

The role in the map T played by the potential is completely described by the function $\Delta\Theta(\varphi)$.

Definition 11. *The function $\Delta\Theta(\varphi)$ is called* ***the rotation function****.*

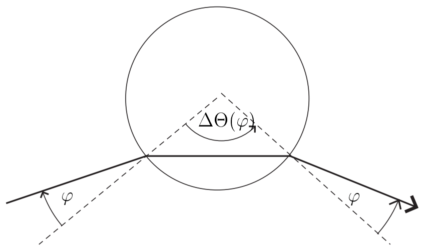


Figure 2: meaning of the rotation function

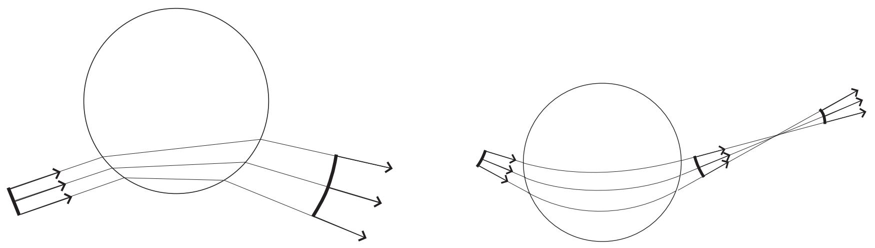
Being mainly interested in the differential aspects of T we introduce one more notation:

$$\kappa(\varphi) = \frac{d\Delta\Theta(\varphi)}{d\varphi}.$$

2.2.1 Mechanisms of hyperbolicity

For soft billiards in 2 dimensions, [10] have shown two possible mechanism for hyperbolicity.

Dispersing case. Assume $\kappa < -2$ or $\kappa \geq 0$ for all φ. The soft billiard is ergodic and hyperbolic. Mechanism of hyperbolicity is, just like in the hard case, related to convex fronts. I.e. incoming convex fronts (those reaching the potential disk) turn into outgoing convex fronts (convex when leaving the disk). See Figure 3.



(a) dispersing case

(b) defocusing case

Figure 3: mechanisms of hyperbolicity

Defocusing case. Assume there is some $\delta > 0$ such that $0 \geq \kappa > -2 + \delta$. In addition, the configuration is such that there is some lower bound $t_{\min}$ on the free flight between two disks which satisfies $t_{\min} > \frac{2R(2-\delta)}{\delta}$. The soft billiard is ergodic and hyperbolic. Nevertheless, the mechanism of hyperbolicity is different from the one in the dispersing case. Incoming convex fronts may turn into outgoing concave fronts. These concave fronts, as there is enough time until the next disk is reached, defocus and turn into convex fronts during free flight. Thus, when entering a potential region for the next time, they are convex again. See Figure 3.

3 Results

3.1 Geometry of singularities in multi-dimensional dispersing billiards

In several papers that appeared, singularities were assumed – either explicitly or implicitly – to consist of smooth one codimensional submanifolds of the phase space. Often, even a uniform bound on the curvature was assumed, independent of the order of the singularity. These assumptions were needed in order to estimate the measures of neighbourhoods of the singularities. Such a uniform bound on curvature does exist in 2-dimensional billiards. However, it does not exist in higher dimensions, as the following theorem – based on a counter example presented in the thesis – shows:

Theorem 12. ***Generally in multi-dimensional dispersing billiards, the curvature of $\mathcal{S}_2$ is not bounded. The unit normal vector is not a continuously differentiable function of its base point.***

For a rigorous proof of some finer properties (such as ergodicity and correlation decay of multi-dimensional dispersing billiards, it seems essential to characterize singularities in a systematic way. Part of this characterization – which is adequate for the proof of local ergodicity – is done in the thesis, see Section 3.2 of this outline. In the following we do not give rigorous proofs; we would like to point out some analogies to and emphasize some interesting features of the irregularities of the singularity set.

3.1.1 The Whitney-umbrella

Consider the one codimensional set in $\mathbb{R}^3$ defined by the polynomial equation

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 z = y^2\},$$

the Whitney-umbrella. 'One half' of this set (its intersection with the quadrants $\{xy \leq 0\}$) is shown in Figure 4. For simplicity we use the notations W_2 for this 'half-umbrella' and W_1 for the $\{z = 0\}$ plane. Clearly

- W_2 terminates on W_1 (in the points of the x axis), thus $W_1 \cap W_2 = \partial W_2$.
- at every point of the x axis where $x \neq 0$ the intersection of W_2 and W_1 is tangential.
- W_2 has smooth manifold structure in its interiour; nevertheless, near the origin its curvature is unbounded as the normal vector changes rapidly (actually, the unit normal vector does not even have a well-defined limit at the origin).

