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EVOLUTION OF RANDOM BOOLEAN NETWORKS

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Introduction In this paper we investigate asymptotic behavior of dynamics of random Boolean networks introduced by Stuart Kauffman. In his seminal book *The origins of order* Kauffman proposes and studies several models describing evolution. Two of them are termed NK–model, we consider here the second one which describes enzymatic interactions. Kauffman's theory is based on the results of Erdős and Rényi on the evolution of random graphs. We initiate similar investigations for random Boolean networks extending the Boolean NK–model of Kauffman. We suggest a Boolean MKPD–model where D refers to the past. The state of the system at time t depend on D previous state from the past. In this model K is the maximal number of connections, and the number of connections is increasing in course of evolution. The new model was investigated by computer simulations in order to find critical state of the systems. In a previous paper we applied Tom Ray's strategy for developing networks. The two types of evolution strategies may help to detect hidden aspects of attractor structures of the systems.

Biological Systems It is a common misbelief that good mother nature is wise. One source of the myth is that in many situations mathematicians were able to built effective algorithms mimicking natural processes. Yet nobody knows what is the real purpose of the quite complex history of life on Earth. May be the best way to think about these questions is simply to learn from the facts.

The human body consists of cells. Cells consist of proteins described by DNA. Proteins are interacting with each other. Typically one protein cuts into two pieces other ones but it may happen that proteins like acrobats form together more complex structures. We may think of proteins as chemical elements, like the hydrogen, oxygen and potassium form a new molecule being amalgamated by chemical powers. In case of proteins, which are themselves molecules, the amalgamating power is supplied by the space structure of the individual proteins which is reshaped by interactions. Just like gentlemen shaking hands. It is typical behavior for one protein to get specified for one activity, say to synthesize one amino acid. Like in the human society there are policemen, physicians, and there are mathematicians. Thus in one respect proteins behave like elements and in another respect they are functioning as overspecialized persons. In this respect they are enzymes.

All of our cells have the same blueprints in the DNA. Yet our body have many different segments mostly because the activation of the enzymes may follow different patterns in them. In special tasks certain cells are united and in the developing system the individual cells are communicating with each other. The cells in our bones are nearly non-living material. The muscles supply the mechanical energy. Lung circulates the oxygen, the stomach digests new proteins. Humans are capable to synthesize all amino acids but histidine. That is digested from food. The nerve system consists of cells which for a while were believed to have only two possible states.

NEURAL NETWORKS Twenty years ago a mathematical model was developed mimicking collaboration of nerve cells. Neural network is a set of binary automatons connected by directed edges. Elements of the system, the nerve cells, are changing their states in discrete time. The most general model is the following.

Let $\mathcal{A}$ be a finite set of binary automatons, and let $a(t),\ a \in \mathcal{A},\quad t = 0, \ldots$ be the state of element a of $\mathcal{A}$ at time t, $a(t)$ may be 0 or 1. A K-tuplet $G_a = (a_1, \ldots, a_{K_a})$ formed by different elements of $\mathcal{A}$ is ordered to each element a of $\mathcal{A}$ where K_a is a non-negative integer which may be different for different elements of $\mathcal{A}$. G_a is the reference group of a which itself may be an element of G_a. At time $t+1$ the state $a(t+1)$ of $a \in \mathcal{A}$ is given by the binary vector consisting of states of elements of the K-tuplet $(a_1, \ldots, a_{K_a})$ at time t:

$$a(t+1) = f_a(a_1(t), \ldots, a_{K_a}(t)), \quad t = 0, 1, \ldots.$$

The functions $f_a, a \in \mathcal{A}$ are the look-up-tables of the system. In case of nerve cells these functions are supposed to be monotonous: the stimuli of reference cells are accumulating and if the sum surpasses a threshold then the cell fires: its state changes from 0 to 1. In the model of associative memory the set of stabile states of the system forms the memories. Learning means to construct the reference groups and to determine the look-up-tables f_a of the system. Surprisingly enough this system has a specific holistic integral feeling of its state what is the cognition. There is a certain border separating the states of nerve cells and other segments of the body what is called the unconscious part or the subconscious. In the human body there is another system, which is capable to learn, the immune system. The two systems, nerve system and immune system are structurally similar but there is nothing what we could place in the role of immune-cognition. Why?

ATTRACTOR STRUCTURE Kauffman proposed a similar model for enzymatic interactions, where bits reflect the active or inactive states of enzymes. For the sake of simplicity here K_a does not depend on a. The model is called *NK–model*, N stands for the number of elements of $\mathcal{A}$ and K refers to the size of reference groups. Having finitely many possible states the whole system returns to some previously visited state after a while and it will repeat a certain cycle. We call the elements of cycles *cyclic* ones and the other elements *transient* ones. Transient elements form the *tail* of the cycle. A path of the dynamics starts with a tail which may be empty and it ends in a cycle which may be trivial if it has only one element. Cycles are the *attractors* of the dynamics. We are interested in

– the number of cycles and

– the distribution of the lengths of the cycles.

These statistics together form the attractor structure of the system. You might readily ask where stochastics enters into this picture. There are two possibilities, which, of course, are very close to each other.

a) We may investigate randomly generated structures. One model was proposed by Bastolla and Parisi: the reference group is purely random, and the bits in the look-up-tables are iid random variables.

b) We may list all of the structures and accumulate the corresponding statistics.

In certain systems of automatons nearly all of the 2^N possible states are visited before repetitions. Such systems are called *chaotic*. In other cases the number of cyclic elements is negligible with respect to N. Such systems are called *convergent*. The two extreme behaviors are separated by a thin layer of special kind of systems called *critical*. The number of cycles and the length of cycles in critical systems are believed to be approximately $\sqrt{N}$.

According to Kauffman's philosophy life developed on the edge of chaos, in the critical state. There are calculations showing that Darwin's theory is not feasible in course of relatively short time of the evolution of life on the Earth. Kauffman extends Darwin's ideas substituting pure random search with more sophisticated dynamics forced to remain in the vicinity of critical behavior. The driving force behind the process may be the simple fact that organisms are unable to survive either with convergent or with chaotic dynamics.

EMERGENT PROPERTIES Simplest neural network structures are directed: they are flexible, they are able to learn quite arbitrary input-output relations, but long time dynamics are not involved in their mechanism. In advanced forms of neural networks there are feed-back connections between nerve cells but look-up-tables remain directed. One wonders how it is possible to model human intelligence with such a simpleminded structure. If we are really thinking, the thoughts are storming in our brain, at least this is our feeling. Yet in the model of associative memory the dynamics are convergent. Especially in spin glass models the dynamics are defined by a quadratic potential and local optima of the potential are the stable states of the system. Storming appears only for random Boolean look-up-tables and in Kauffman's theory the attractors correspond to life cycles of cells. In cells the enzymes may be active or inactive and in young cells other enzymes are activated than in old ones. The whole sequence of activation dynamics characterize specific clusters of possible behaviors for cells coded by the same DNA.

That is why rich attractor structures are advantageous in evolution. However it is not clear what is the mathematical definition of “richness”. The complete enzymatic interaction scheme will not be known perhaps in 50 years. There are some evidences showing that both attractor sizes and the number of attractors are of order $\sqrt{N}$. Nevertheless we do not know any fitness function resulting in systems with this property.

