ON THE ORBITAL DIAMETER OF CLASSICAL GROUPS IN STANDARD ACTIONS

ATTILA MARÓTI AND KAMILLA REKVÉNYI

Abstract. Let G be a primitive permutation group acting on a finite set X. The orbital diameter $\mathrm{diam}(X, G)$ is defined to be the supremum of the diameters of the (connected) orbital graphs of G after disregarding the directions of all edges in the graphs. This invariant is studied in the case when G is an almost simple group in a standard action. A lower bound is given for $\mathrm{diam}(X, G)$ and we provide a partial classification of pairs (X, G) for which the orbital diameter is at most 2.

1. Introduction

Let G be a permutation group acting on a set X. An *orbital* is an orbit of G on $X \times X$. The *orbital graph* associated with an orbital E is the directed graph with vertex set X and edge set E. If G is transitive and E consists of loops, then the orbital graph is called *diagonal*. The criterion of Higman [3] (see also [2, Theorem 1.9]) states that a transitive permutation group is primitive if and only if all non-diagonal orbital graphs are connected. (A directed graph is said to be *connected* if the associated undirected graph is connected.) In this paper we suppose that G is a finite transitive permutation group (and X is a finite set). In this case a connected orbital graph is strongly connected (see [2, Theorem 1.10]). (A directed graph is said to be *strongly connected* if every vertex may be reached from any other vertex along a path with directed edges.)

In this paper an (undirected) orbital graph for (X, G) is a graph with vertex set X whose edge set is an orbit of G on the collection of unordered 2-element subsets of X. When G is primitive on a finite set X, we shall write $\mathrm{diam}(X, G)$ for the supremum of the diameters of the undirected, non-diagonal orbital graphs for (X, G) and call it the *orbital diameter* of G acting on X. We will focus on the orbital diameter of classical groups in their standard actions.

Liebeck, Macpherson and Tent [6] classified the infinite families of primitive permutation groups with bounded orbital diameter. These results and their proof methods were motivated by model theory and hence [6] contains no explicit bounds. Since then, the orbital diameter has been extensively studied from a purely group theoretical standpoint. Some explicit bounds in the case of almost simple groups

2020 *Mathematics Subject Classification.* 20B15, 5E15, 51N30.

Key words and phrases. finite classical group, orbital graph, primitive permutation group.

The first author was supported by the National Research, Development and Innovation Office (NKFIH) Grant No. K153681 and Grant No. K138828.

with alternating socles are provided by Sheikh [10]. Such groups with orbital diameter at most 5 are also described. Similarly, explicit bounds and descriptions of groups with small orbital diameters are given by the second author [12] for the case of primitive groups of simple diagonal type and in [13] for primitive affine groups. For upper bounds in the affine case, see [7], [8] and [11].

For an integer $n \geq 2$ and a prime power q, let $\mathrm{Cl}_n(q)$ denote any of the groups $\mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$, $\mathrm{PSU}_n(q)$, $\mathrm{P}\Omega_n^{\pm}(q)$. Let G be an almost simple primitive permutation group acting on a set X of size m. Let the socle of G be isomorphic to A_n or $\mathrm{Cl}_n(q)$. Let t be a positive integer at most $n-1$. We now introduce the definition of a standard t-action taken from [6, Section 1.2].

Definition 1.1. *Let t be a positive integer and let X be a finite set. Let G be an almost simple primitive permutation group acting on X. Let the socle of G be G_0 (a non-abelian simple group). We say that the group G acting on X has a standard t-action if any of the following holds.*

