Introduction to Extremal graph theory

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This is a cleaned up version of my three "Tallinn extremal lectures", on the Tallinn homepage and on my homepage

www.renyi.hu/~miki

covering my lectures in March 2026 in Estonia and also some of my similar lectures in China in 2025,. . .

- I shall repeat some important facts several times
- Updating: In the next weeks I may update this file occasionally, check my homepage

Sources

Extremal Graph Theory is one of the fastest developing area in Discrete Mathematics. In the last approximately 20 years I have given several lectures or series of lectures as introduction to extremal graph theory. For these lectures a good source on many details, e.g. survey papers, can be found on my homepage,

http://www.renyi.hu/˜miki

I also recommend the reading of the papers of Paul Erdős on his homepage, up to 1989:

http://www.renyi.hu/˜p_erdos

From the many sources I would recommend – among others – my first survey [54], and the Komlós-Simonovits paper [40] on the Regularity Lemma, further, the Füredi-Simonovits paper [34] on Degenerate Extremal graph problems.

- Here I avoid providing too many (several hundreds!) sources, and mostly I concentrate on the more introductory papers.

- Also I tried to emphasize some computer science aspects.

Explanation

"My Tallinn Lectures, March 2026" covered a small part of this extensive, beautiful and deep theory. It covers

- Introduction to extremal graph theory
- Random graphs, modified random graphs,
  - quasi-random graphs
- Regularity Lemma [63], [40], Removal Lemma [15]

Remark:

In extremal graph theory one needs among others, the Regularity Lemma, the Random methods, and the use of $O(f)$, $o(g)$.

Left out, being too difficult the first time

Most of the supersaturated graphs, the hypergraph results, e.g. hypergraph regularity, quasi-randomness, and graph limits.

My professors, 1969, Lake Louse

Kató Rényi, Paul Turán, Vera T. Sós, Paul Erdős

Two cultures

The slides where the title is in "Purple" were not mentioned in my "Tallinn lectures", yet I feel it is worth mentioning them. Originally the expression "two cultures" came from an English writer C.P. Snow, in connection with his friend, the famous mathematician G.H. Hardy. Nowadays we apply the expression "Two Cultures" to mathematicians

(a) going from general theories to particular results or

(b) from particular results to general theories.

Tim Gowers has a nice paper on it [35].

Luca/Rota: Theorems or Proof-techniques?

- Hashing? This is an interesting data-base technique in theoretical computer science
  - Ajtai-Komlós-Szemerédi [1]
- Property testing
  - Alon and Shapira [6]
- Delta systems Erdős-Rado [20], Razborov lower bound [49].

Erdős, Turán, Rényi . . .

Mostly the Budapest Number Theory and Discrete Math School, founding a problem, started from the simplest cases, solved several cases of it, and at least pin-pointed the first really difficult subcase of it.

Examples:

- Erdős-Turán, $r_k(n)$: arithmetic progressions in sets of positive density.
  - Green-Tao: arithmetic progressions of primes
- Turán hypergraph problems (still unsolved)
- Brown-Erdős-Sós [14]: (6,3)-problem, led to the [40] removal lemma.
  - (7,4) is still unsolved

Property testing and Erdős

Erdős formulated several several questions, problems, conjectures which may seem surprising at first sight, however later it turned out to be much more important, far reaching. One of them is strongly connected to

Problem (Property Testing, via Erdős)

If G_n cannot be changed into a bipartite graph H_n by deleting $\le \varepsilon n^2$ edges, then it contains a short odd cycle: an odd C_k of $\le f(\varepsilon)$ vertices.

Actually, Bollobás, Erdős, Simonovits, and Szemerédi proved this in [12] with $f(\varepsilon) = \frac{2}{\varepsilon}$.

Theorem

Let t be a natural number and let $\varepsilon > 0$. Then there is an n_0 such that if $n > n_0$ and if G has n vertices, and $C_m(t) \not\subset G_n$ for $m = 3, 5, \ldots, 2\ell + 1$, where $2\ell + 1 > c - 1$, then G_n can be turned into a bipartite H_n by deleting fewer than εn^2 edges. 1

Remark (With and without Regularity Lemma)

The above theorem is proved in [12] in two distinct ways. One of them could be regardes as using a simple form of the Regularity Lemma, the other solution is more elementary.

Searching for algorithmic solutions

There were more and more existence proofs where one felt that a construction for the objects would be important. Such cases are e.g. the

- the Ramsey graphs,
- expander graphs,
- some applications of the Regularity Lemma,
- Some applications of the Lovász Local Lemma

In some of these cases the original existence proofs were replaced by algorithmic existence proofs. In some of these cases the reader will find later the corresponding references.

Here I primarily introduce Extremal graph theory

- These slides do not cover my 4th "Tallinn lecture" on deterministic and randomized estimation of the volume and the diameter, i.e. the approach by deterministic and randomized algorithms,

  – the application of Monte Carlo Markov Chain methods. They are posted separately. 2

- Since it is difficult to follow lectures where the slides are very near to what are explained in the lecture, therefore many details mentioned below were mentioned in my lecture but not seen on my original slides.
- I shall also add a list of related papers, or other sources, however, I wanted to keep the list relatively short, so many excellent items were left out.
- On the last slide before the references I shall include a short list of topics which I left out, though I would have included if I had twice as many lectures.

