Patterns of the winning positions in the chips taking game and other discrete dynamical systems

Research proposal, 2025 Spring

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1 Problem description

The chips-taking game is a two-player combinatorial game which proceeds as follows. There are n chips on a table and a finite set of positive integers, A is fixed. Two players take chips alternating. In each round, each player can take any $x \in A$ chips. The game terminates when the current number of chips is smaller than $\min(A)$ and thus no moves are possible. The player who makes the last move is the winner. Who has a winning strategy? The winning strategy depends on n and A. For example, if $A = \{1, 2, \ldots, k-1\}$, then the second player has a winning strategy if and only if $n \equiv 0 \mod k$.

The winning positions can be computed for an arbitrary A by a dynamic programming algorithm. Let $f_A(n)$ be 1 if the first player has a winning strategy when the game starts with n chips, and let $f_A(n)$ be 0 if the second player has a winning strategy. Then it is easy to see that

$$f(n) = 1 - \min_{x \in A} \{f(n-x)\}. \tag{1}$$

It is also easy to show that the winning positions eventually become periodic, and the period length is at most $2^{\max(A)}$. The typical period length is a linear function of $\max(A)$, and it is an open question if any exponential period length is possible. In a previous research class in 2022 Spring, we proved the existence of a superpolynomial period length in a slight modification of the chips-taking game, see [3].

Proving or disproving the existence of exponential period lengths might be too hard. However, there are many other interesting (and hopefully easier) questions. The main goal of the research class is to make significant progress in at least one of the following three directions.

1.1 The pattern of the winning positions of 3-move rulesets

In the previous research class, we completely characterized the pattern of the winning positions when $|A| \leq 3$, and solved the problem for some special $|A| = 3$. We call such cases 3-*move rulesets.* Experimental results suggest the emergence of a fractal like structure of possible

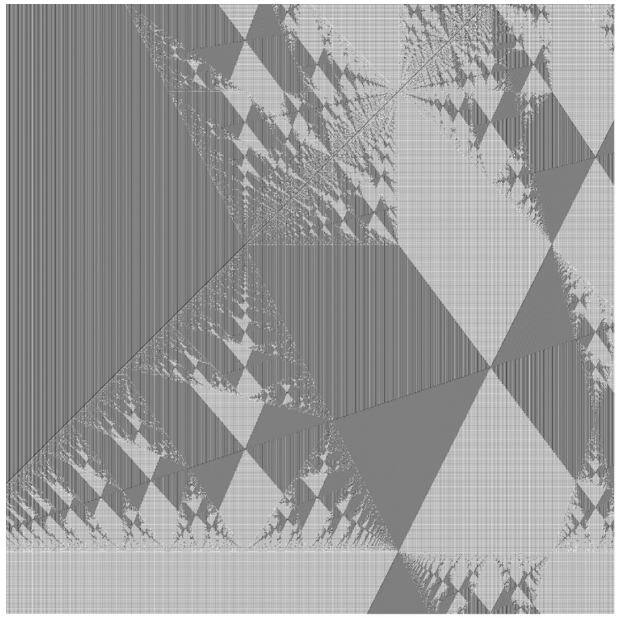

Figure 1: The fractal structure of the period lenghts in 3-move rulesets. From [1]. See also the text.

period lenghts in 3-move rulesets. Particularly, fix a large number s_3, and consider all possible rulesets $A = \{s_1, s_2, s_3\}$ with $s_1, s_2 < s_3$. Then the period lengths in the majority of the cases will be $s_1 + s_3$ or $s_2 + s_3$ or $s_1 + s_2$. It is conjectured that the density of cases with other periods tends to 0 as s_3 tends to infinity [1]. Furthermore, the previously mentioned 3 possible period lengths form a fractal structure on the (s_1, s_2) 2D plane, and this fractal structure does not seem to depend on s_3, see Figure 1 and also [1]. We would like to prove the existence of this fractal structure.

1.2 Periodicity in the 2D chips taking game

The chips taking game can be extended to any dimension. The 2D game considers two piles of chips, and a finite set of ordered pairs of non-negative integers $A \subset \mathbb{N} \times \mathbb{N} \setminus \{0, 0\}$. It is easy to extend the dynamic programming recursion (Equation 1) to 2D (and in general, to an arbitrary dimensional) game. However, the winning positions rarely form a single periodic pattern for the whole positive quarter of the starting positions. Instead, there might be a few periodic segments (infinite strips or cones) of the patterns of the winning positions, see for example, [2] and Figure 2. It is conjectured that there are at most $|A| + 1$ periodic strips, and also, the middle cone might not be periodic [2]. We would like to investigate this problem.