- *(a) $G_0 = \mathrm{A}_n$ and $X = I^{\{t\}}$, the set of t-subsets of $I = \{1, \ldots, n\}$ with the natural action of A_n.*
- *(b) $G_0 = \mathrm{Cl}_n(q)$ and X is an orbit of subspaces of dimension or codimension t in the natural module $V_n(q)$; the subspaces are arbitrary if $G_0 = \mathrm{PSL}_n(q)$, and otherwise are totally singular (here we denote X by $\mathcal{S}_t$), non-degenerate (here we denote X by $\mathcal{N}_t$), or, if G_0 is orthogonal and q is even, are non-singular 1-spaces (in which case $t = 1$ and X is denoted by $\mathcal{N}_1$).*
- *(c) $G_0 = \mathrm{PSL}_n(q)$, G contains a graph automorphism of G_0, and X is an orbit of pairs of subspaces $\{U, W\}$ of $V = V_n(q)$, where either $U \subseteq W$ or $V = U \oplus W$, and $\dim U = t$, $\dim W = n - t$.*
- *(d) $G_0 = \mathrm{Sp}_n(q)$, q even, and a point stabilizer in G_0 is $\mathrm{O}_n^{\pm}(q)$ (here we take $t = 1$).*

Remark 1.2. Note that if G_0 is orthogonal and t is even, then $\mathcal{N}_t$ has two types which we will denote by O_t^+ and O_t^-.

Let $k = \min\{t,\ n-t\}$. Note that in [6, p. 229] it is remarked that it is easy to establish the bound $\mathrm{diam}(X, G) \geq k$ in case $G_0 = \mathrm{PSL}_n(q)$. In this paper we provide a proof for this fact. In fact we show that in case (b) for $G_0 = \mathrm{PSL}_n(q)$, $\mathrm{diam}(X, G) = k$. In [6, p. 230] it is proved that if G_0 is a classical group different from $\mathrm{PSL}_n(q)$ and $\mathrm{diam}(X, G)$ is bounded, then k is bounded.

Our first main result is the following.

Theorem 1.3. *Let G be a primitive permutation group acting on a finite set X such that G has a standard t-action for some positive integer t as in (a)-(d) of Definition 1.1. Put $k = \min\{t, n-t\}$. If $(G_0, X) \neq (\mathrm{P}\Omega_n^+(q), \mathcal{S}_{n/2})$, then $\mathrm{diam}(X, G) \geq k$, otherwise $\mathrm{diam}(X, G) = \lfloor k/2 \rfloor$.*

We now turn to the classification of primitive permutation groups G acting on finite sets X in standard t-actions for integers t such that $\mathrm{diam}(X, G) \leq 2$.

Let G be a primitive permutation group acting on a finite set X such that $\mathrm{diam}(X, G) = 1$. In this case G is a 2-homogeneous permutation group. It follows that G is 2-transitive or is of odd order. Let G have a standard t-action for some positive integer t. Since G has even order, G must be a 2-transitive permutation group. The finite 2-transitive almost simple groups have been classified in [1, Theorem 5.3] (see also Note 2 after [1, Theorem 5.3]). The 2-transitive groups G in standard t-actions are the groups in (a) of Definition 1.1 with $t \in \{1, n-1\}$, the groups in (b) of Definition 1.1 with $G_0 = \mathrm{PSL}_n(q)$ and $t \in \{1, n-1\}$, the groups in (b) of Definition 1.1 with $G_0 = \mathrm{PSU}_3(q)$ $(q > 2)$ and $X = \mathcal{S}_t$, and the groups in (d) of Definition 1.1 (in the latter case see also [4, Theorem 2]).

Now let G be a primitive permutation group acting on a finite set X such that $\mathrm{diam}(X, G) = 2$. Suppose that G has a standard t-action for X for some positive integer t. The pairs (X, G) satisfying (a) of Definition 1.1 are completely described in [10, Theorem 1.3 (1)]. The pair (X, G) cannot satisfy (d) of Definition 1.1 by the previous paragraph. The pairs (X, G) satisfying (b) of Definition 1.1 with $X = \mathcal{S}_t$ are completely described in [9, Theorem 6.2.1 (1)].