Some notation

G_n, H_n, S_n, $T_{n,p}$ are n-vertex graphs. In general, (almost always) if a graph is denoted by a capital letter with some subscripts, then THE FIRST SUBSCRIPT DENOTES THE NUMBER OF VERTICES.

- $e(G)$, $v(G)$, $\chi(G)$ denote the number of edges, vertices, and the chromatic number, respectively.

A vertex set X is independent if no two vertices of X are connected by an edge. $\alpha(G)$ is the maximum number of independent vertices in G.

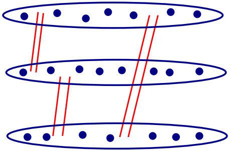
Chromatic number $\chi(G)$: is the minimum number of independent sets covering the vertices of G.

Exceptions: Listing graphs, like $L_1, L_2, \ldots, L_m$.

- Also the complete p-partite graph $K_p(n_1, \ldots, n_p)$ is an exception, having $\sum n_i$ vertices.

Turán theorem

$T_{n,p}$: n vertices are partitioned into p classes as uniformly as possible, and two vertices are joined if and only if they belong to distinct classes.



Another important way to define $T_{n,p}$ is to say that it is the $\leq p$-chromatic n-vertex graph of the maximum number of edges.

Theorem (Turán, 1941 [65])

Among the graphs G_n (on n vertices) not containing a K_{p+1}, the Turán graph $T_{n,p}$ below has the most edges.

Turán type problems

For a while here we consider simple graphs: no loops, no multiple edges. In the general Turán type problems we have a family $\mathcal{L}$ of excluded graphs and wish to maximize $e(G_n)$ under the conditions that G_n contains no $L \in \mathcal{L}$.

- The maximum will be denoted by $\mathbf{ex}(n, \mathcal{L})$, the graphs attaining this maximum will be called Extremal Graphs.
- The important step was that Turán asked to determine this maximum in the general case, e.g., for the Platonic Bodies, for the path P_k, . . .

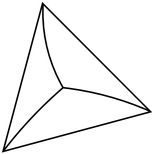
Outline

If I had much more time, I would follow the following outline:

- Stability in general,
- Extremal graph theory, short description,
- Progressive induction [53]: the beginning of stability method
  - Examples when Progressive Induction worked
- **Erdős-Simonovits stability**
- Cases when we do not have stability
  - e.g. multigraphs and
  - (3-uniform) hypergraphs
- **Regularity Lemma** and stability
- Supersaturated graphs
- Lovász-Sim. stability, and its applications
  - connection to Razborov flag algebras
  - how to prove Erdős-Sim. using Lovász-Sim.

Outline, continuation

- Stability, extremal subgraphs of Random Graphs,
  - and the Babai-Sim.-Spencer theorem
  - Extremal subgraphs of Random Graphs
- Stability and the Erdős-Frankl-Rödl theorem:
  How many L-free graphs G_n are there
  - Hidden stability results
- Stability for well-structured Ramsey problems:
- Stability in Kohayakawa-Sim.-Skokan theorems
- Stability in the Erdős-Sós Conjecture on trees?
  - and related newer results
- Hidden stability results (Ramsey-Turán problems)



Extremal Graph Theory

Extremal graph theory is one of the oldest areas of Graph Theory. In the 1960's it started evolving into a large, deep, connected theory.

This is a short version of a sequence of survey talks, which describe some major areas in the classical extremal graph theory. Here we shall also concentrate on some basic tools that turned out to be very useful in it, primarily

- On the stability method
- Random Graphs
- Regularity Lemma

Is extremal graph theory useful?

Turán thought YES,

because it is a generalization of Pigeon hole principle

- Yes, because it leads to important tools.
- Ramsey theory and Turán type extremal theorems are strongly connected to each other.
- They led to Random graphs, random and modified "random constructions"
- Erdős Magic: This is how Spencer calls the first applications of Random methods
- See also Shannon's random codes!
- They led to Szemerédi Regularity Lemma
- They led to Lovász Local Lemma, Janson inequality
- They led to graph limits
- Gowers methods?

Expander graphs

Expander graphs are important

in computer networks partly because they can replace randomness:
Randomness is difficult to check and people in algorithms do not like randomness

- Construction of expander graphs
- Lubotzky-Phillips-Sarnak, Ramanujan graphs

Why is this problematic?

Because it contains deep mathematics!

Why is this (CCA) good?

Because it provides several extremal objects: it is good in constructions

See also Wigderson+ . . . , zigzag

Methods applicable in many cases

- Parallel sorting, Ajtai-Komlós-Szemerédi [3]
- Sorting is important in algorithms, see e.g Donald E. Knuth: Art of computer programming
- Hashing, Ajtai-Komlós-Szemerédi [1] the importance of double hashing

Extremal graphs

We consider simple graphs: no loops, neither multiple edges G_n. S_n, $T_{n,p}$: n-vertex graphs

Given a family $\mathcal{L}$ of forbidden graphs

Subchromatic number:

$$p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1. \tag{1}$$

Extremal function:

$$\mathbf{ex}(n, \mathcal{L}) = \max \left\{ e(G_n) \ : \ \text{If } L \in \mathcal{L}, \text{ then } L \text{ is not contained in } G_n \right\}.$$

Extremal graphs: S_n attaining this maximum

- Later we shall have a short look at multigraphs and at hypergraphs

Turán problems, historical remarks

Formally the first extremal problem was an "exercise" which is mostly called today Mantel Theorem [46]. Perhaps the first real extremal result was that of Erdős [19] on the Multiplicative Sidon Problem. This had two problems, in one of which the extremal result on $\mathbf{ex}(n, C_4)$ was needed.