One attractor in a random Boolean network is given by three different components:

i) the set of look-up-tables $\mathcal{F} = (f_a, a \in \mathcal{A})$

ii) the set of reference groups $\mathcal{G} = (G_a, a \in \mathcal{A})$ and

iii) the set of initial states $(a(0), a \in \mathcal{A})$ of the system.

Initial values resulting in the same attractor form the *basin of attraction* of the attractor. We may imagine one attractor with its basin as a lake with creeks flowing into the lake. (All similes are faulty: the water in lakes is placid while the dynamics keep cycling the bits in the attractor.) At first sight there are no evidences explaining the empirical fact that usually the majority of attractors of one system have the same size. Why would be all the lakes of the same size in Finland? The situation may be understood perhaps in some vague statistical sense: according to the law of large numbers the sizes are clustered around the expected value. In random Boolean networks stability of attractor size is somehow more pronounced. Like a whistle has a fundamental tone with joining overtones, attractor sizes typically exhibit sophisticated inner structures with one leading term.

One possible explanation of the phenomenon is a transformation in the role of cycles: cycles of states of a system may emerge from cycles formed by automatons of the system. Like in a circular railway trains go round and round, changes in state of automatons may circulating in the system as long as they return to the initiating automatons. Then the cycling beguns. The picture may be valid only in statistical sense. Attractors with small basin of attractions may be different but they usually do not appear in sampling experiments. In this sense we may speak about attractor structure of a system defined by the couple $\mathcal{F}$ and $\mathcal{G}$.

Fixing one set of look-up-tables $\mathcal{F}$ we can sample attractor structures for different reference groups $\mathcal{G}$. The question is, are there any straightforward operators depending on $\mathcal{F}$ and ordering the attractor structure to $\mathcal{G}$? Or to be a bit less ambiguous: are there some subclasses of look-up-tables with such a property? Bastolla and Parisi postulated a critical probability in their construction. This conjecture is a special case of our program: characterizing critical structures the size of cycles is given at least statistically.

Kauffman originally conjectured that criticality is caused by special Boolean functions called *canalysing* ones. A function is canalysing if a single input being in a fixed state is sufficient to force the output to a fixed state, regardless of the state of any other inputs (see point

5.6 in Lucas(1999)). There are evidences showing that look-up-tables of enzyme systems are canalysing.

At the end of our paper we present a simple example for emergent properties in a three variable binary recursion. We have limited experiences with neural networks. If the dynamics are asymmetric then it is possible that non-trivial cycles appear in a monotonous system. We present an example for this phenomena, it is the look-up-table of majority voting (Table 6). But we think the phenomena is not typical. The distinction, that neural networks are convergent and random Boolean networks may be chaotic, is perhaps also an emergent property.

Evolution of Kauffman's Models Kauffman constructs models for evolution in biology with the help of the mathematical evolution of random graphs of the Erdős-Rényi paper. We are pretty sure that Kauffman find the Erdős-Rényi paper because in its title evolution was explicitly mentioned. What is good enough for graphes will work for animals perhaps he thought. Now we turn the direction of the argument upside down. What happens if the mathematical structure of the system evolves?

The structure consists of two parts:

- the set $\mathcal{G}$ of reference groups and

- the set $\mathcal{F}$ of look-up-tables.

The first part is essentially a directed graph with ordered in-edges. In this respect the original Erdős-Rényi idea applies: we may add step by step randomly chosen new directed edges to the graph. Doing so the edges will be already ordered by the construction. But how should we extend the K-valued function

$$f_a(a_1(t), \ldots, a_K(t))$$

to a $(K+1)$-valued function

$$f'_a(a_1(t), \ldots, a_K(t), a_{K+1}(t))?$$

The procedure is the following. Initially all of the K_a-s are equal to zero. Step by step we choose randomly one a in $\mathcal{A}$, and extend its reference groups G_a with one random element of $\mathcal{A}$. Simultaneously the corresponding look-up-table is split up into two parts: it does not change if the state of the new automaton is equal to zero and a genuinely new look-up-table is generated if it is equal to 1. Formally

$$f'_a(a_1(t), \ldots, a_K(t), 0) = f_a(a_1(t), \ldots, a_K(t)),$$

$$f'_a(a_1(t), \ldots, a_K(t), 1) = g_a(a_1(t), \ldots, a_K(t)),$$

where g_a is a new K-variable binary function with a look-up-table generated independently of f_a.

Our experience is that in course of the process the sizes of cycles are vaguely increasing. First the system is convergent, cycles consist of one point in the state space, and eventually all bits become frozen. In course of evolution step by step bits are melting, they form cycles and sizes of cycles in the state space become polynomial functions of the system parameters for a while. This is the critical state. Is it possible to generate directly critical structures? Of course one theoretical possibility is conditioning, to use the conditional distribution of a system with a critical attractor structure. Unfortunately this procedure is not constructive, we have to use some heuristic model here. The simplest model is the following. It is a special case of the original Boolean NK–model with $K = N$. The prototype section of our paper (Rejtő-Tusnády 1999) gives a more detailed description of the system.

Memoryless Systems Let $n = 2^N$, let S be the set of the first n positive integers and let $X_1, \ldots, X_n$ be iid S-valued random variables.

Definition The random subset $\mathcal{C}$ of S is the cyclic set of $X_1, \ldots, X_n$ if

i) for each $i \in \mathcal{C}$ we have $X_i \in \mathcal{C}$ and there is a $j \in \mathcal{C}$ such that $i = X_j$ (i and j are not necessarily different)

ii) $\mathcal{C}$ is maximal among subsets with property i).

Definition The random subset $\mathcal{B}$ of S is the birthday set of $X_1, \ldots, X_n$ if

$$\mathcal{B} = \{X_1, \ldots, X_\tau\},$$

where τ is the largest index t such that all of $(X_1, \ldots, X_t)$ are different.

The birthday lemma The random sets $\mathcal{B}$ and $\mathcal{C}$ have the same distribution.

The lemma is due to Burtin and Ross. In Kauffman's model the lemma explains the factor $\sqrt{N}$ in a loose sense. In the chaotic case the set of possible states is the whole S, thus the sizes of cycles are of exponential order. In critical systems the majority of elements of the system get frozen and the power set of not frozen states takes the role of S. To get a closer look on the critical behavior we made long evolutionary runs of a collection of Kauffman's model. We used a fitness function evaluating the attractor structure and applied the Tierra strategy in developing of structures. Analyzing the evolution of individual system we learned that in each case there is a core process consisting of not frozen elements of the system. Thus one possible way to understand asymptotics is to define directly such a core process.

Classification of the Edges Critical dynamics may be investigated with the help of the following directed graph. The vertices are the elements of $\mathcal{A}$. For each element a of

$\mathcal{A}$ and each element b of its reference group G_a a directed edge is formed starting from b and ending in a. Let us say that an edge e is *basic* if there is a directed cycle containing e. Let us define the basic order L_a for each element a of $\mathcal{A}$ as the number of basic edges ending in a. According to our experiences the attractor structure of a system is more or less established by the number of vertices with basic order larger then 1. Let us call such vertices intrinsic, the others will refer as negligible. Negligible vertices form chains and and the role of these chains is to ensure communications between intrinsic vertices. The length of the chain is a time-delay in the dynamics, it is the time of the communication.

Let us call an edge e *subsidiary* if it is not basic and there is a directed path starting from an intrinsic vertex and containing e. In course of evolution of a system first all the vertices are negligible, and the dynamics are getting frozen in all elements of $\mathcal{A}$. The first melting elements are the elements of the first cycle and its subsidiaries. At this stage of evolution there are no intrinsic vertices. The developing structure in the next steps is the origin of the following model. The chains of negligible automatons form the memory puffer of the model.