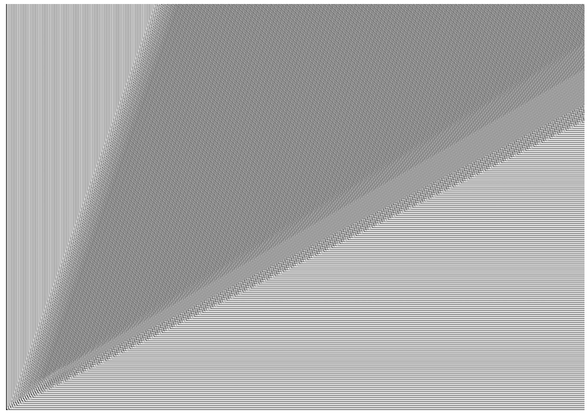

Figure 2: Periodic strips of the winning positions in a 2D subtraction game. From [2].

1.3 Patterns in similar discrete dynamics

Equation 1 can be seen as a not-AND logical operation. Indeed, consider the mapping φ : $\{0,1\} \to \{TRUE, FALSE\}$ with $\varphi(0) = FALSE$ and $\varphi(1) = TRUE$. Then for the k-ary function f mapping from $\{0,1\}^k$ to $\{0,1\}$ with rule $f(X) = 1 - \min_{i=1}^{k}\{x_i\}$, it holds that

$$\varphi(f(X)) = \overline{\varphi(x_1) \wedge \varphi(x_2) \wedge \ldots \wedge \varphi(x_k)}.$$

This motivates the study of the discrete dynamics of 0-1 functions or in general any finite (symmetric) functions with the following setup. Let S be a finite set, and let f be a k-ary (symmetric) function from S^k to S. Further, let A be a finite set of k positive integers, and let $\mathcal{S} = (s_1, s_2, \ldots, s_{\max(A)})$ be a finite vector from $S^{\max(A)}$. Define $w(n) := s_i$ for $n \leq \max(A)$, and for all $n > \max(A)$ we define

$$w(n) = f(w(n - a_1), w(n - a_2), \ldots, w(n - a_k)).$$

What can we say about $w(n)$? For which f, A and initial values $\mathcal{S}$ it will have an exponential period length?

2 Qualifying problems

Please, solve the first three exercises and at least one of the other exercises.

1. Prove the following statement. In the 1D chips taking game, if $A = \{1, 2, \ldots, k-1\}$, then the second player has a winning strategy if and only if $n \equiv 0 \mod k$.

2. Prove the correctness of the dynamic programming recursion (Equation 1). Give the extension of this recursion for an arbitrary d dimensional chips taking game.

3. Prove that in the 1D chips taking game, the winning positions become periodic for any A, and the period length is at most $2^{\max(A)}$. Also prove the periodicity of $w(n)$ given in subsection 1.3, and give an upper bound on the period length.

4. Extend the discrete dynamics in subsection 1.3 to 2D. Let $A = \{(0,1),(1,0),(1,1)\}$. Fix a finite set S, a finite 3-ary function $f : S^3 \to S$. Chose an $s_0 \in S$, and for each i, let $w(i,0) = w(0,i) = s_0$. Then for each $i, j > 0$, let

$$w(i,j) = f(w(i,j-1), w(i-1,j), w(i-1,j-1)).$$

   Prove that for any fixed row i, the row will be eventually periodic. That is, there exists a j_0 and p such that for all $j > j_0$, $w(i,j) = w(i,j+p)$.

5. In the 1D chips taking game, find an A of size 3 such that the pattern of the winning positions has an irregular start (also called preperiod) before becoming periodic.

6. In the 2D chips taking game, find a ruleset such that the winning positions form a single periodic pattern.

References

[1] Larsson, U., Saha, I. (2024) A brief conversation about subtraction games, `https://arxiv.org/abs/2405.20054`.

[2] Larsson, U., Saha, I., Yokoo, M. (2024) Subtraction games in more than one dimension, `https://arxiv.org/abs/2307.12458`.

[3] Miklós, I., Post, L. (2024) Superpolynomial period lengths of the winning positions in the subtraction game. International Journal of Game Theory, `https://link.springer.com/article/10.1007/s00182-024-00911-5`