Problem (Multiplicative Sidon Problem)

How many integers $a_1, \dots, a_m$ can be chosen in $[1, n]$ so that the pairwise products are all different: if $a_i a_j = a_k a_\ell$ then $\{i, j\} = \{k, \ell\}$?

Here Erdős needed the lemma that if G_n is a C_4-free bipartite graph on n, n vertices, then $e(G_n) \le 3n\sqrt{n}$.

Historical remarks continued

Remark

Here later Erdős complained that he had not noticed that this is a beautiful area: he missed to discover it.

Actually, Erdős, Ko, and Rado had slightly earlier their important theorem, however, they published it only much later.

Theorem (Erdős-Ko-Rado [21])

Consider k-element subsets of an n-element set S, e.g. of the integers: $A_1, \dots, A_m \subset [1, n]$. If any two of them have non-empty intersection, then $m \le \binom{n-1}{k-1}$.

The sharpness can be seen by considering all the k-subsets containing a fixed element.

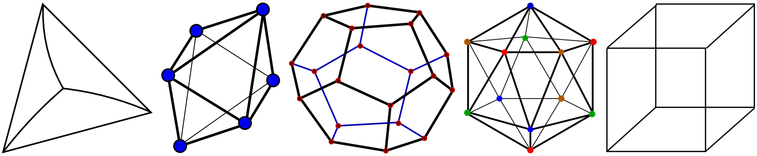
Turán problems

Theorem

$$\mathbf{ex}(n, \mathcal{L}) = o(n^2),$$

iff $\mathcal{L}$ *contains some bipartite graph. If it does not, then* $\mathbf{ex}(n, \mathcal{L}) > \left\lfloor \frac{n^2}{4} \right\rfloor$.

Given an arbitrary excluded graph L, e.g. the graph of a Platonic body, determine $\mathbf{ex}(n, L)$.



Central topic

Nearly extremal graphs are also in structure near to $T_{n,p}$:

Theorem (Erdős-Sim. Stability Theorem)

Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number

$$p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1. \tag{1}$$

For every $\varepsilon > 0$, there exist a $\delta > 0$ and an n_ε such that, if G_n contains no $L \in \mathcal{L}$, and if, for $n > n_\varepsilon$,

$$e(G_n) > \mathbf{ex}(n, \mathcal{L}) - \delta n^2, \tag{2}$$

then G_n can be obtained from $T_{n,p}$ by changing[a] at most εn^2 edges.

[a] deleting and adding

Critical edge theorem

Theorem (Sim. Inverse theorem)

The following assertions are equivalent:

(a) $T_{n,p}$ *is extremal for* $n > n_0$.

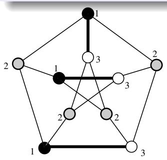
(b) $\chi(L) \geq p+1$ *for each* $L \in \mathcal{L}$ *and some* $L \in \mathcal{L}$ *has an edge e for which* $\chi(L-e) = p$.

(a) $T_{n,p}$ *is the only extremal graph for* $n > n_1$.

This implies that for any odd cycle C_k $T_{n,2}$ is the (only) extremal graph, if n is sufficiently large. It also implies that if for $n > n_0$ $T_{n,p}$ is extremal, then for $n > n_1$ $T_{n,p}$ is the only extremal graph.

This cannot be applied to the Petersen graph

since the 4-chromatic Petersen graph has no critical edges.



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Inverse extremal problems

Sometimes we change the Turan type extremal problems to Inverse Problems where we have a sequence of graphs S_n and ask, for which (finite) $\mathcal{L}$ are these graphs the Extremal Graphs for $\mathcal{L}$ and $n > n_0$.

Which objects: The universe

Generally we choose some combinatorial objects, (we could call their family the Universe, e.g.

- Simple graphs: no loops, no parallel edges
- multigraphs where the loops are excluded and two vertices can be joined by at most r parallel edges, and the loops are excluded
- r-uniform hypergraphs
- Subsets of integers
- some subsets of groups

Large or small excluded subgraphs?

In many extremal results we try to ensure the existence of a fixed subgraph L, or some $L \in \mathcal{L}$. In some others we consider spanning or almost sunning subgraphs, e.g. Hamiltonian cycles.

Theorem (Gábor Dirac [28])

If

$$d_{\min}(G_n) > n/2$$

then G_n is Hamiltonian.

Theorem (Hajnal–Szemerédi [36])

If the maximum degree of the graph G_n, $d_{\max}(G_n) < k$, then $V(G_n)$ can be partitioned into k independent sets of sizes $\leq \lceil n/k \rceil$[a].

[a] i.e. the sizes differ by at most one

Turán hypergraph problem, the simplest case

Hypergraph exremal problems are often hopelessly difficult.
Turán formulated his very famous hypergraph conjecture but had not realized that it is very difficult. Here we restrict ourselves to the simplest case of 3-uniform hypergraphs, and to the case of excluded complete 4-graph or 5-graph.

Problem

How many hyper-edges can a 3-uniform hypergraph $H_n^{(3)}$ without containing a complete 4-uniform sub-hypergraph

Unsolved, even in this simplest case!