Thus in order to classify all primitive permutation groups G acting on finite sets X in standard t-actions for integers t such that $\mathrm{diam}(X, G) \leq 2$, we may suppose that $\mathrm{diam}(X, G) = 2$ and that (b) or (c) of Definition 1.1 is satisfied with the assumption that $X \neq \mathcal{S}_t$ provided that (b) holds.

Our second main theorem is a partial classification of primitive permutation groups G acting on finite sets X in standard t-actions for integers t such that $\mathrm{diam}(X, G) \leq 2$.

Theorem 1.4. *Let G be a primitive permutation group acting on a finite set X such that G has a standard t-action. Let $k = \min\{t, n-t\}$. Suppose that $n \geq 8$. Let $\mathrm{diam}(X, G) = 2$ and that (b) or (c) of Definition 1.1 is satisfied with the assumption that $X \neq \mathcal{S}_t$ provided that (b) holds. One of the following holds.*

(1) (X, G) is as in (b) of Definition 1.1 and $k = 1$.
(2) (X, G) is as in (b) of Definition 1.1 and $k = 2$, moreover $G_0 = \mathrm{P}\Omega_n^{\epsilon}(q)$ with $q \equiv 3 \pmod 4$ and the elements of X are of type O_2^-.

Conversely, if (1) holds, then $\mathrm{diam}(X, G) = 2$.

Remark 1.5. In part (2) of Theorem 1.4 we have not been able to determine whether the orbital diameter is 2. We conjecture that it is at least 3.

2. Background

In this section we recall the background and notation which we will need.

Let us start with forms. Let V be a finite vector space over a finite field F. Let f be a map from $V \times V$ to F. The map f is called *non-degenerate* if for each non-zero vector v in V the maps from V to F given by $x \to f(x, v)$ and $x \to f(v, x)$ are non-zero. A quadratic form Q is called *non-degenerate* if its associated (symmetric) bilinear form f is non-degenerate. The map f is called *symmetric* if $f(u, v) = f(v, u)$ for all u, v in V. The map f is called *skew-symmetric* if $f(u, v) = -f(v, u)$ for all u, v in V. If the characteristic of F is odd and f is skew-symmetric, then $f(v, v) = 0$ for all $v \in V$. The map f is said to be *symplectic* if f is skew-symmetric, bilinear, and $f(v, v) = 0$ for all $v \in V$. The map f is said to be *unitary* if F admits an involutary field automorphism α and f is left-linear and $f(u,v) = f(v,u)^{\alpha}$ for all u, $v \in V$.

Let f be a bilinear or a sesquilinear form (or a unitary map) on V. Vectors u and v in V are said to be *orthogonal* or *perpendicular* if $f(u,v) = 0$. For a subspace U in V, let $U^{\perp}$ be the subspace of V consisting of all vectors v in V such that $f(u,v) = 0$ for all $u \in U$. The form f is called *non-singular* if $V^{\perp} = \{0\}$ and *singular* otherwise. A subspace U of V is called *totally singular* if $f(u,v) = 0$ for all u, $v \in U$ and is called *non-degenerate* if $U \cap U^{\perp} = \{0\}$.

