# Easy and hard puzzles related to graph degree sequence problems

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## 1 Problem description

A degree sequence $D = d_1, d_2, \ldots d_n$ is a series on non-negative integers. A degree sequence is graphical if there is a vertex labeled simple graph $G$ in which the degrees of the vertices are exactly $D$. Such graph $G$ is called a realization of $D$. Similarly, a bipartite degree sequence is a pair of series of non-negative integers, $D = \{d_{1,1}, d_{1,2}, \ldots, d_{1,n}\}, \{d_{2,1}, d_{2,2}, \ldots, d_{2,m}\}$, which is graphical if a vertex labeled bipartite graph realization exists whose degrees are exactly the given degrees.

A graph is a multigraph if more than one edges might go between two vertices.

Deciding if a simple graph or a bipartite graph realization exists for a degree or a bipartite degree sequence is an easy algorithmic problem. That is, polynomial running time algorithms exists to decide if a realization exists, and if so, the algorithm also constructs a realization. See the Appendix for such an algorithm. Deciding if a multigraph realization exists for a degree sequence is an even easier problem.

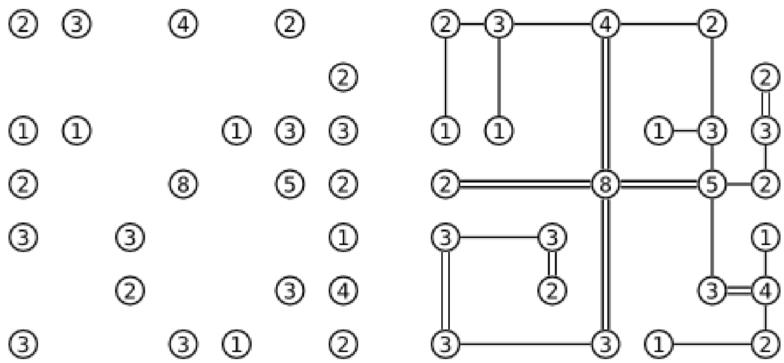
However, if constraints are added to the degree sequence problems, then these problems might become hard. This makes some puzzles challenging, of which we introduce three of them.

### 1.1 Hashi or Hashiwokakero

There are islands at some points on a $2D$ grid. Each island has an integer number between 1 and 8, including. The aim of the game is to add bridges such that

1. Each bridge connects two islands, travelling a straight line either horizontally or vertically.
2. Bridges must not cross any other bridges or islands
3. There can be at most two (parallel) bridges between two islands.
4. The number of bridges connected to each island must match the number on that island.
5. The bridges must connect the islands into a single connected group.

An example puzzle with its solution is shown below:

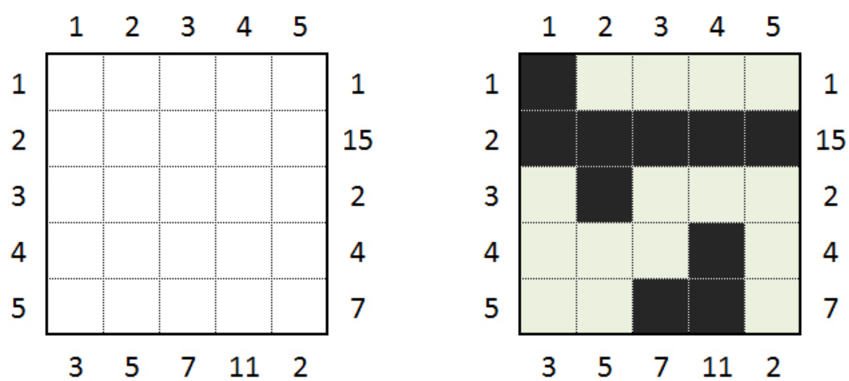


*By Val42 at English Wikipedia, CC BY-SA 3.0,*
*https://commons.wikimedia.org/w/index.php?curid=27632733*

Clearly we can consider a solution as a multigraph with maximum two parallel edges. The numbers on the islands are the degrees. However, it is restricted which vertices might be adjacent, furthermore, the edges cannot cross each other and we are looking for a connected realization.

## 1.2 Kakurasu

Kakurasu is played on a rectangular grid. For each row and each column, a sum of weights is given. The aim of the game is to make some of the cells black such that the sum of their weight is the given row and column sums. The weight of a black cell at position $(i, j)$ has a row weight $j$ and a column weight $i$, that is, it adds a weight $j$ to the row sum in row $i$, and it adds a weight $i$ to the column sum in column $j$. An example puzzle with solution is shown below:

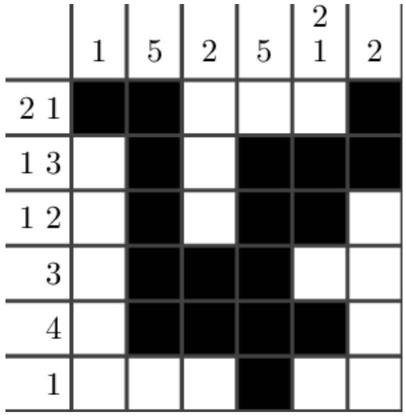



*from http://www.curiouscheetah.com/Museum/Puzzle/Kakurasu*

If each weight was uniformly 1 instead of a value based on the row and column index, then the prescribed row and column sums would form a bipartite degree sequence, and any solution would be the adjacency matrix of a realization with black cells equal 1 and white cells equal 0. Recall that the adjacency matrix $A(G)$ of a bipartite graph $G = (U, V, E)$ with $|U| = n$ and $|V| = m$ is an $n \times m$ matrix of 0's and 1's such that $a_{i,j} = 1$ if and only if $(u_i, v_j) \in E$.