It turned out that if Turán's conjecture holds, there exist many extremal subgraphs. Turán's conjectured extremal hypergraph had 3 classes, W.G. Brown has found an equally good hypergraph with 6 classes, [15], which was extended by Kostochka [41], . . . No stability!
If we have many conjectured extremal construction, that explains the difficulties.

Erdős-Ko-Rado (1938?) (1961)

Problem

How many hyper-edges can have a k-uniform hypergraph $H_n^{(k)}$ have without containing two independent edges?

Theorem (Erdős-Ko-Rado (1961))

The maximum number of hyperedges $H_n^{(k)}$ can have without containing two independent hyperedges edges is $\binom{n-1}{k-1}$ if $n > n_0(k)$.

Extremal problems in general

Problem ($\mathbf{ex}(n, \mathbb{P})$, $\mathbf{ex}(n, \mathbb{P}, x_f, x_g)$)

Given a Universe $\mathbb{U}$, a property $\mathbb{P}$ and some parameters $f(G)$, $g(G)$, $e(G)$ on the Universe, (for $G \in \mathbb{U}$) and we try to maximize $e = e(G)$ on $\mathbb{U} \cap \mathbb{P}$ under the assumption that $f(G) = x_f$, $g(G) = x_g$.

Universes:

- Simple graphs: no loops, neither multiedges
- Hypergraphs: primarily r-uniform hypergraphs
- Subsets of a set: Erdős-Ko-Rado, Sperner theorem, ...
- Integers: Erdős-Turán problem on Arithmetic Progressions, Sidon problem, Multiplicative Sidon Problem
- **Subsets of a group, e.g. Multiplicative Sidon Problem**

Examples

Problem (Chromatic perturbation)

What is the maximum of $e(G_n)$ *if* $K_4 \not\subset G_n$ *and* $\chi(G_n) > 120$*?*

Problem (Ramsey-Turán)

What is the maximum of $e(G_n)$ *if* $K_4 \not\subset G_n$ *and* $\alpha(G_n) < \frac{n}{\log\log n}$.

Problem (Hamiltonicity, G. Dirac)

What is the maximum of $d_{min}(G_n)$ *if* G_n *has no Hamiltonian cycle?*

Erdős-Simonovits Limit theorem

Subchromatic Number:

$$p := p(\mathcal{L}) = \min\{\chi(L) - 1 \ : \ L \in \mathcal{L}\}$$

Theorem (Erdős-Sim. Limit theorem)

$$\mathbf{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2)$$

An easy consequence of Erdős-Stone Theorem

Theorem (Erdős-Stone, 1946)

$$\mathbf{ex}(n, K_{p+1}(t, \dots, t)) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2)$$

Erdős-Simonovits Stability

Stability method:
Considering an optimization problem first we prove that the structure of almost optimal graphs is almost the same as the structure of the (conjectured) extremal objects.

Theorem (The First Stability Theorem)

Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number p. For every $\varepsilon > 0$, there exist a $\delta > 0$ and an n_ε such that, if G_n contains no $L \in \mathcal{L}$, and if, for $n > n_\varepsilon$,

$$e(G_n) > \mathbf{ex}(n, \mathcal{L}) - \delta n^2, \tag{3}$$

then G_n can be obtained from $T_{n,p}$ by changing[a] at most εn^2 edges.

[a] deleting and adding

How to prove the general Stability Thm?

There are several nice proofs, we concentrate on three proofs.

- The original Simonovits proof
- Using the Lovász-Simonovits Stability
- Using the Regularity Lemma and the Füredi Stability

How to prove the general Stability Thm?

There are several nice proofs, we concentrate on three proofs.

- The original Simonovits proof
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Trees

If $d_{\min}(G_n) > k-2$, then G_n contains all the k-vertex trees.

Complete bipartite subgraphs excluded

Lemma (Kővári-Sós-Turán)

$$\mathbf{ex}(n, K(a,b)) = O(n^{2-(1/a)})$$

Bondy-Simonovits

If G_n contains no C_{2k}, then $e(G_n) < c_k n^{1+(1/k)}$.

- Motivation: ...

Erdős-Sim., Cube = Q_8 thm

Theorem (Erdős-Sim)

$$\mathbf{ex}(n, Q_8) = O(n^{8/5})$$

See a much more general (recursion) theorem.
Missing any reasonable lower bound.

Trivial lower bound:

$$\mathbf{ex}(n, Q_8) > \mathbf{ex}(n, C_4) \approx \frac{1}{2} n\sqrt{n}.$$

since $C_4 \subset Q_8$.

Lower bound: Is it true that

$$\mathbf{ex}(n, Q_8) > n\sqrt{n} \log\log n \ ?$$

Lemma (Erdős, hypergraph version)

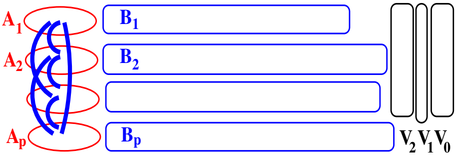
$$\mathbf{ex}_r(n, K_r(a, b, \dots)) = O(n^{r-(1/a^{r-1})})$$

Sidon problem

Lemma

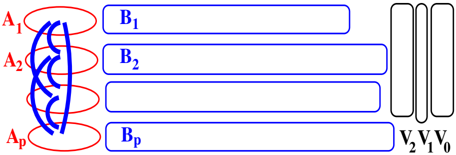
If $G(n,n)$ is a bipartite graph not containing C_4, then $e(G(n,n)) \leq 3n\sqrt{n}$

How to prove the general Stability Thm?