Let n denote the dimension of V over F. Let f be a non-degenerate unitary form on V. In this case V admits an orthonormal basis (see [5, Proposition 2.3.1]), that is, a basis $\{e_1, \ldots, e_n\}$ such that $f(e_i, e_i) = 1$ for all i with $1 \leq i \leq n$ and $f(e_i, e_j) = 0$ whenever i and j are different indices between 1 and n. There is another useful basis called the *unitary basis* $\mathcal{B}$ for V. If $n = 2m$ is even, then V has a basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ and if $n = 2m+1$ is odd, then V has a basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m, x\}$ such that $f(u,v) = 0$ whenever u, $v \in \mathcal{B}$, except when $(u,v) = (x,x)$ or $\{u,v\} = \{e_i, f_i\}$ for some index i with $1 \leq i \leq n$, in which case $f(u,v) = 1$. See [5, Proposition 2.3.2]. Let f be a non-degenerate symplectic form on V. In this case $n = 2m$ is even and V admits a so-called *symplectic basis* $\mathcal{B} = \{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ such that $f(u,v) = 0$ for all u, $v \in \mathcal{B}$ unless $\{u,v\} = \{e_i, f_i\}$ for some i with $1 \leq i \leq m$ in which case $f(e_i, f_i) = -f(f_i, e_i) = 1$. See [5, Proposition 2.4.1]. Let Q be a non-degenerate quadratic form on V. Let f be the associated bilinear form. This is a non-degenerate symmetric bilinear form on V. In this case V admits a *standard basis* $\mathcal{B}$ defined in the following way (see [5, Proposition 2.5.3]). First let $n = 2m+1$ be odd. In this case $\mathcal{B} = \{e_1, \ldots, e_m, f_1, \ldots, f_m, x\}$ such that $Q(e_i) = Q(f_i) = 0$ for all i with $1 \leq i \leq m$, the vector x is non-singular, and $f(u,v) = 0$ for all u, $v \in \mathcal{B}$ except when $\{u,v\} = \{e_i, f_i\}$ for some index i with $1 \leq i \leq m$, in which case $f(u,v) = 1$. When $n = 2m$ is even, then there are two cases. In the first case the basis $\mathcal{B}$ is $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ such that $Q(e_i) = Q(f_i) = 0$ for all i with $1 \leq i \leq m$ and $f(u,v) = 0$ for all u, $v \in \mathcal{B}$ unless $\{u,v\} = \{e_i, f_i\}$ for some i in which case $f(u,v) = 1$. We will refer to $\langle e_1, \ldots, e_m, f_1, \ldots, f_m \rangle$ as type O_n^+. In the second case $\mathcal{B}$ has the form $\{e_1, \ldots, e_{m-1}, f_1, \ldots, f_{m-1}, x, y\}$ with the following property. We have $Q(e_i) = Q(f_i) = 0$ for all i with $1 \leq i \leq m$, $Q(x) = 1$, $Q(y) = \zeta$ where ζ is such that the polynomial $t^2 + t + \zeta$ (in the variable t) is irreducible over F, and $f(u,v) = 0$ for all u, $v \in \mathcal{B}$ unless $\{u,v\} = \{e_i, f_i\}$ for some i or $\{u,v\} = \{x,y\}$, in which case $f(u,v) = 1$. We will refer to $\langle e_1, \ldots, e_{m-1}, f_1, \ldots, f_{m-1}, x, y \rangle$ as type O_n^-.

We recall Witt's Lemma from [5, Proposition 2.1.6].

Theorem 2.1 (Witt's lemma)**.** *Assume that (V_1, κ_1), (V_2, κ_2) are two isometric classical geometries and that W_i is a subspace of V_i for $i \in \{1,2\}$. Further assume that there is an isometry φ from (W_1, κ_1) to (W_2, κ_2). Then φ extends to an isometry from (V_1, κ_1) to (V_2, κ_2).*

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3.

3.1. **Case (a).** Let (G, X) satisfy (a) of Definition 1.1. Recall that in case (a) $G_0 = \mathrm{A}_n$ and $X = I^{\{t\}}$, the set of t-subsets of $I = \{1, \ldots, n\}$ with the natural action of A_n. In this case we have that $\mathrm{diam}(X, G) = k$ by [10, Theorem 1.1] provided that $k < n/2$. Note that the fact that k is a lower bound for $\mathrm{diam}(X, G)$ can be found in [6, p. 229].

3.2. **Case (d).** Let (G, X) satisfy (d) of Definition 1.1. Recall that in this case $G_0 = \mathrm{Sp}_n(q)$, q even, and a point stabilizer in G_0 is $\mathrm{O}_n^{\pm}(q)$. Here $\mathrm{diam}(X, G) \geq 1$ is obvious.