THE MKPD–MODEL The dynamics of the automatons in the model are defined by the following elements

– M is the number of automatons (it is typically smaller then the original number of elements, M stands for "moving")

– K is the number of the elements of the reference groups (here we use the same notation as in he original model but in the core process the corresponding number actually might be smaller and it is not necessarily the same for different automatons)

– P is the probability in the look-up-tables (the situation is similar to the case of K)

– D is the depth of memory, it is the new element of the model (in time series analysis this quantity is called LAG, we shall use this name in the sequel).

The dynamics of the model are the following

$$a_i(t) = f_i(a_{\nu(i,1)}(t-\Delta(i,1)),\ldots,a_{\nu(i,K)}(t-\Delta(i,K))),$$

where $i = 1\ldots,M$, $G_i = (\nu(i,1),\ldots,\nu(i,K))$ is the reference set of the i-th automaton, and $(\Delta(i,1),\ldots,\Delta(i,K))$ is the set of LAGs of the same automaton. For the sake of simplicity the LAGs are supposed to have independent uniform distribution on the interval $(1,D)$ of integers.

Evolution of the MKPD–models may be generated similarly to that of the NK–models. We fixed a structure and activated many times one by one its connections. In course of this process the sum of the sizes of reference groups were increased from zero up to $M*K$ in random order. The number of connections is referred as the number of steps. In Table 1 parameters of the model investigated in details are presented. As it is seen in the table, there are identical elements in reference groups and LAGs, even there is an identical pair (6, 31) for $a = 5$. This coincidence reduces the complexity of the system, while we planned $K_a = 4$ for all $1 \leq a \leq 10$, in our system $K_5 = 3$ effectively. In Table 2 the cumulated statistics of cycle lengths are given as the number of connections of the system increases from 1 to 39 step by step. First all cycles are of size 1: the system is convergent. Non-trivial cycles appear from step number 3 up to step number 28 and sometimes there are definite decreases in cycle-sizes. The most frequent cycle length is 59: the system behaves like a whistle, it has a fundamental tone. In other occasions cycle length are doubled. The number of steps is bounded by 39 because of our computation capacity. It would be interesting to know, what are the cycle lengths in the complete system. In theory they should be around 2^{320}. Our point is that in most cases the evolution of cycle lengths is nearly continuous. Accordingly the logarithm of the t-th cycle lengths given in Table 2 is S-shaped as a function of t with turning points 59 and 508 respectively. Below 59 the increase in sizes is fast, between 59 and 508 it slows down and above 508 it is speeding up again. In course of this final phase chaotic regime is reached. Roughly speaking cycle lengths are doubled in each step.

Computer runs resulting in Table 2 are censored: we dropped the initializations without cycles in 1 000 000 steps. The total number of successive initializations was 292. In Table 2 only the most frequent cycles are given. Other detected cycle lengths are the followings: 11, 14, 20, 44, 48, 51, 52, 66, 78, 104, 119, 120, 132, 135, 142, 151, 156, 170, 176, 180, 196, 200, 216, 221, 230, 249, 272, 292, 336, 340, 342, 351, 352, 364, 392, 420, 432, 464, 528, 531, 540, 560, 584, 590, 615, 627, 649, 660, 664, 672, 677, 700, 812, 816, 840, 868, 870, 936, 938, 1 003, 1 008, 1 020, 1 029, 1 085, 1 120, 1 128, 1 168, 1 176, 1 180, 1 206, 1 260, 1 441, 1 560, 1 608, 1 770, 1 876, 1 960, 2 016, 2 108, 2 124, 2 244, 2 324, 2 352, 2 520, 2 552, 2 604, 2 832, 2 876, 3 320, 3 408, 3 540, 4 488, 5 546, 5 664, 5 782, 6 077, 6 324, 6 565, 8 847, 10 502, 10 920, 12 700, 13 244, 13 924, 14 280, 15 204, 17 272, 21 680, 22 860, 24 640, 33 807, 37 044, 44 781, 45 312, 58 056, 91 922, 104 112, 124 896, 141 718, 170 805, 231 162, 267 716. The distribution of terminating step numbers is the following: 19 (1), 20 (1), 21 (1), 23 (1), 24 (2), 25 (1), 26 (3), 27 (1), 28 (6), 29 (6), 30 (3), 31 (4), 32 (6), 33 (7), 34 (6), 35 (7), 36 (3), 37 (3), 39 (1). The total number of investigated evolution histories was 63. In course of evolution usually 5 cycles were detected in one system, that is why the number of initializations was 292.

A strange evolution history is presented in Table 3. The order of activation of automatons resulting in this strange history is the following: 7, 4, 6, 3, 4, 6, 3, 3, 10, 8, 10, 9, 8, 4, 8, 9, 2, 2, 1, 8, 3, 4, 2, 6, 7, 2, 9, 6, 5, 5, 10, 10, 7. After 33 steps the system blows up. We call this history strange because in the 19-th step it nearly blows up. Even the largest detected length (267

716) appeared here. We made a more thorough search for cycle lengths in this case and found that all the lengths are multiples of 34. The ratios with frequencies are the followings 1 (37), 2 (21), 3 (1), 4 (16), 6 (86), 8 (10), 12 (149), 14 (22), 24 (14), 28 (15), 42 (8), 56 (1), 62 (18), 84 (20), 124 (4), 186 (11), 248 (1), 254 (46), 372 (3), 434 (2), 508 (5), 762 (25), 1016 (5), 1524 (4), 1778 (3), 7874 (2). The system is bubbling: cycles of size 34 amalgamate. The strange phenomenon is that during the next steps this bubbling collapses and cycles of size 12 become dominant. The history shows that the transition between convergent and chaotic regime does not follow monotonously the qualitative parameters of the model. Structural properties are more important.

Automatons 2, 5, and 6 are frozen in the 19th step of the evolution. Automatons 7 and 10 are exciting each other, 7 is exciting 9, 9 and 10 together excite 3 which excites 1, finally 1 excites 4 which excites itself, too. Automatons 1, 3 and 7 together excite 8 which sometimes also become frozen. Fundamental tune comes from $\Delta(7,1) = 5$ and $\Delta(10,2) = 29$. Under normal circumstances two LAGs result in a simple circle and the size of the cycle is equal to the sum of the LAGs. The relatively complicated chain structure forming a one-way dynamics running from this couple up to 4 modulates the cycle. On the effect of next steps the chain is feedbacked to the couple (7, 10) and the resonance is finished.

There is another evolution history which is perhaps stranger. It does not blow up even in the 39-th step. The edges were activated in the following order: 1, 8, 9, 6, 2, 5, 6, 8, 3, 4, 8, 10, 2, 5, 5, 2, 7, 1, 8, 1, 10, 2, 10, 6, 10, 1, 4, 9, 3, 6, 3, 5, 9, 3, 4, 7, 7, 7, 9, 4. In the 39-th step only the 4th automaton's reference group was of size 3, the others were of size 4. Fundamental tone of the system changed from 59 to 174 and 102. It is really hard to understand that all the other evolutions are terminated earlier. It shows that the unexpected returning of the dynamics from a nearly chaotic stage to a convergent one is typical: attractor sizes jump up and down in course of evolution.