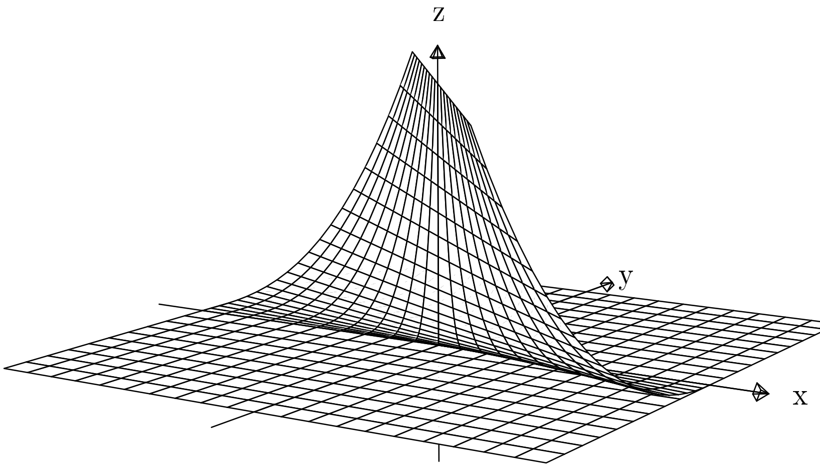


Figure 4: the Whitney umbrella

By these properties the geometry of singularities in the counter example is analogous to Figure 4. [2] W_1 corresponds to $\mathcal{S}_1$, W_2 corresponds to $\mathcal{S}_2$ while the origin corresponds to the set of those doubly tangential reflections where the two radii are orthogonal (this set is one codimensional in $\mathcal{S}_1 \cap \mathcal{S}_2$).

[2] To be precise, the situation in Figure 4. has one dimension less – in contrast to W_2 the singularities are 3-dimensional manifolds – but this has little significance to the analogy.

3.2 Fundamental Theorem for multi-dimensional semi-dispersing billiards

This section is devoted to the Fundamental Theorem – or local ergodicity theorem – for semi-dispersing billiards. The two-dimensional case had been settled in Sinai's celebrated work [18]. Seventeen years had elapsed until the multi-dimensional generalization given by Chernov and Sinai appeared [9]. It offered a quite involved, but in essence very transparent formulation of the theorem and a delicate proof. A self-contained exposition of the original ideas with detailed conditions and arguments was provided in [13] where a slightly more general, the so called 'transversal' version of the Fundamental Theorem was announced – mainly with its application to three-billiards in mind. Several other papers have appeared in the 90's with nice expositions of the theorem, even for classes of dynamical systems more general than the original semi-dispersing billiard setting (eg. Hamiltonian systems with singularities in [14]). However, all of these papers assumed that for all powers of the dynamics the singularity set is a finite collection of one codimensional smooth and compact submanifolds. Since, as Theorem 12 shows, this is not the case, it became utmost necessary to replace this assumption.

We first need to define the regularity property that should replace the smoothness of singularities. This property, called 'Lipschitz decomposability' will be defined for subsets of $\mathbb{R}^m$ here. This definition can be transferred to subsets of Riemannian manifolds through coordinate charts.

Definition 13. *Consider $H \subset \mathbb{R}^m$, and $L \in \mathbb{R}$. H will be called a 'Lipschitz decomposable' (one codimensional) subset with constant L if it can be decomposed into a finite number of open locally Lipschitz graphs and a small remainder set in the following way: There exist H^* and $H_1, ..., H_K$ such that:*

- $H \subset \bigcup_{i=1}^{K} \bar{H}_i \bigcup H^*$,
- $H_i \cap H_j = \emptyset$ *for any* $i \neq j$,
- *every H_i is a one codimensional open locally Lipschitz graph (with constant L) – that is, if viewed in an appropriate coordinate system, it is the graph of a locally Lipschitz function over an open domain,*
- $(\bigcup_{i=1}^{K} \partial H_i) \cup H^*$ *is more than one codimensioinal (in some measure-theoretic sense).*

The set H^* is only included in the decomposition for technical reasons.

The precise property that we expect the singularities of semi-dispersing billiards to have is formulated in the form of a conjecture:

Conjecture 14. *For any semi-dispersing billiard with a finite horizon there exists an $L \in \mathbb{R}$ such that for any integer N the set $\bigcup_{|n| \leq N} \mathcal{S}_n$ of singularities of order up to N is 'Lipschitz decomposable' with constant L.*

It is worth noting that by introducing 'transparent walls' (cf. [9]) any semi-dispersing billiard can be reduced to one with a finite horizon.