- V_0 : Delete small degree vertices, $d_G(x) < (1 - \frac{1}{p} - 2\sqrt{\delta})n$.

How to prove the general Stability Thm?



- V_0 : Delete small degree vertices, $d_G(x) < (1 - \frac{1}{p} - 2\sqrt{\delta})n$.

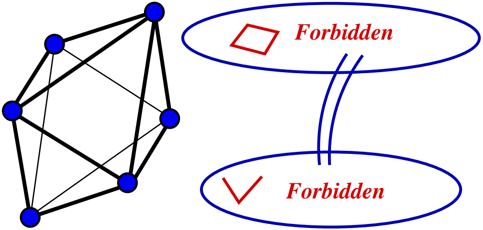
$|V_0| < 2\sqrt{\delta}n$. Otherwise $G \setminus V_0$ is too dense, contains a $T_{t(p+1),p+1}$.

- Locate a $T_{mp,p}$ with parts $A_1, \dots, A_p$. $m = m(p, \delta)$.
- $V_1 :=\{x : |N(x, A_i)| \geq t$ for all $i\}$. $|V_1| \leq \binom{m}{t}^p$.
- $B_i :=\{x : |N(x, A_i)| < t$ and for all $j \neq i$ $|N(x, A_i)| > \frac{t}{t+1}m\}$.
- $V_2 :=$ rest of the vertices. $|V_2| < O(pt(|V_0| + |V_1|))$.

Lemma

B_i does not contain a $K_{t,t}$ so $|E(G[B_i])| = O(n^{2-1/t})$.

Octahedron Theorem, Erdős-Sim.:



See Erdős-Rényi-T. Sós, and Füredi for C_4-free graphs, ...
A similar result was obtained by Griggs, Simonovits, and Rubin Thomas [36]
The paper [29] contains a much more general results, on $\mathbf{ex}(n, K_{p+1}(a_1, a_2, \ldots, a_{p+1}))$ however we shall not discuss those results here.

Stability, ε-δ form

Theorem (Stability, Erdős-Sim.)

For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ and an $n_0(\varepsilon)$ such that if $n > n_0$ and

$$e(G_n) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - \delta n^2$$

and G_n does not contain any $L \in \mathcal{L}$ then one can change $\leq \varepsilon n^2$ edges of G_n to get $T_{n,p}$.

Erdős magic

- Turán conjecture
- Erdős counter-example

Hypergeometric model

The random graph distribution used here is that in which, for each n, an integer E_n between 0 and $\binom{n}{2}$ is fixed, and each graph G_n with E_n edges has the same probability p_n, given by

$$p_n = \binom{\binom{n}{2}}{E_n}^{-1}.$$

Binomial model

In the binomial model we have an edge-probability p_n for each $n > n_0$. We choose the $\binom{n}{2}$ possible edges independently (!)

Equivalence of the two models?

- Only for "good properties":
  - monotone properties
  - convex properties

Corresponding threshold parameters:

$$p_n = \frac{E_n}{\binom{n}{2}}.$$

Theorem

If Q is a monotone property, then the thresholds in these two models correspond to each other.

- See Bollobás, yellow (Springer) book.

Stopping rule

We drop down (i.e. add the edges one by one) and stop when the minimum degree becomes 2.

Theorem (Ajtai-Komlós-Szemerédi [2], Bollobás [10] ???)

If we stop when the minimum degree becomes 2, then almost surely the obtained graph will be Hamiltonian.

Explanation?

- On the evolution of Random Graphs:
  - as long as the number of edges is small, we have trees, then some cycles appear, then larger and larger components, then one large component and many isolated vertices.
- The isolated vertices disappear around $E = (\frac{1}{2} \pm \varepsilon) n \log n$.

In details: Consider a random graph with edge-probability p.

- If $p = (\frac{1}{2} - \varepsilon)\frac{\log n}{n}$, then the obtained random graph R_n will have isolated vertices almost surely.
- If $p = (\frac{1}{2} + \varepsilon)\frac{\log n}{n}$, then the obtained random graph R_n will be connected almost surely.
- Even more:

If $p = (\frac{1}{2} + \varepsilon)\frac{\log n}{n}$, then the obtained random graph R_n will be Hamiltonian almost surely.

Importance

The theory of random graphs is an interesting, important, and rapidly developing subject. Applications of probabilistic methods have proved effective not only in graph theory, but in coding theory, analysis, and other areas of mathematics. We shall not go into details. Rather, we restrict ourselves to simple applications of random graph methods to extremal theory, concluding with a brief description of what we call pseudo-random graphs.

The Erdős-Rényi Threshold Theorem

Theorem (The Erdős-Rényi Threshold Theorem [26])

Let $\mathcal{L} = L_1, \ldots, L_t$ be a family of graphs, and let

$$c = c(\mathcal{L}) = \min_j \min_{H \subseteq L_j} \frac{v(H)}{e(H)}.$$

Further, let $\{E_n\}$ be a sequence of integers with $0 \le E_n \le \binom{n}{2}$ and let G_n be a graph of order n with E_n edges. Then the probability that G_n contains a member of $\mathcal{L}$ tends to

(a) 0, if $E_n = o(n^{2-c})$;
(b) 1, if $E_n / n^{2-c} \to \infty$.