The space where the evolution is investigated is formed by 10-dimensional vectors with integer coordinates between 0 and 4. Thus there are 5^{10} possible systems. We investigate monotone pathes in the space: one system is followed with another having strictly one larger coordinate and the increase is 1. Before our computer experiences our expectation was that there was a critical line separating convergent and chaotic systems in the space. Critical line was expected to be formed by vectors with sum of coordinates is around 20. We were mistaken in two points. The critical sum is 30 and the border between convergent and chaotic regime is not as simple as we expected it. Our evolution space is a rugged landscape containing many holes formed by convergent systems which are separated by walls formed by chaotic systems.

A Three Variable Binary Recursion Let $A > B > C > 0$ be integers, and consider the binary recursion

$$X(t) = f(X(t-A), X(t-B), X(t-C)),$$

where f is an arbitrary three-variable binary function. This system is a special MKPD–model with $M = 1, K = 3$. Let us code the investigated functions according to the 2-base system. For example the code of the function with look-up-table given in Table 4 is $64 + 32 + 8 + 1 = 105$.

Let us define the first marginal of a function $f(x, y, z)$ by

$$MA = \sum_{y,z} f(0, y, z),$$

and similarly set

$$MB = \sum_{x,z} f(x, 0, z),$$

$$MC = \sum_{x,y} f(x, y, 0).$$

For the above function we have $MA = MB = MC = 2$.

Let us define the dual $\tilde{f}$ of a function f by the relation

$$\tilde{f}(x, y, z) = 1 - f(1-x, 1-y, 1-z),$$

in the example $\tilde{f} = f$. Here we consider only such functions for which the number of 1-s in the look-up-table is equal to 4, and $1 \leq MA, MB, MC \leq 3$. Tables 5A,B,C,D characterize stochastically the attractor structure of these functions. In the tables invertibility means *partial invertibility*: we say that a function f is A-invertible, if $f(x, y, z)$ determines x for given y and z. The function f is B-invertible, if $f(x, y, z)$ determines y for given x and z, and it is C-invertible, if $f(x, y, z)$ determines z for given x and y.

For functions given in Table 5A the rules are the following. All memory-parameters A, B, C are possible cycle-lengths but if the marginal is equal to 3 then the corresponding cycle-length is multiplied by 2.

For functions given in Table 5B the rules are the following. Only parameters with marginal value 1 are possible cycle lengths. If there is only one such parameter then the sum of the other two memory-parameters is also a possible cycle-length. If there are no parameters with marginal value 1, then the sum of marginals with value 3 is a possible cycle length.

For functions given in Table 5C the rules are the following. It is easy to see that marginal invertibility coincides with parameters 1 or 3 in the same marginal. If this marginal is equal to one, then in the majority of cases the cycle length is a multiple of the corresponding memory-parameter (the multiplier usually is a power of 2). This is shown by a, b, c. In the table "tail" means the number of steps before entering the cycle. Tail = 0 means that from an arbitrary initial value we reach immediately the cycle. It is equivalent to invertibility with respect to A.

We have not find rules for functions given in Table 5D. Tail = ? means that Tail > 0, or rather that we do not know. The function of code 105 is in a certain sense the simplest here: it orders simply the parity code to its variables, thus it is linear, and it is a member of a wide family of binary operators called shift-registers. If $A = B + C$ then cycle length is $B * C$ or $2 * B * C$. In other cases the cycles are exponentially large and all the prim factors of their size are divisors of $2^s - 1$ with some $s \leq A$. In some cases there are only four cycles which form dual pairs with respect to switch 0 to 1 and vice versa. One cycle consists of one point: the sequence of all zeros is stable point of the function, the size of other cycle is $2^{A-1} - 1$. We found no rule giving these extreme triplets (A, B, C). This is the only linear function among the investigated ones. It is strange that it has the longest attractors.

The above rules come from our computer experiments. Some of them are trivial others are not. Now we are working on the explanation of the second part. For example the function with code number 23 given in the first line of Table 5A is the majority voting rule: if the sum of input variables is less then 2 then the output is zero, otherwise it is 1. This system is a neural network thus it serves an example where non-trivial cycles appear. In Table 6 some cases are presented showing the cycles for this function. In the table $N1$ stands for the number of tested initial states resulting in convergence regardless of the number of different limit stable states detected. Similarly NC, NB, NA stand for number of initializations resulting in cycle length C, B, A respectively, the column headed by "other" gives the sizes of cycles different from A, B, C (these cycle lengths are divisors of A) and "not conv" gives the number of initializations resulting in runs without cycles in 32 000 steps. As it is seen in the table, cycle length $1, A, B, C$ are only possibilities, their expression depends on number theoretical relations between the numbers A, B, C. In some cases other cycles appear, and in some cases our computing capacity was not enough for detecting any cycle.

THE SIMPLEST CASE Let A be an arbitrary positive integer and let B be another integer between 0 and A. Let $X(1), \ldots, X(A)$ be arbitrary bits and for $t > A$ set

$$X(t) = \begin{cases} 1 & \text{if } X(t-A) = X(t-B), \\ 0, & \text{otherwise.} \end{cases}$$

This system has a trivial fix point: $X(t) = 1$ for all A and B. It may happen that all of the other cycles have equal sizes. We say that these cycles are *uniform*. In Table 7 the number of uniform cycles are given up to $A = 18$ for pairs (A, B) having only uniform cycles. Other pairs are marked by '-'. The dual of this dynamics is the two-variables version of the function of code number 105 given in Table 5D. Tails are again empty and the systems of parameter (A, B) and $(A, A-B)$ are equivalent with respect to time reflection. We could not extend the table much further. For example we are not able to decide whether for $A = 100,\ B = 59$ there are only two cycles (one is the trivial stable point and the other is joining all the other states).

Let us denote by $L(A, B)$ the cycle length for initial values all 0. In Table 8 the 2-base logarithms of $L(A, B) + 1$ are given. It is easy to see that for any positive integer k

$$L(k * A, k * B) = k * L(A, B)$$

holds true, that is why pairs with a common divisor larger then 1 are marked by '-' in the Table. The cases are denoted with a star where there are only two cycles: the trivial fixed point and an other joining all the other states. A plus sign refers to the cases where the figure is larger then the integer given in Table. The fact that $L(18, 13) = 2^{15} - 1$ is shown in Table with 15 with an '=', and the cycle lengths $L(21, 2) = 2097151 = 2^{21} - 1$, and $L(21, 10) = 2088705$ are represented by 21 = and 20+ respectively. (For $A = 21, B = 10$ the remaining states form two other cycles, one is of size 255 and it goes through the state 110010111111101110001, the other one is of size 8 191 and it goes through the state 100010111111111111011.) The sign '>' refers to the fact that the size of the cycle is larger then the number in Table: as in all of our computer works long cycles were not detected.

