Semiring Theory in Algebraic Dynamic Programming

Research proposal, 2020 Spring

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Problem description

The key idea of algebraic dynamic programming is to separate how to build the search space and what to compute on it. Essentially, any dynamic programming algorithm can be described in this way, furthermore, these dynamic programming algorithms all compute partition functions over different semirings. For example, the semiring can be the natural numbers when the aim is to compute the number of objects with given constraints or the tropical semiring when the aim is to find a minimum scored object. The examples are numerous: one can find appropriate semirings to compute the partition function of objects with additive score, to compute moments of a distribution, to do a Pareto front optimization, to find the widest path in a directed graph, etc.

The aim of this research class is to explore the possible semirings applicable in dynamic programming, develop systematic ways to construct novel semirings for dynamic programming and generalize the results.

A detailed description of algebraic dynamic programming can be found here: `https://users.renyi.hu/~miklosi/2020SpringRES/ComputationalComplexityOfCountingAndSampling_Chapter2.pdf`

Assignment for the first week

You have to solve at least five of the following exercises by the end of the first week to join this research class.

1. There are n biased coins, and the coin with index i has probability p_i for a head. Give a dynamic programming algorithm that computes the probability that there will be exactly k heads when all the coins are tossed once. Give a time analysis of the algorithm.

2. Give a dynamic programming algorithm that takes a sequence of numbers and calculates the total sum of sums that can be obtained by inserting + symbols into the text. For example, if the input is 123, then the number to be calculated is

$$168 = 123 + (1 + 23) + (12 + 3) + (1 + 2 + 3)$$

   Give a time analysis of the algorithm.

3. A subsequence of a sequence is *isolated* if does not contain consecutive characters. Let $a_1, a_2, \ldots a_n$ be a series of real numbers, and treat each number as a character. Give a dynamic programming algorithm that calculates for this series

   (a) the largest sum of isolated subsequences and

   (b) the sum of the products of isolated subsequences of length k.

   Give a time analysis of the algorithm.

4. The following operation on symbols a, b, c is defined according to the following table:

	a	b	c
a	b	b	a
b	c	b	a
c	a	c	c

Notice that the operation defined by the table is neither associative nor commutative. Give a dynamic programming algorithm that counts how many parenthesizations of a given sequence of symbols $\{a, b, c\}$ there are that yield a. Give a time analysis of the algorithm.

5. Give a dynamic programming algorithm that calculates for each k how many increasing subsequences of length k are in a given permutation. Give a time analysis of the algorithm.

6. In a semiring $(R, +, *)$, let $a \leq b$ if $a + b = a$ or $a = b$. Show that $\leq$ is a partial ordering, furthermore, $(R, *, \leq)$ is a partially ordered semigroup, that is for all $a, b, c \in R$ if $b \leq c$, then $a * b \leq a * c$ and $b * a \leq c * a$.

7. Show that $(\mathbb{R} \cup \{\infty, -\infty\}, \max, \min)$ is a semiring. What is the additive and multiplicative unit?