Always graphic uniform hypergraphic degree sequences

Research proposal, 2024 Spring

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1 Problem description

A hypergraph $H = (V, E)$ is a generalization of graphs, where $E(H) \subseteq 2^V \setminus \emptyset$, that is, a hyperedge is a non-epmpty subset of the vertices. Clearly, a simple graph is a hypergraph where each hyperedge is a subset of vertices of size 2. A hypergraph is a *k-uniform hypergraph* if each hyperedge is a subset of size k (that is, simple graphs are 2-uniform hypergraphs). The *degree* of a vertex is the number of hyperedges incident with it. We denote the degree of v by $d(v)$.

We are interested in the following problem

Problem 1 (k-UNIFORM HYPERGRAPHIC DEGREE SEQUENCE REALIZATION)**.**
INPUT *degree sequence, $D := (d_1, d_1, \ldots, d_n)$ and a positive integer k.*
OUTPUT *"Yes" if there is a k-uniform hypergraph $H = (V, E)$ such that for all i $d(v_i) = d_i$, and "No" otherwise.*

When the answer is "Yes", such an H hypergraph is called the *realization* of the degree sequence D. If D has a realization, we say that D is a *graphic* degree sequence ("graphical degree sequence" is an obsolate definition).

Surprisingly, the problem whether or not there is a 3-uniform hypergraph with prescribed degrees is already an NP-complete problem [1]. However, there are special degree classes for which it is easy to decide if a partite, 3-uniform hypergraph exists with those given degreees. Even more notably, we can give degree sequence classes wich are *always graphic*, that is, any degree sequence in that class is graphic.

In the 2023 Summer research class, we had the following result:

Theorem 1. *[3] If D is a hypergraphic degree sequence on n vertices such that each degree is at least $\frac{2\left\lfloor\frac{n}{3}\right\rfloor^2}{7} + 4 * \left\lceil\frac{n}{5}\right\rceil + 1$ and at most $\frac{5\left\lfloor\frac{n}{3}\right\rfloor^2}{7}$, $n \geq 45$ and the sum of the degrees can be divided by 3, then D has a 3-uniform hypergraph realization.*

The proof is constructive and provides an efficient algorithm to construct a realization. We feel that this bound is far from being tight. The main goal of the proposed research class is to find wider bounds in which every degree sequence with degree sum 0 modulo 3 is always graphic. We will start focusing on 3-uniform hypergraphs, and we would also like to generalize results to k-uniform hypergraphs.

2 Qualifying problems

2.1 Learning the background theory

- The key concept seems to be the *hinge flip operations* and their relation to the graphicality of degree sequences. Please read our manuscript (`https://arxiv.org/pdf/2312.00555.pdf`) from the beginning till Theorem 2.2., and make sure you have understood the concept of hinge flips.

- Consider the neighborhood of a vertex in a 3-uniform hypergraph. It is a set of pair of vertices, and we can look at it as the edge set of a simple graph. This explains why simple graphs and their degree sequences are considered in the research of 3-uniform hypergraphs.

  The Erdős-Gallai theorem on the graphicality of simple graph degree sequences might also play a central role in our research that we state here.

  Theorem 2. *[2] Let $D := d_1 \geq d_2 \geq \ldots \geq d_n$ be a degree sequence. Then G is graphic if and only if*

  1. $\sum_{i=1}^{n} d_i$ *is even and*
  2. *for all* $k = 1, 2, \ldots n-1$,

  $$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{d_j, k\}.$$

- The Tripathy-Vinay theorem improves the Erdős-Gallai theorem in the sense that it shows that is is sufficient to check only a certain subset of the inequalities [4]. Most notably, assume that a degree sequence consists of k number of degree $d_{\max}$ and $n-k$ number of degree $d_{\min}$. Then it is graphic if and only if the sum of the degrees is even and

  $$kd_{\max} \leq k(k-1) + (n-k)\min\{k, d_{\min}\}.$$

  Furthermore, if a degree sequence contains k number of degree $d_{\max}$ and $n-k-1$ number of degree $d_{\min}$ and exactly one degree d_x, $d_{\min} < d_x < d_{\max}$, then it is graphic if and only if the sum of the degrees is even and the following two inequalities hold

  $$kd_{\max} \leq k(k-1) + (n-k-1)\min\{k, d_{\min}\} + \min\{k, d_x\},$$

  $$kd_{\max} + d_x \leq (k+1)k + (n-k-1)\min\{k, d_{\min}\}.$$

2.2 List of problems

Please, solve at least the first 4 of the following exercises. Solutions to the remaining exercises will be considered in case of competition.