Linear Systems Let n be an arbitrary positive integer and $\mathbf{A}$ an $n \times n$ matrix of $0, 1$ elements. Let $X(0)$ be an arbitrary n-dimensional vector of $0, 1$ elements and let us define the sequence $X(t),\ t > 0$ by

$$X(t) = \mathbf{A}X(t-1), \quad t = 1, 2, \ldots,$$

where the operations (summation and multiplication) are the mod 2 ones:

$$0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0,$$

$$0 * 0 = 0, \quad 0 * 1 = 0, \quad 1 * 0 = 0, \quad 1 * 1 = 1.$$

This system is a special Boolean network with linear look-up-tables. Shift registers are special linear systems. It is a special property of linear systems that cyclic elements together form a linear subspace of the space of binary vectors. Thus the total number of cyclic elements is 2^k where k is the ultimate rank of the matrices $\mathbf{A}^s$ with s going to infinity. If z is the first integer such that the rank of $\mathbf{A}^z$ is equal to $\lim_{s \to \infty} \text{rank}(\mathbf{A}^s)$ then z is the maximal size of tails. In Table 9 the distribution of the ultimate rank of random linear systems is given for $n = 15$. In Table 9 "No" means the number of 1-s in $\mathbf{A}$. The Table shows that the distribution is sickle-shaped. The sizes of cycles were the followings (with frequencies in brackets) 1 (1728), 2 (693), 3 (512), 4 (348), 5 (100), 6 (261), 7 (413), 8 (67), 9 (30), 10 (38), 11 (1), 12 (103), 13 (2), 14 (200), 15 (231), 16 (3), 17 (6), 18 (11), 20 (15), 21 (169), 23 (1), 24 (18), 28 (70), 30 (102), 31 (270), 34 (3), 35 (21), 36 (7), 40 (1), 42 (92), 45 (6), 48 (1), 51 (28), 56 (18), 60 (42), 62 (123), 63 (150), 65 (1), 70 (8), 72 (2), 73 (19), 84 (30), 85 (41), 89 (2), 90 (1), 91 (1), 93 (89), 102 (9), 105 (58), 112 (1), 119 (1), 120 (7), 124 (70), 126 (68), 127 (231), 146 (7), 153 (1), 155 (13), 168 (8), 170 (20), 186 (47), 195 (1), 204 (6), 210 (33), 217 (59), 219 (5), 238 (1), 240 (1), 248 (12), 252 (30), 254 (97), 255 (109), 273 (2), 279 (3), 292 (2), 310 (9), 315 (27), 340 (6), 341 (32), 357 (8), 372 (20), 381 (75), 390 (1), 408 (2), 420 (11), 434 (29), 438 (4), 455 (8), 465 (29), 504 (4), 508 (46), 510 (42), 511 (141), 527 (2), 546 (1), 558 (1), 585 (7), 595 (7), 620 (3), 623 (1), 630 (8), 635 (12), 651 (31), 682 (10), 714 (3), 744 (3), 762 (42), 765 (1), 819 (9), 840 (2), 868 (13), 876 (1), 889 (52), 910 (2), 930 (17), 1016 (6), 1020 (14), 1022 (53), 1023 (92), 1054 (1), 1071 (2), 1095 (2), 1116 (1), 1143 (2), 1170 (1), 1190 (4), 1260 (1), 1270 (3), 1302 (19), 1335 (1), 1364 (4), 1365 (13), 1524 (13), 1533 (38), 1581 (4), 1638 (1), 1705 (1), 1736 (4), 1778 (23), 1785 (35), 1860 (7), 1905 (34), 1953 (21), 2040 (1), 2044 (25), 2046 (34), 2047 (89), 2263 (2), 2380 (1), 2387 (4), 2540 (2), 2555 (11), 2604 (11), 2635 (4), 2667 (30), 2728 (1), 2730 (2), 3048 (1), 3066 (16), 3162 (1), 3255 (5), 3556 (8), 3570 (17), 3810 (9), 3906 (6), 3937 (31), 4088 (4), 4092 (14), 4094 (29), 4095 (41), 4445 (2), 4599 (4), 4681 (4), 4774 (1), 5110 (3), 5115 (10), 5334 (10), 5355 (8), 5461 (12), 6132 (4), 6141 (23), 6477 (1), 6510 (2), 7112 (1), 7140 (4), 7161 (18), 7620 (2), 7665 (8), 7812 (2), 7874 (11), 7905 (12), 8001 (12), 8184 (2), 8188 (12), 8190 (7), 8191 (74), 9765 (2), 10235 (4), 10668 (1), 10795 (2), 11811 (6), 12282 (4), 13335 (2), 14322 (5), 14329 (10), 15748 (5), 15810 (3), 15841 (14), 16002 (1), 16380 (5), 16382 (13), 16383 (24), 24573 (8), 27559 (1), 30705 (3), 32385 (4), 32767 (20).

In majority of cases attractor sizes are of the form $\prod_{i=1}^{s} d_i$, where s is an integer, d_i is a divisor of $(2^{\nu_i}-1)$, $\sum_{i=1}^{s}\nu_i = k$ and $(\nu_1,\ldots,\nu_s)$ are different. Let us call a system *primitive* if $s=1$. In primitive systems non-trivial cyclic elements form one cycle. The systems given in Table 7 with frequency number 1 are primitive ones. In Table 8 primitive systems are marked by a star. Linear combinations of elements of one attractor form again an attractor which is in the subspace spanned by the elements of the original attractor. That is why in many cases attractors are uniform. Linear systems may serve as a prototype of Boolean networks. In one respect they are simpler, easier to investigate and they may have special properties but in another respect they may have all the characteristic properties of the more complicated systems. Especially the number of 1-s in $\mathbf{A}$ may control the transition from convergent to chaotic regimes in random systems.

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Table 1. Parameters of an MKPD–model ($M = 10, K = 4, P = 0.5, D = 32$)

Reference groups

3	10	6	1	6	5	10	1	7	6
5	6	10	4	6	6	9	7	6	7
9	2	9	4	10	10	2	3	9	1
3	6	6	3	8	4	7	1	9	9

LAGs

17	19	29	29	31	28	5	25	29	5
24	15	20	12	31	28	31	22	28	29
19	17	22	28	12	20	23	26	7	20
5	23	26	26	24	3	22	4	3	14

Look-up-tables

1	1	0	0	1	0	1	0	1	0
0	0	0	0	0	1	0	0	1	1
1	1	1	0	0	0	0	1	0	1
1	0	1	1	0	1	1	0	1	1
1	0	1	1	0	1	0	0	0	1
1	1	0	1	0	1	0	0	1	0
0	1	0	1	1	1	0	0	0	0
0	1	0	0	1	0	1	1	1	1
1	0	0	1	0	0	1	0	0	1
1	0	1	0	1	0	1	0	0	1
1	1	0	0	1	0	0	1	0	1
0	1	0	0	1	1	0	1	0	1
1	1	1	0	0	1	1	0	1	1
1	1	0	0	0	1	1	0	0	1
1	0	0	0	1	0	0	1	1	0
1	1	0	0	1	1	0	0	0	0