## 1.3 Nonogram

Nonogram is a picture logic puzzle. On a rectangular grid, a sequence of positive integers are given for each row and each column. The aim of the game is to make some cells black such that for each row and each column, the runs of consecutive black cells have lengths as the prescribed sequence of integers. An example puzzle with solution is shown below:




If we replace each black cell by 1 and each white cell by 0, we get the adjacency matrix of a bipartite graph whose degree sequence can be obtained by adding the sequence of numbers for each row and column. That is, we can look at a nonogram puzzle as asking for a special realization of a bipartite degree sequence in which the edges are grouped in a prescribed way.

### 1.4 Previous results and research plan

Hashi and nonogram puzzles are both known to be hard (NP-complete to decide if a solution exists) [1, 3]. Kakurasu is also suspiciously NP-complete. A special case of the nonogram puzzle problem where each row and each column contains exactly one run of black cells is solvable in polynomial time [2].

The aim of this research class is to study what happens with these puzzle problems if some of the constraints are modified: relaxed or even extended. Will the problems become easy or remain hard? For example:

1. What makes Hashi hard? The connectivity? The forbidden crossing bridges? Parallel bridges? What happens if bridges might cross and a solution might contain disconnected islands? What happens if parallel bridges are not allowed?

2. What happens if the definition of the weights are modified in Kakurasu? For example, what happens if each entry in the grid have some arbitrary prescribed weights, however, these weights are bounded?

3. Consider the weighted nonogram puzzle problem where instead of black cells, positive integers should be written into the grid. The row and column sequences prescribes the sum of consecutive positive integers. Is it a simpler or even harder problem?

## Qualifying problems

Please, solve the following exercises:

1. Construct nonogram puzzles with multiple solutions. Particularly, construct a nonogam puzzle with multiple solutions where each row and each column contains exactly one run of consecutive black cells. Also, construct a nonogram puzzle with multiple solutions where some of the rows and/or columns contain multiple runs of consecutive black cells.

2. ~~Using pigeonhole principle,~~ show that there are impossible $n \times n$ Kakurasu puzzles for any sufficiently large $n$, that is, with no solution. We assume that the prescibed weights are between 1 and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. (No need to use the PHP in this exercise!)

3. Construct an impossible nonogram puzzle for which the sum of the given integer sequences form a graphical bipartite degree sequence.
4. Construct a Hashi puzzle with multiple solutions.
5. Construct an impossible Hashi puzzle that could have a solution if crossing bridges were allowed.

# References


[1] Andersson, D. (2009) Hashiwokakero is NP-complete. Information Processing Letters, 109(19):1145-1146.

[2] Chrobak, M, Dürr, C. (1999) Reconstructing hv-convex polyominoes from orthogonal projections, Information Processing Letters, 69(6):283–289.

[3] Hoogeboom, H. J., Kosters, W., van Rijn, J. N., Vis, J. K. (2014). Acyclic Constraint Logic and Games. ICGA Journal. 37 (1): 3–16.


# Appendix

**Theorem 1** (Havel-Hakimi)**.** *A degree sequence $D = d_0 \geq d_1 \geq d_2 \geq \ldots \geq d_n$ is graphical if and only if the degree sequence $D' = d_1 - 1, d_2 - 1, \ldots, d_{d_0} - 1, d_{d_0+1}, \ldots, d_n$ (with some possible reordering) is graphical.*

*Proof.* The backward direction is trivial: if $D'$ is graphical, take a realization of it, and extend it with one vertex, call it $v$, and add edges between $v$ and the first $d_0$ vertices. Then we get a graph whose degrees are $D$, thus $D$ is also graphical.

Proving the forward direction is done in an iterative way. Let the vertices be indexed by their degree indices, namely, $v_i$ is the vertex with degree $d_i$. We show if $D$ is graphical then such a realization also exists in which vertex $v_0$ is adjacent to the vertices $v_1, v_2, \ldots, v_{d_0}$ . Assume that in a realization of $D$, there is an index $i$ such that $v_0$ is not adjacent to $v_i$, although $i \leq d_0$. Let $i$ be the smallest such index. Then there must be an index $j$ such that $j > i$, and $v_0$ is adjacent to $v_j$. We know that $d_i \geq d_j$, therefore amongst the neighbors of $v_i$, there must be a vertex which is not a neighbor of $v_j$. Let this vertex be $v_k$. Then edges $(v_0, v_j)$ and $(v_i, v_k)$ exist in the realization, and $(v_0, v_i)$ and $(v_i, v_k)$ do not exist. If we delete the before mentioned existing edges and add the not existing edges, we get a realization of $D$ in which $v_0$ is adjacent to $v_i$, thus the first index $i$ for which $v_0$ is not adjacent to $v_i$ is greater than $i$. We can repeat this alteration such that eventually $v_0$ is adjacent to $v_1, v_2, \ldots, v_{d_0}$. Then deleting $v_0$ and its incident edges leads to a realization of $D'$. $\square$

The proof is constructive, namely, it is also possible to construct a realization of $D$ if such exists by following the proof: take $n+1$ vertices, index it with $v_0, v_1, \ldots v_n$. Add edges between $v_0$ and $v_1, v_2, \ldots, v_{d_0}$. Then take the sequence $d_1 - 1, d_2 - 1, \ldots, d_{d_0} - 1, d_{d_0+1}, \ldots, d_n$, reorder it, moving the vertices together with the degrees, so we get another degree sequence $D'$. Take the corresponding $v_0$, connect it to the next $d_0$ vertices, modify the degrees accordingly, rearrange them, etc. In this way, either we construct a graph with the prescribed sequence or at some point, $d_0$ will be greater than the number of remaining vertices with non-zero degrees, and thus, the degree sequence is not graphical.

Similar theorem is true for bipartite graphs and it is left as an exercise.