(In extremal graph theory, only part (a) of this theorem is used.)

Lovász Local Lemma

Bollobás has a book on Random Graphs [11] and Janson, Luczak and Ruczinski also have a book on this [37].
One of the best sources on the application of Random methods is the Alon-Spencer book [7].

- One of the important tools in this area is the Lovász Local Lemma [30]. The meaning of it is that in some cases the joint occurrence of an event of very small probability is very difficult to prove. This is easy if they are independent events. The Lovász Local Lemma helps to extend this when they are not independent but only almost independent.

Lovász Local Lemma 2

Theorem (see e.g Alon-Spencer [7])

Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. A directed graph D is called a dependency graph for $A_1, \ldots, A_k$ on the vertex set $\{1, 2, \ldots, n\}$ if for each i in $[1, n]$ A_i is mutually independent from the events A_j for which (i, j) is not an arc of D. Assume that for some $x_1, \ldots, x_n$ in $[0, 1)$

$$\mathbb{P}(A_i) \leq x_i \prod_{(i,j)\in E(D)} (1 - x_j)$$

for $i = 1, 2, \ldots, n$. Then

$$\mathbb{P}(\bigcap_{i=1}^{n} \overline{A}_i) > \prod_{i=1}^{n}(1 - x_i).$$

In particular, with positive probability none of A_i holds.

Computer Science aspect

- The primary goal of Lovász Local Lemma was [30] to prove the existence of certain objects. However, the existence did not provide any of them.
- Later algorithms were devised to find such objects, e.g. by Moser and Tardos [40]
- many further important references

Generalized random graph

Definition

We have an $r \times r$ probability matrix $P = (p_{i,j})$ and r classes $U1, \ldots, Ur$ we join any $x \in Ui$ and $y \in Uj$ with probability p_{ij}, independently.

Notation (Edge-density)

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

Definition (ε-regularity)

Given a graph G_n and two subsets of vertices, $A, B \subseteq V(G_n)$, they are ε-regularly connected if for any $X \subseteq A$ and $Y \subseteq B$ with

$$|X| > \varepsilon|A| \text{ and } |Y| > \varepsilon|B|, \qquad |A|, |B| > \varepsilon n$$

we have $|d(X, Y) - d(A, B)| < \varepsilon$.

The meaning of the Regularity Lemma

Any graph G_n can be approximated by generalized random graphs in the following sense:

4 ↯

Regularity Lemma

As soon as Szemerédi has proved his Regularity Lemma, several aspects of the extremal graph theory have completely changed.

Most people would think it originates from the proof of the famous Erdős-Turán conjecture

Theorem (Szemerédi [63])

$r_k(n) = o(n)$.

No: This form was needed for Chvatál and Szemerédi [16] to prove a Bollobás-Erdős-Sim. conjecture from [13] on the quantitative Erdős-Stone.

Clusters, Reduced Graph

The classes V_i will be called groups or clusters.
Given an arbitrary graph $G = (V, E)$, a partition P of the vertex-set V into $V_1, \dots, V_k$, and two parameters ε, d, we define the Reduced Graph (or Cluster Graph) R_k as follows: its vertices are the clusters $V_1, \dots, V_k$ and V_i is joined to V_j if (V_i, V_j) is ε-regular with density more than d.

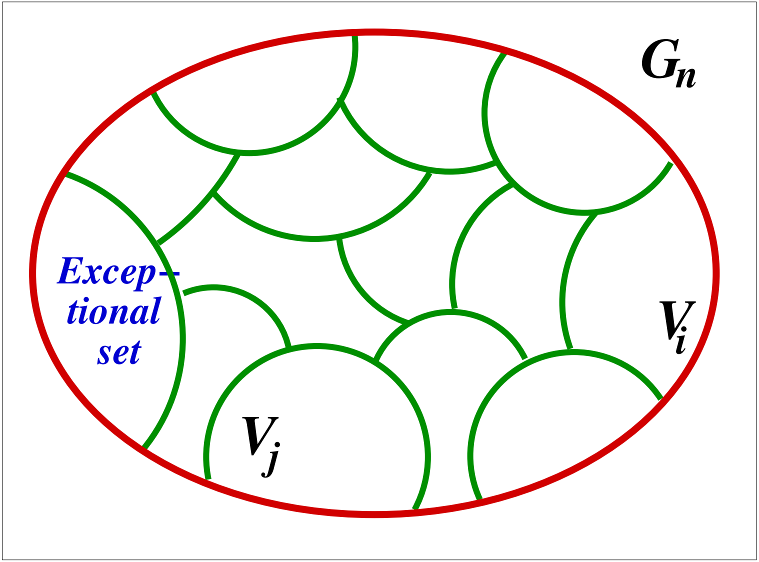
Most applications of the Regularity Lemma use Reduced Graphs, and they depend upon the fact that many properties of R_k are inherited by G.