Table 2. Results of evolution of the model given in Table 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	292	292	291	291	276	249	230	205	174	150	109	110	103	83	68	55	47	32	21	18
2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	2	3	2	-	-
3	-	-	-	-	-	-	-	-	-	3	3	5	3	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4	3	2	3	6	-
8	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1
12	-	-	-	-	-	-	-	-	-	-	10	5	5	5	6	12	7	12	4	16
17	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	5	5
22	-	-	-	-	-	-	-	-	-	-	-	-	4	5	5	5	4	4	4	1
24	-	-	1	1	1	11	25	35	45	41	30	27	32	22	27	22	22	15	20	27
25	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
28	-	-	-	-	-	-	4	4	5	5	5	5	5	16	25	25	21	19	13	7
34	-	-	-	-	5	10	10	15	30	45	60	40	45	52	52	42	18	16	13	7
40	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	-	5	10	1
45	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
50	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
56	-	-	-	-	-	-	-	-	-	3	4	5	4	-	4	9	9	15	14	9
59	-	-	-	-	10	22	22	22	27	27	40	49	49	60	58	54	75	58	71	76
68	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3	2	-	1	2
73	-	-	-	-	-	-	-	-	-	2	-	-	-	-	1	-	1	-	-	1
84	-	-	-	-	-	-	-	-	-	-	-	-	-	4	5	-	3	4	4	3
87	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	4
102	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4	8	3	3	-
112	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
118	-	-	-	-	-	-	-	-	-	-	-	-	1	5	4	7	9	6	7	6
136	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	1	-
168	-	-	-	-	-	-	-	-	-	-	-	5	5	5	-	3	1	1	1	1
174	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5
177	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	9	9	4
204	-	-	-	-	-	-	-	-	-	-	-	5	5	5	5	-	1	1	1	6
224	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3	1	1	2
236	-	-	-	-	-	-	-	-	-	-	-	-	-	3	-	-	4	-	1	2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
252	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	2	1	1
264	-	-	-	-	-	-	-	-	-	-	5	5	1	-	-	-	-	-	-	-
280	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2	2	4	1	-
295	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
300	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	-	-	-
308	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	5	5
354	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-
360	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
408	-	-	-	-	-	-	-	-	-	-	5	5	5	10	5	6	-	5	3	7
413	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
448	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
472	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	1	1	-	2	5
476	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
508	-	-	-	-	-	-	1	11	11	16	21	21	15	4	-	10	10	12	12	16
680	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	-
708	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	10	5	6
944	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
952	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
984	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	1	-	-
1416	-	-	-	-	-	-	-	-	-	-	-	-	-	10	-	-	5	5	5	5
1524	-	-	-	-	-	-	-	-	-	-	-	-	-	2	-	-	-	-	-	-
2006	-	-	-	-	-	-	-	-	-	-	-	5	5	-	-	-	-	-	-	-
2152	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2	2	1	-	-	-
2360	-	-	-	-	-	-	-	-	-	-	-	-	-	-	6	-	-	10	10	10
3556	-	-	-	-	-	-	-	-	-	-	-	-	5	-	-	-	-	-	-	-
7080	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7112	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	-	-	-	-	-
8636	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2	7	12	13	15	13
25908	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3	3	3	7	5	9
29972	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
89916	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 2. (continued)

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
1	21	19	18	18	9	3	1	-	-	2	-	-	-	-	1	-	2	-	1
2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	4	3	3	4	2	-	4	3	-	-	-	-	-	-	-	-	-	-	-
6	-	-	1	1	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-
8	-	3	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
12	16	22	25	26	24	29	33	32	20	12	9	5	3	1	-	-	-	-	-
17	5	5	5	10	5	-	-	-	-	-	-	-	-	-	1	1	3	-	-
22	1	1	1	1	1	1	1	1	1	-	-	-	1	-	-	-	-	-	-
24	29	17	28	23	23	15	15	4	-	5	5	6	6	5	-	-	-	-	-
25	4	3	5	4	8	4	5	-	-	-	-	-	-	-	-	-	-	-	-
28	6	15	14	15	20	22	15	13	8	11	10	13	9	6	7	1	-	2	-
34	9	5	1	-	-	2	-	1	-	-	-	1	-	2	-	2	-	-	-
40	3	1	2	1	-	-	2	-	-	-	1	4	-	-	-	-	-	-	-
45	-	-	-	-	-	2	2	-	-	3	4	5	2	1	1	-	-	-	-
50	1	2	-	1	2	1	-	-	-	-	-	-	-	-	-	-	-	-	-
56	-	-	-	-	-	-	-	4	2	3	2	2	-	-	-	-	-	1	-
59	81	82	75	72	73	71	59	66	59	39	42	23	21	14	12	6	5	-	-
68	-	-	-	-	-	2	-	-	4	-	-	-	-	-	-	2	-	-	-
73	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
84	2	3	1	2	-	1	5	4	3	2	3	2	-	7	6	1	-	-	-
87	5	5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
102	2	-	-	-	2	1	-	-	-	1	-	-	-	-	2	-	-	-	4
112	-	-	1	-	-	1	1	-	1	2	2	3	-	-	-	-	-	1	-
118	-	1	-	-	2	1	1	3	-	2	1	-	-	2	1	-	2	-	-
136	-	6	7	2	2	1	2	-	-	-	-	-	5	-	-	-	-	-	-
168	2	1	3	2	-	-	1	4	1	4	-	-	-	-	-	-	-	-	-
174	5	5	1	5	10	10	10	4	15	25	25	18	18	15	-	-	-	-	-
177	5	-	5	4	14	17	20	14	12	14	13	10	14	10	1	-	-	-	-
204	9	6	1	2	3	5	1	2	-	-	1	-	-	-	-	-	-	-	-
224	-	2	1	-	-	1	2	4	5	2	7	2	1	-	1	-	-	-	-
236	2	4	9	4	2	8	9	10	13	9	9	8	7	5	5	6	1	-	-

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
252	4	2	1	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-
264	-	-	1	1	-	-	1	1	-	1	1	2	-	-	-	-	-	-	-
280	-	-	1	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-
295	-	-	-	-	1	1	3	1	1	-	2	-	-	-	-	1	-	-	-
300	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
308	5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
354	-	-	-	-	-	1	-	-	1	1	-	2	1	2	-	-	-	-	-
360	-	-	-	-	-	-	-	-	-	1	1	2	-	1	-	-	-	-	-
408	3	6	1	3	2	4	1	-	1	-	-	1	2	-	1	2	-	-	-
413	-	-	-	-	-	-	-	-	-	-	-	1	-	4	-	-	-	-	-
448	-	-	-	-	-	1	-	3	-	-	-	-	-	-	1	-	-	-	-
472	2	4	5	5	8	12	10	17	24	18	17	13	15	8	9	5	5	-	-
476	1	-	-	-	-	-	1	-	-	-	1	1	-	-	1	-	-	-	-
508	11	8	8	16	9	7	-	1	-	-	-	-	1	-	-	-	-	-	-
680	-	1	-	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-
708	7	1	6	8	9	14	22	12	7	7	7	6	6	4	5	3	1	-	-
944	-	-	1	-	-	-	1	1	1	2	1	2	1	1	1	1	1	-	-
952	-	-	-	-	-	-	-	-	-	-	-	2	-	-	3	3	-	-	-
984	-	-	-	-	-	-	-	-	-	-	-	-	-	2	-	-	-	-	-
1416	13	8	8	2	11	9	14	8	11	2	1	2	1	1	-	-	-	-	-
1524	1	5	-	4	7	9	-	5	-	-	-	-	-	-	-	-	-	-	-
2006	-	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-
2152	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2360	10	10	5	6	5	2	-	-	-	-	-	-	-	-	-	-	-	-	-
3556	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7080	-	-	5	4	5	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7112	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
8636	6	1	6	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
25908	5	9	4	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
29972	5	15	10	5	-	-	-	-	-	2	2	2	-	-	-	-	-	-	-
89916	-	-	-	5	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 3. A strange evolution history

Step No	Size of cycles
1 - 10	1
11 - 18	34(5)
19	34, 408, 2108, 6324, 267716
20	408(2), 8636, 17272, 25908
21	204, 408(2), 476, 8636
22	136, 136, 272, 204, 408
23	136(2), 204, 272, 408
24	136, 204(2), 408, 680
25	102, 204(2), 408, 680
26	204(2), 340, 408, 816
27	136(2), 204,1020(2)
28	12(4), 204
29	12(4), 408
30	12, 168(3), 264
31	12(3), 84, 204
32	6, 12(2), 264