The statement of this conjecture will appear word by word among the conditions of the modified version of the Fundamental Theorem for semi-dispersing billiards. The conjecture will be proven for the utmost important special case of billiards with algebraic scatterers.

Before its formulation, it is important to point out the **conditions** under which the modified proof of the theorem works.

Condition 15. *(Chernov-Sinai Ansatz, Condition 3.1 from [13]). For almost every singular phase point x, the trajectory of x is only once singular, and, moreover, the positive semi-trajectory of x is sufficient.*

What follows below is our new condition – Lipschitz decomposability – on singularities.

Condition 16. *There exists an $L \in \mathbb{R}$ such that for every $N \in \mathbb{N}$ the singularity set $\bigcup_{|n|\leq N} \mathcal{S}_n$ is 'Lipschitz decomposable' with constant L (fc. Conjecture 14).*

Now we are ready to formulate the main result of this section:

Theorem 17. ***(The Fundamental Theorem) Suppose Conditions 15 and 16 are satisfied and x is a sufficient phase point. Then, there is a neighbourhood U of x such that every point of U belongs to the same ergodic component.***

With this theorem, the proof of ergodicity boils down to showing that there are many sufficient phase points – which is not a topic of this thesis.

3.2.1 The case of algebraic scatterers

The following theorems show that the singularity submanifolds of algebraic semi-dispersing billiards satisfy the Lipschitz decomposability property formulated in Conjecture 14. Fortunately, the most important examples of semi-dispersing billiards are algebraic. Consequently, the algebraicity condition does not essentially restrict the applicability of the Fundamental Theorem.

For definiteness we will say that the zero-set of a system of polynomial equations is an *algebraic variety*. Any (measurable) subset of a k-dimensional algebraic variety will be denoted as a k-dimensional SSAV (for 'subset of an algebraic variety'). As for the dimension of an algebraic variety, see [16]. We also use the following definition.

Definition 18. *A billiard is* algebraic *if it has finitely many scatterers and the boundary of each of these scatterers is a finite union of one codimensional SSAV-s (as subsets of $\mathbb{T}^d \subset \mathbb{R}^d$).*

Theorem 19. *In an algebraic billiard, $\mathcal{S}_n$ is a finite union of one codimensional SSAV-s in $\mathbb{R}^{2d}$.*

Theorem 20. *Any one codimensional algebraic variety H is Lipschitz decomposable (in the sense of Definition 13) with any constant $L > 0$.*

Corollary 21. ***(The Fundamental Theorem for algebraic semi-dispersing billiards) Suppose Condition 15 is satisfied and the semi-dispersing billiard is algebraic. Let x be a sufficient phase point. Then, there is a neighbourhood U of x such that every point of U belongs to the same ergodic component.***

3.3 Correlation decay in soft billiards

Our aim now is to prove exponential decay of correlations in certain ergodic soft billiards. Based on these results we conjecture that the rate of mixing is exponential in essentially all the (finite horizon) cases for which [10] obtained hyperbolic ergodicity. Nevertheless, in contrast to [10], we do not have many – actually, we have only two classes of – specific potentials for which exponential decay of correlations can be explicitly shown.

The small number of specific examples is related to the necessity of understanding the 'second derivative of the dynamics'. To obtain curvature and distortion bounds – which is inevitable for the application of methods from [8] and [19] – we need to check the derivative of $\kappa(\varphi)$.

Below two properties are defined, in terms of which our main theorem is formulated.

Definition 22. *The soft billiard system satisfies property H (H stands for 'hyperbolicity') in case*

1. *there is some positive constant c such that $|2+\kappa(\varphi)| > c$ for all φ;*
2. *the configuration of scatterers is such that the distance of any two circles is bounded below by $\tau_{\min}$ where*
$$\tau_{\min} \gneqq \max_{\varphi} \left\{ -2R\kappa(\varphi) \frac{\cos\varphi}{2+\kappa(\varphi)} \right\}.$$

Remark 23. *Although a bit otherwise formulated, it was essentially proven in [10] that soft billiard systems with property H are hyperbolic and ergodic.*

Definition 24. *The rotation function is termed regular in case the following properties hold.*