Counting Lemma

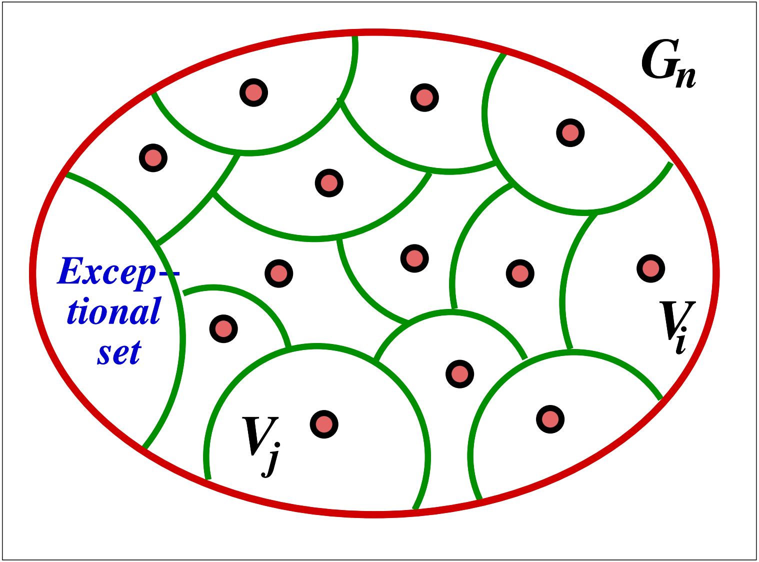
Through a simplified example:

- If the Generalized Random Graph corresponding to G_n contains many copies of L, then G_n also contains many (approximately the same number of copies of L)
- If the reduced graph contains an L then G_n contains at least $cn^{v(L)}$ copies of L.

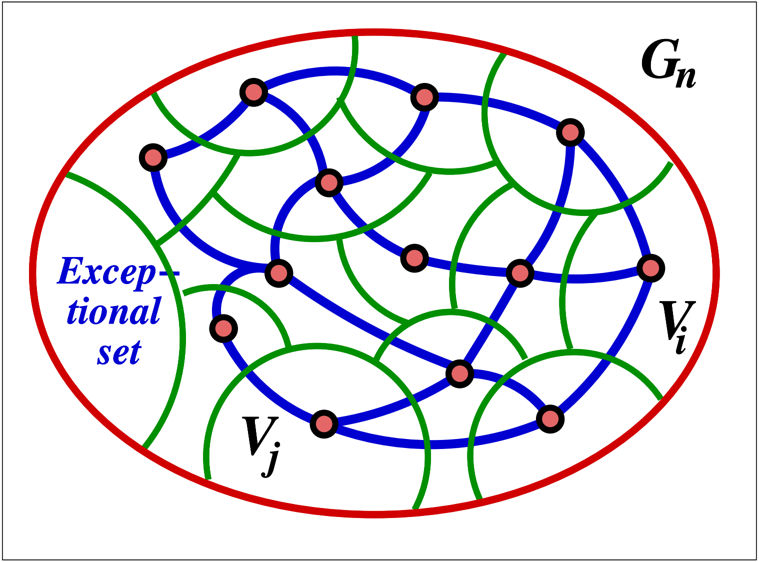
The Cluster graph, illustrated:



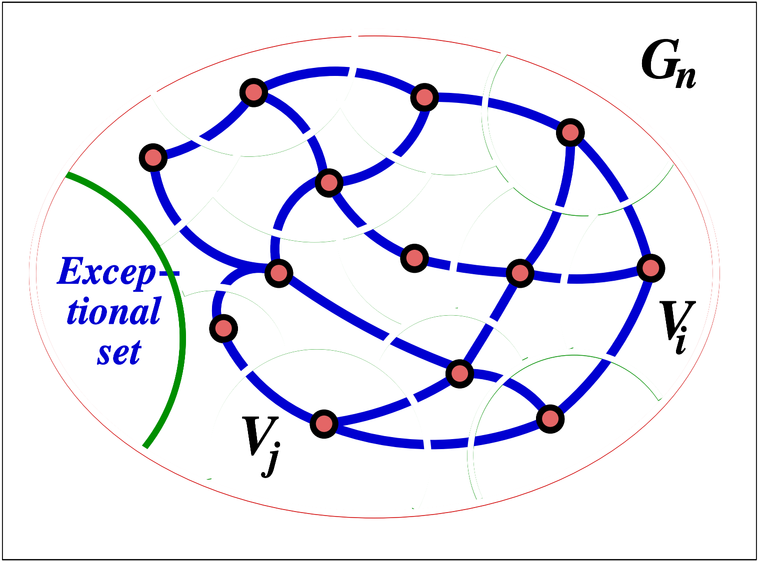
The Cluster graph, illustrated:



The Cluster graph, illustrated:



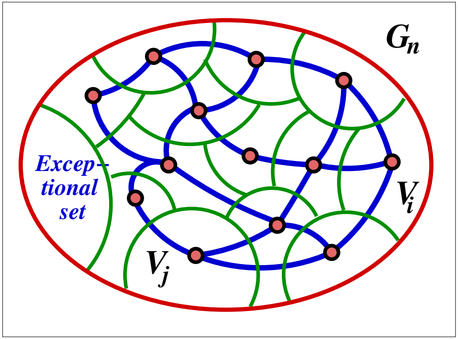
The Cluster graph, illustrated:



Distributing the exceptional vertices

- One can forget about V_0: its vertices can be distributed evenly, at random, in the other V_i's. Originally it was introduced to make the groups V_i exactly of the same size, to make the calculation easier (???)
-

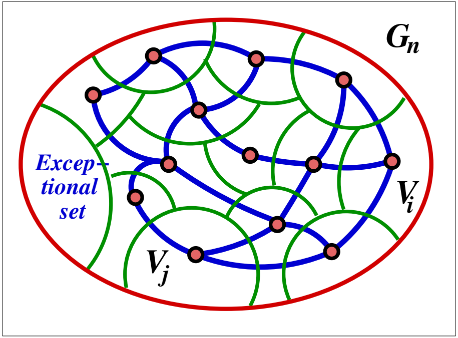
How to prove Erdős-Stone?