Table 4. Coding three variable binary functions

000	>>>	0	*	128
001	>>>	1	*	64
010	>>>	1	*	32
011	>>>	0	*	16
100	>>>	1	*	8
101	>>>	0	*	4
110	>>>	0	*	2
111	>>>	1	*	1

Table 5A. Attractor structure of three variable binary recursions

There is no marginal of value 2

Primal's look-up-table								In ver- tibi- lity	Code		M	M	M	Attractors			
1 2 8	6 4	3 2	1 6	8	4	2	1		Pri- mal	Du- al	A	B	C				
0	0	0	1	0	1	1	1	-	23	23	1	1	1	A	B	C	1
0	0	1	0	1	0	1	1	-	43	43	1	1	3	A	B	$2C$	-
0	1	0	0	1	1	0	1	-	77	77	1	3	1	A	$2B$	C	-
0	1	1	1	0	0	0	1	-	113	113	3	1	1	$2A$	B	C	-
1	0	0	0	1	1	1	0	-	142	142	1	3	3	A	$2B$	$2C$	-
1	0	1	1	0	0	1	0	-	178	178	3	1	3	$2A$	B	$2C$	-
1	1	0	1	0	1	0	0	-	212	212	3	3	1	$2A$	$2B$	C	-
1	1	1	0	1	0	0	0	-	232	232	3	3	3	$2A$	$2B$	$2C$	-

Table 5B. Attractor structure of three variable binary recursions
There is one marginal of value 2

Primal's look-up-table								In										
1								ver-	Code		M	M	M	Attractors				
2	6	3	1					tibi-	Pri-	Du-	A	B	C					
8	4	2	6	8	4	2	1	lity	mal	al								
0	0	0	1	1	0	1	1	-	27	39	1	1	2	A	B	-	1	
0	0	0	1	1	1	0	1	-	29	71	1	2	1	A	-	C	-	
0	0	1	0	1	1	1	0	-	46	139	1	2	3	A	-	-	-	$B+C$
0	0	1	1	0	1	0	1	-	53	83	2	1	1	-	B	C	1	
0	0	1	1	1	0	1	0	-	58	163	2	1	3	-	B	-	-	$A+C$
0	1	0	0	1	1	1	0	-	78	141	1	3	2	A	-	-	-	$B+C$
0	1	0	1	1	1	0	0	-	92	197	2	3	1	-	-	C	-	$A+B$
0	1	1	1	0	0	1	0	-	114	177	3	1	2	-	B	-	-	$A+C$
0	1	1	1	0	1	0	0	-	116	209	3	2	1	-	-	C	-	$A+B$
1	0	1	0	1	1	0	0	-	172	202	2	3	3	-	-	-	-	$B+C$
1	0	1	1	1	0	0	0	-	184	226	3	2	3	-	-	-	-	$A+C$
1	1	0	1	1	0	0	0	-	216	228	3	3	2	-	-	-	-	$A+B$

Table 5C. Attractor structure of three variable binary recursions

There are two marginals of value 2

Primal's look-up-table								In										T
1								ver-	Code		M	M	M	Attractors				a
2	6	3	1					tibi-	Pri-	Du-	A	B	C					i
8	4	2	6	8	4	2	1	lity	mal	al								l
0	0	0	1	1	1	1	0	A	30	135	1	2	2	a	-	-	-	0
0	0	1	0	1	1	0	1	A	45	75	1	2	2	a	-	-	-	0
0	0	1	1	0	1	1	0	B	54	147	2	1	2	-	b	-	-	?
0	0	1	1	1	0	0	1	B	57	99	2	1	2	-	b	-	-	?
0	1	0	1	0	1	1	0	C	86	149	2	2	1	-	-	c	-	?
0	1	0	1	1	0	0	1	C	89	101	2	2	1	-	-	c	-	?
0	1	1	0	1	0	1	0	C	106	169	2	2	3	-	-	-	-	?
0	1	1	0	1	1	0	0	B	108	201	2	3	2	-	-	-	-	?
0	1	1	1	1	0	0	0	A	120	225	3	2	2	-	-	-	-	0
1	0	0	1	1	0	1	0	C	154	166	2	2	3	-	-	-	-	?
1	0	0	1	1	1	0	0	B	156	198	2	3	2	-	-	-	-	?
1	0	1	1	0	1	0	0	A	180	210	3	2	2	-	-	-	-	0

Table 5D. Attractor structure of three variable binary recursions

There are three marginals of value 2

Primal's look-up-table								Invertibility	Code					Attractors	Tail
128	64	32	16	8	4	2	1		Primal	Dual	M A	M B	M C		
0	0	1	1	1	1	0	0	AB	60	195	2	2	2	- - - -	0
0	1	0	1	1	0	1	0	AC	90	165	2	2	2	- - - -	0
0	1	1	0	0	1	1	0	BC	102	153	2	2	2	- - - -	? (>0)
0	1	1	0	1	0	0	1	ABC	105	105	2	2	2	- - - -	0
1	0	0	1	0	1	1	0	ABC	150	150	2	2	2	- - - -	0

Table 6. Cycle lenghts in majority voting

	$N1$	C	NC	B	NB	A	NA	Other	Not Conv
1	780	23	1	59	180	137	200	-	-
1	0	274	10	458	0	865	0	-	-
1	8	9	0	110	0	598	2	-	-
1	2	128	8	146	0	421	0	-	-
1	0	162	0	516	0	621	8	3	1
1	1	548	0	558	1	947	1	-	1
1	1	28	0	93	0	100	9	-	-
1	7	15	0	59	1	216	2	-	-
1	6	8	0	17	4	52	0	-	-
1	0	17	0	154	10	194	0	-	-
1	8	3	0	13	0	211	2	-	-
1	0	167	0	213	10	282	0	-	-
1	6	26	0	54	1	169	1	13	-
1	0	17	0	21	0	287	10	-	-
1	7	9	0	10	0	107	3	-	-
1	0	15	0	27	0	207	10	-	-
1	3	75	0	347	2	398	5	-	-
1	8	85	0	152	0	323	0	17, 19	-
1	1	5	0	34	0	283	9	-	-
1	1	42	0	88	1	112	2	2, 14	-
1	10	1	10	2	0	3	0	-	-
1	5	109	0	244	3	303	2	-	-
1	8	32	0	152	1	329	1	-	-
1	1	66	1	74	1	102	4	2	-
1	1	80	9	114	0	303	0	-	-
1	0	116	10	159	0	232	0	-	-
1	3	3	7	18	0	94	0	-	-
1	2	197	0	235	7	345	1	-	-
1	0	14	10	28	0	127	0	-	-
1	7	29	3	74	0	184	0	-	-
1	5	11	0	73	4	301	1	-	-
1	7	248	0	252	0	353	3	-	-

Table 7. Number of uniform cycles in the simplest case

	B																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
A																	
4	1	-	1														
5	-	1	1	-													
6	1	-	7	-	1												
7	1	-	1	1	-	1											
8	-	-	-	-	-	-	-										
9	7	-	-	1	1	-	-	7									
10	-	-	1	-	-	-	1	-	-								
11	-	1	-	-	-	-	-	-	1	-							
12	-	-	91	-	5	-	5	-	91	-	-						
13	-	-	-	-	-	-	-	-	-	-	-	-					
14	-	-	-	-	3	-	-	-	3	-	-	-	-				
15	1	-	-	1	-	-	1	1	-	-	1	-	-	1			
16	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		
17	-	-	1	-	1	1	-	-	-	-	1	1	-	1	-	-	
18	-	-	1387	-	-	-	1	-	9709	-	1	-	-	-	1387	-	-