1. Prove the $\Rightarrow$ direction of the Erdős-Gallai theorem, that is, if G is graphic then the conditions necessarily hold.

2. Prove or disprove the following statement. Let $D = d_1, d_2, \ldots, d_n$ be a hypergraphic degree sequence, such that for some $0 \leq k < \binom{n-1}{2}$ each d_i is k or $k+1$, further, $\sum_{i=1}^{n} d_i \equiv 0 \mod 3$. The claim is that D has a 3-uniform hypergraph realization.

3. Generalize the previous exercise for k-uniform hypergraphs (and of course, solve it).

4. Given $0 < c_1 \leq c_2 < 1$, let $\mathcal{D}(c_1, c_2)$ denote the (infinite) class of degree sequences (of simple graphs) such that for each degree sequence $D \in \mathcal{D}(c_1, c_2)$ of length n the following holds:

   (a) The sum of the degrees is even in D, and

   (b) each degree d in D is at least $c_1 n$ and at most $c_2 n$ (equalities are allowed).

   Find $0 < c_1 < c_2 < 1$ such that $\mathcal{D}(c_1, c_2)$ contains finite number of non-graphic degree sequences. Suggestred strategy: first find some c_1 and c_2 for wich $\mathcal{D}(c_1, c_2)$ is always graphic. Use hinge-flips and the Tripathy-Vinay theorem. Find some c_1 and c_2 such that $\mathcal{D}(c_1, c_2)$ contains infinitely many degree sequences that are not graphic. What happens at the boundary?

5. Let $\mathcal{D}_{3-H}(c_1, c_2)$ denote the class of 3-uniform hypergraphic degree sequences such that for each $D \in \mathcal{D}(c_1, c_2)$ the following holds:

   (a) The sum of the degrees in D is $0 \mod 3$, and

   (b) each degree d in D of length n is at least $c_1\binom{n-1}{2}$ and at most $c_2\binom{n-1}{2}$ (equalities are allowed).

   Find $0 < c_1 < \frac{1}{2}$ such that $\mathcal{D}_{3-H}(c_1, 1-c_1)$ contains a non-graphic degree sequence, that is, a degree sequence that has no 3-uniform hypergraph realization.

6. Let $\mathcal{D}_{3-H}(c_1, c_2)$ denote the class of 3-uniform hypergraphic degree sequences such that for each $D \in \mathcal{D}(c_1, c_2)$ the following holds:

   (a) The sum of the degrees in D is $0 \mod 3$, and

   (b) each degree d in D of length n is at least $c_1\binom{n-1}{2}$ and at most $c_2\binom{n-1}{2}$ (equalities are allowed).

   Find $0 < c_1 < \frac{1}{2}$ such that $\mathcal{D}_{3-H}(c_1, 1-c_1)$ is always graphic.

References

[1] Deza, A., Levin, A., Meesum, S.M., Onn, S. (2019) Hypergraphic degree sequences are hard. `https://arxiv.org/pdf/1901.02272.pdf`

[2] Erdős, P., Gallai, T. (1960) Graphs with vertices of prescribed degrees (in Hungarian). Matematikai Lapok, 11:264–274.

[3] Li, R. Miklós, I. (2023) Dense, irregular, yet always graphic 3-uniform hypergraphic degree sequences, `https://arxiv.org/abs/2312.00555`

[4] Tripathi, A., Vijay, S. (2003) A note on a theorem of Erdős and Gallai, Discrete Math. 265:417-420.