- No K_{p+1} in the Reduced graph H_k
- Apply Turán's theorem
- Estimate the edges of the original graph: **5**

$$e(G_n) \le e(H_k)m^2 + 3\varepsilon n^2.$$

How to prove Stability?



- No K_{p+1} in the Reduced graph H_k
- Apply Turán's theorem with stability (Füredi)
- Estimate the edges of the original graph

How to get rid of Regularity Lemma? And why?

Stability and the Regularity Lemma

If we have stability for a graph, we have stability for its blown-up versions.

- What would we need to get the Erdős-Sim. Stability?

The stability of the K_p-problem.

- Füredi's solution: Improving Zykov's symmetrization method, Füredi gives a simple, elementary proof for the stability of the K_p-extremal problem.

Importance:

This is interesting, but not too simple...

Füredi stability theorem

First: **Large p-chromatic subgraphs**

Theorem (Füredi 2015/2006)

Suppose $K_{p+1} \not\subset G$, $|V(G)| = n$ and

$$e(G) \geq e(T_{n,p}) - t.$$

Then there exists a p-chromatic subgraph H_0, $E(H_0) \subset E(G)$ such that

$$e(H_0) \geq e(G) - t.$$

Lovász-Simonovits inequality

Theorem

Let $v(G) = n$, $e(G_n) = E$. *Define* t *by*

$$E = e(G_n) = \left(1 - \frac{1}{t}\right)\binom{n}{2}.$$

Then

$$k_p(G_n) \geq \binom{t}{p}\left(\frac{n}{t}\right)^p. \tag{4}$$

Lovász-Simonovits stability theorem

Theorem

Let C be an arbitrary constant. There exist positive constants δ and C' such that if $0 < k < \delta n^2$ and G_n is a graph on n vertices for which

$$k_p(G_n) \le \binom{t}{p}\left(\frac{n}{t}\right)^p + Ckn^{p-2}, \tag{5}$$

then there exists a $K_d(n_1, \ldots, n_d)$ such that $\sum n_i = n$, $|n_i - \frac{n}{d}| < C'\sqrt{k}$, and G_n can be obtained from this $K_d(n_1, \ldots, n_d)$ by adding fewer than $C'k$ edges to it and then deleting less than $C'k$ edges from it.

Early graph applications

- Ramsey-Turán, see the Sim-Sós survey [59]
- (6,3)-problem (Ruzsa-Szemeredi [51])
- Sharper Erdős-Stone

There are many cases in extremal graph theory where a problem can be proved with and without Regularity Lemma.

Among these cases there are many ones where the solution with Regularity Lemma is much more transparent. On the other hand, the disadvantage of the application of the Regularity Lemma is that it works only for very-very large number of vertices.

Property testing and Regularity Lemma

It turned out that many problems on Property testing primarily depends on the Regularity Lemma. So, among others, several results of Alon and Shapira [6] and e.g. [5] can be summarized in some sense by saying that Property Testing and the applicability of Regularity Lemma are basically the same.

Ruzsa-Szemerédi thm

Consider the case when for fixed k and ℓ the family of excluded graphs is the family of k-vertex graphs with ℓ edges. In some sense a paper of Gábor Dirac was the first paper into this direction. Next Erdős and also I proved several results in this area. The breakthrough was when W.G. Brown, Erdős, and Vera T. Sós wrote two papers into this direction, but for hypergraphs. The paper [14] considered the case of 3-uniform hypergraphs. Some of its results were basically the same as mine on hypergraph problems connected to the structure of colour-critical graphs.

The first very difficult problem was the case when

we consider 3-uniform hypergraphs and the excluded subhypergraphs have 6 vertices and 3 hyperedges. The question was is $\mathbf{ex}_3(n, \mathcal{H}^{(3)}_{(6,3)}) = o(n^2)$ or not.

Szemerédi, using a Regularity type argument, proved that YES: $\mathbf{ex}_3(n, \mathcal{H}^{(3)}_{(6,3)}) = o(n^2)$.

Removal Lemma

This was the first case, when the Removal Lemma occurred, for K_3, in its simplest form:

Theorem

If a graph G_n contains $o(n^3)$ copies of K_3, then we can delete $o(n^2)$ edges of it to obtain a triangle-free graph.

This can be extended to arbitrary excluded graphs L, and also to hypergraphs.

Theorem (???)

Consider r-uniform hypergraphs. If $\mathcal{H}_n^{(r)}$ is an r-uniform hypergraph containing $o(n^v)$ copies of $L_v^{(r)}$, then we can delete $o(n^r)$ hyperedges of it to obtain an $L_v^{(r)}$-free hypergraph.

What was left out?

The answer is simple: very many things:

- The product conjecture
- Using the Decomposition class
- Erdos-Frankl-Rödl theory
- Open questions for the Degenerate Case.

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