In proving Theorem 1.3 in cases (b) and (c) of Definition 1.1 we will need a lemma.

Let X be a set of vector spaces each of dimension k. Let G be a primitive permutation group acting on X in standard t-action. Let U, $U' \in X$ be in the same G-orbit such that $U' \neq U$ and $\dim(U \cap U')$ is maximal. Let $\mathcal{O}$ be the orbital graph containing the edge $\{U, U'\}$. For two elements A and B in X let $d(A, B)$ denote the distance between A and B in the graph $\mathcal{O}$.

Lemma 3.1. *Let* $\dim(U \cap U') = k - l$ *with* $l \geq 1$*. If there exists* U'' *in* X *such that* $\dim(U \cap U'') = k - rl$ *for some positive integer* r*, then* $d(U, U'') \geq r$*. In particular, if* $\dim(U \cap U') = k - 1$ *and there is* U'' *such that* $\dim(U \cap U'') = 0$*, then the diameter of* $\mathcal{O}$ *is at least* k*.*

Proof. Let $A, B \in X$. We claim that if $d(A, B) < r$, then $\dim(A \cap B) > k - rl$. Hence if $\dim(A \cap B) \leq k - rl$, then $d(A, B) \geq r$. We will prove the claim by induction on r. The base case $r = 1$ is clear. For $r = 2$ we may suppose that $d(A, B) = 1$ and so $\dim(A \cap B) = k - l > k - 2l$.

Suppose that $r \geq 3$ and the claim is true for $r - 1$, that is, $d(A, B) < r - 1$ implies that $\dim(A \cap B) > k - (r - 1)l$. We wish to prove it for r. For this let A and B be members of X such that $d(A, B) = r - 1$. The vector space B has a neighbor C such that $d(A, C) < r - 1$. We have $\dim(B \cap C) = k - l$ and $\dim(A \cap C) > k - (r - 1)l$ by the induction hypothesis. It follows that

$$k = \dim(C) \geq \dim((A \cap C) + (B \cap C)) \geq$$

$$\geq \dim(A \cap C) + \dim(B \cap C) - \dim(A \cap B) > 2k - rl - \dim(A \cap B),$$

giving $\dim(A \cap B) > k - rl$. □

Now we prove Theorem 1.3 for the cases (b) and (c) of Definition 1.1.

3.3. **Case (b).** Let (G, X) satisfy (b) of Definition 1.1. Recall that in this case $G_0 = \mathrm{Cl}_n(q)$ and X is an orbit of subspaces of dimension or codimension t in the natural module $V_n(q)$; the subspaces are arbitrary if $G_0 = \mathrm{PSL}_n(q)$, and otherwise are totally singular, non-degenerate, or, if G_0 is orthogonal and q is even, are non-singular 1-spaces (in which case $t = 1$).

Note that for $k = 1$ the bound immediately follows so we can assume that $t \geq 2$.

Let U be a vector space in X. We may assume in all cases that $\dim(U) \leq n/2$. For $G_0 = \mathrm{PSL}_n(q)$, this follows by the use of the inverse transpose automorphism.

Let $G_0 \neq \mathrm{PSL}_n(q)$. For U totally singular, this follows by [5, Corollary 2.1.7] and for U non-degenerate, by [5, Lemma 2.1.5 (ii), (iii), (v)].

We begin with the case when $G_0 = \mathrm{PSL}_n(q)$.

Lemma 3.2. *Let $G_0 = \mathrm{PSL}_n(q)$ and X the set of all subspaces of $V_n(q)$ of dimension k. In this case $\dim(U \cap U') = k-1$ and the diameter of $\mathcal{O}$ is k.*

Proof. We will prove that for $G_0 = \mathrm{PSL}_n(q)$ and X the set of all subspaces of $V_n(q)$ of dimension k we have $\dim(A \cap B) = r$ if and only if $d(A,B) = k-r$ for any A, B in X. We proceed by induction on r. This is clear for $r = k$ and it is also clear by transitivity for $r = k-1$. Assume that $r \leq k-2$ and that the claim is true for larger values of r.