Table 8. Two base logarithms of cycle sizes in the simplest case

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	2∗	::	::	::	::	::	::	::	::	::	::	::	::	::	::	::	::
3	3∗	3∗	::	::	::	::	::	::	::	::	::	::	::	::	::	::	::
4	4∗	-	4∗	::	::	::	::	::	::	::	::	::	::	::	::	::	::
5	4+	5∗	5∗	4+	::	::	::	::	::	::	::	::	::	::	::	::	::
6	6∗	-	-	-	6∗	::	::	::	::	::	::	::	::	::	::	::	::
7	7∗	6+	7∗	7∗	6+	7∗	::	::	::	::	::	::	::	::	::	::	::
8	6=	-	7+	-	7+	-	6=	::	::	::	::	::	::	::	::	::	::
9	6+	8+	-	9∗	9∗	-	8+	6+	::	::	::	::	::	::	::	::	::
10	9+	-	10∗	-	-	-	10∗	-	9+	::	::	::	::	::	::	::	::
11	10+	11∗	10+	10+	9+	9+	10+	10+	11∗	10+	::	::	::	::	::	::	::
12	11+	-	-	-	9+	-	9+	-	-	-	11+	::	::	::	::	::	::
13	12+	10+	12+	12+	12+	12+	12+	12+	12+	12+	10+	12+	::	::	::	::	::
14	13+	-	12+	-	12+	-	-	-	12+	-	12+	-	13+	::	::	::	::
15	15∗	12+	-	15∗	-	-	15∗	15∗	-	-	15∗	-	12+	15∗	::	::	::
16	8=	-	15+	-	14=	-	15+	-	15+	-	14=	-	15+	-	8=	::	::
17	8+	16+	17∗	10=	17∗	17∗	12+	15+	15+	12+	17∗	17∗	10=	17∗	16+	8+	::
18	17+	-	-	-	15=	-	18∗	-	-	-	18∗	-	15=	-	-	-	17+
19	18+	16+	18+	16+	18+	18+	18+	15+	17+	17+	15+	18+	18+	18+	16+	18+	16+
20	19+	-	20∗	-	-	-	19+	-	16+	-	16+	-	19+	-	-	-	
21	12+	21∗	-	18+	12+	-	-	20+	-	20+	20+	-	20+	-	-	12+	18+
22	22∗	-	21+	-	21+	-	21+	-	–	-	-	-	–	-	21+	-	21+
23	–	22+	–	20+	23∗	21+	20+	21+	23∗	16+	–	–	16+	23∗	21+	20+	21+
24	21=	-	-	-	23+	-	–	-	-	-	22+	-	22+	-	-	-	–
25	23+	23>	–	21+	-	–	–	22+	23>	-	–	–	–	–	-	23+	22+
26	21+	-	–	-	–	-	–	-	22+	-	22+	-	-	-	22+	-	22+
27	25+	18+	-	22+	23+	-	23>	–	-	25>	23+	-	25+	25>	-	23>	25+
28	24+	-	25>	-	24+	-	-	-	–	-	23+	-	–	-	–	-	23>
29	–	23>	23+	–	21+	25>	–	25>	23+	–	–	23>	23>	23+	23>	23+	23>
30	23+	-	-	-	-	-	23>	-	-	-	25=	-	–	-	-	-	–
31	21=	–	23>	–	–	–	23+	24>	–	25>	–	23>	–	23>	–	–	23+

Table 9. The distribution of the ultimate rank

No	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
10	12	12	9	3	2	-	-	-	-	-	-	-	-	-	-	-
15	5	6	10	11	6	6	5	2	-	1	-	-	-	-	-	-
20	1	2	1	11	8	9	7	3	2	-	1	1	-	-	-	-
25	-	-	3	3	-	1	4	5	12	5	6	3	1	-	-	-
30	-	-	-	4	-	3	3	2	8	6	7	4	4	-	-	-
35	-	-	-	1	-	-	1	-	6	4	5	7	6	7	3	1
40	-	1	-	-	1	-	-	-	4	2	7	2	11	10	1	1
45	-	-	-	-	-	-	2	-	3	2	4	6	7	9	16	5
50	-	-	-	-	-	-	-	-	2	-	1	5	5	12	12	3
55	-	-	-	-	-	-	-	-	-	-	-	3	8	11	11	7
60	-	-	-	-	-	-	-	-	2	-	3	2	9	5	11	8
65	-	-	-	-	-	-	-	-	-	2	1	4	5	6	10	10
70	-	-	-	-	-	-	-	-	1	-	2	5	5	7	9	9
75	-	-	-	-	-	-	1	1	2	-	4	2	2	12	19	9
80	-	-	-	-	-	-	-	-	-	1	2	3	5	7	7	13
85	-	-	-	-	-	-	-	2	1	-	-	3	1	3	17	11
90	-	-	-	-	-	-	-	-	1	-	-	2	5	5	10	15
95	-	-	-	-	-	-	-	-	-	-	1	1	5	7	11	13
100	-	-	-	-	-	-	-	-	-	-	1	-	5	8	12	12

Table 9. (continued)

No	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
105	-	-	-	-	-	-	-	-	-	-	1	3	10	9	14	14
110	-	-	-	-	-	-	-	1	-	1	1	2	4	7	9	13
115	-	-	-	-	-	-	-	-	1	1	1	2	5	9	11	8
120	-	-	-	-	-	-	-	-	-	-	2	2	2	8	13	11
125	-	-	-	-	-	-	-	-	-	-	1	2	7	8	9	11
130	-	-	-	-	-	-	1	1	1	-	-	3	3	6	16	7
135	-	-	-	-	-	-	-	-	-	1	1	2	5	10	19	13
140	-	-	-	-	-	-	-	-	-	-	6	1	6	5	9	11
145	-	-	-	-	-	-	-	1	-	-	-	2	5	7	12	11
150	-	-	-	-	-	-	1	-	-	1	-	2	4	8	13	9
155	-	-	-	-	-	-	-	-	-	1	-	2	6	15	10	4
160	-	-	-	-	-	-	-	-	-	-	1	-	5	10	13	9
165	-	-	-	-	-	-	-	-	-	1	1	6	4	7	6	13
170	-	-	-	-	-	-	-	-	1	-	-	3	6	10	12	6
175	-	-	-	-	-	-	-	-	-	1	1	4	5	8	10	9
180	-	-	-	-	-	-	-	-	-	1	-	1	3	13	14	6
185	-	-	-	-	-	-	-	-	1	1	5	3	5	4	13	6
190	-	-	-	-	-	-	-	-	1	1	6	5	7	9	8	1
195	-	-	-	-	-	-	1	1	2	3	4	6	8	8	4	1
200	-	-	-	-	-	-	2	6	3	8	5	8	4	2	-	-
205	-	-	-	-	1	-	3	9	4	11	3	3	2	2	-	-
210	1	1	1	1	1	10	5	7	5	3	2	-	-	-	-	-
215	-	2	2	6	10	10	5	1	-	1	-	-	-	-	-	-
220	5	2	8	13	7	2	-	-	-	-	-	-	-	-	-	-