1. $\Delta\Theta(\varphi)$ *is piecewise* uniformly *Hölder continuous. I.e. there are constants $C < \infty$ and $\alpha > 0$, and furthermore, $[-\frac{\pi}{2}, \frac{\pi}{2}]$ can be partitioned into finitely many intervals, such that for any φ_1 and φ_2 (from the interiour of one of the intervals):*
$$|\Delta\Theta(\varphi_1) - \Delta\Theta(\varphi_2)| \leq C|\varphi_1 - \varphi_2|^{\alpha}.$$
2. $\Delta\Theta(\varphi)$ *is a piecewise C^2 function of φ on the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, in the above sense. (Note, however, that κ, in contrast to $\Delta\Theta$, can happen to have no finite one-sided limits at discontinuity points.)*
3. *There is some finite constant C such that*
$$|\kappa'(\varphi)| \leq C|(2+\kappa(\varphi))^3|$$
where $\kappa'(\varphi)$ is the derivative of κ with respect to φ.
4. *For the final property consider any discontinuity point φ_0 where $\kappa(\varphi)$ (in contrast to $\Delta\Theta(\varphi)$) has no finite limit from the left. Of course, in case there is no finite limit from the right, the analogous property is similarly assumed.*

   Restricted to some interval $[\varphi_0 - \epsilon, \varphi_0)$; $\omega(\varphi) = \frac{2+\kappa(\varphi)}{\cos\varphi}$ is a monotonic function of φ.

Now we are ready to formulate our main theorem.

Theorem 25. ***Suppose that the soft billiard system*** (M, T, μ) ***satisfies property H and the rotation function is regular. Suppose furthermore that there are no corner points and the horizon is finite (***$0 < \tau_{min}, \tau_{max} < \infty$***).***

Then, the dynamics enjoys, in addition to ergodicity and hyperbolicity, exponential decay of correlations and the central limit theorem for Hölder-continuous functions.

3.3.1 Specific potentials

As corollaries of Theorem 25, exponential decay of correlations can be established for certain specific potentials. To prove such corollaries we need to calculate the rotation function $\Delta\Theta(\varphi)$ from the potential $V(r)$.

Corollary 26. *Consider the case of a constant potential,*

$$V(r) = V_0 \text{ for any } r \in [0, R).$$

Correlations decay with an exponential rate in case

- $V_0 > 0$ *and the configuration is arbitrary,*
- $V_0 < 0$ *and the configuration is such that* $\tau_{\min} > \frac{2R}{\sqrt{1-2V_0}-1}$.

Corollary 27. *Given constants* $A > 0$ *and* $\beta > -2$*, consider the potential*

$$V(r) = A\left(1 - \left(\frac{r}{R}\right)^{\beta}\right).$$

Correlations decay at an exponential rate in case:

- $A = \frac{1}{2}$, $0 > \beta(> -2)$ *and the configuration is arbitrary,*
- $A = \frac{1}{2}$, $\beta > 0$ *and the configuration is such that* $\tau_{\min} > \frac{2R}{\beta}$.

4 Outlook

As described in the Introduction, much of this work is aimed at exponential decay of correlations in high-dimensional systems. Moving in that direction, a first step is already taken: hyperbolicity is proven for a class of multi-dimensional soft billiards [5]. To proceed towards decay of correlations, we plan to follow the way described here.

1. As to the possibly most direct challenge, we conjecture that there exist rapidly mixing potentials for which the condition $|\kappa + 2| > c$ (i.e. property H from Definition 22) is not satisfied for nearly tangential trajectories. Thus these systems are not covered by Theorem 25, even more, at least to our knowledge, there is no result in the literature on the ergodicity or hyperbolicity of such soft billiards either. Thus we make the following

Remark 28. *It is possible that κ tends to -2 as $\varphi \to \frac{\pi}{2}$, nevertheless, $|\frac{2+\kappa}{\cos\varphi}| > c$ and the system can be hyperbolic (possibly ergodic or exponentially mixing). This question is under investigation.*

The difficulty with the treatment of this case is that the separate investigation of motion inside and outside the disks seems not to work at several arguments.

For such a system, even a proof of hyperbolicity in two dimensions would be interesting in itself.

2. In the higher dimensional case, that is, softenings of multi-dimensional dispersing billiards (motivated e.g. by the three dimensional Lorentz process with spherical scatterers) even ergodicity seems to be difficult. That is because the algebraic approach to handle singularities does not work for soft systems – so no fundamental theorem is known at the moment. That is why a soft system where $|\frac{2+\kappa}{\cos\varphi}|$ is bounded away from both zero and infinity may be easier to handle. In this case the pathology in the singularities does not appear.

Another direction of future research, motivated mainly by applications to Physics, could be the further investigation of those systems for which rapid mixing is already established. For example, as mathematical evidence on the existence of diffusion and other transport coefficients is given, it would be interesting to understand the dependence of these on certain parameters like the full energy level.

Last but not least, in contrast to the generality of Theorem 25, it is striking how narrow the class of specific potentials is for which we could apply the result in Section 3.3.1. It would be desirable to establish – at least numerically – our reasonable regularity properties for as wide a class of potentials as possible.

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