Let $\dim(A \cap B) = r$. We have $d(A,B) \geq k-r$ by induction. Let $\{e_1, \ldots, e_k\}$ be a basis for A and $\{e_1, \ldots, e_r, f_{r+1}, \ldots, f_k\}$ be a basis for B. For i with $1 \leq i \leq k-r$ put $A_i = \langle e_1, \ldots, e_r, e_{r+1}, \ldots, f_{k-(i-1)}, \ldots, f_k \rangle$. It is clear that

$$A, A_1, \ldots, A_{k-r} = B$$

is a path in $\mathcal{O}$ of length $k-r$. Thus $d(A,B) \leq k-r$ and so $d(A,B) = k-r$.

Assume that $d(A,B) = k-r$. We have $\dim(A \cap B) \leq r$ by induction. There exists a vector space C in X with $d(A,C) = k-r-1$ and $d(C,B) = 1$. We have $\dim(A \cap C) = r+1$ and $\dim(B \cap C) = k-1$ by induction. Now

$$k = \dim(C) \geq \dim((A \cap C)+(B \cap C)) = \dim(A \cap C)+\dim(B \cap C)-\dim(A \cap B \cap C) =$$

$$= k + r - \dim(A \cap B \cap C).$$

It follows that $r \geq \dim(A \cap B) \geq \dim(A \cap B \cap C) \geq r$. □

In fact, if $G_0 = \mathrm{PSL}_n(q)$ and X is the set of all k-dimensional subspaces of $V_n(q)$, then $\mathrm{diam}(X,G) = k$, provided that $k < n/2$.

Lemma 3.3. *If $G_0 = \mathrm{PSL}_n(q)$ and X is the set of all k-dimensional subspaces of $V_n(q)$, then $\mathrm{diam}(X,G) = k$, provided that $k < n/2$.*

Proof. The lower bound $\mathrm{diam}(X,G) \geq k$ follows from Lemma 3.2. The upper bound can be seen as follows. Let $\mathcal{G}$ be an arbitrary (non-diagonal) orbital graph containing the edge $\{W, W'\}$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for $V_n(q)$. Since $k \leq n/2$, by applying an element of G_0 to W and W' if necessary, we may assume that both W and W' have a basis, say $\mathcal{W}$ and $\mathcal{W}'$ respectively, which are subsets of $\mathcal{B}$. Consider the natural action of A_n on $\mathcal{B}$. (This group may be viewed as a subgroup of G_0.) This defines an action of A_n on the set of k-element subsets of $\mathcal{B}$. Consider the orbital containing the edge $\{\mathcal{W}, \mathcal{W}'\}$. This has diameter k by [10, Proposition 3.1], provided that $k < n/2$. Now let U and U' be arbitrary vertices in X of maximal distance apart in $\mathcal{G}$. By applying an element of G_0 if necessary, it may be assumed that the bases $\mathcal{U}$ and $\mathcal{U}'$ of U and U' respectively are subsets of $\mathcal{B}$. Since $\mathrm{diam}(\mathcal{B}, \mathrm{A}_n) = k$, provided that $k < n/2$, the distance between $\mathcal{U}$ and $\mathcal{U}'$ and thus the distance between U and U' in $\mathcal{G}$ is at most k, provided that $k < n/2$. □

The corresponding exact diameter for classical groups with socle a simple symplectic group is not known. See [9, Conjecture 6.2.3].

Let $\mathcal{S}$ be the set of all totally singular or non-degenerate subspaces of $V_n(q)$ which are isomorphic to U.

Lemma 3.4. *The orbit X is equal to $\mathcal{S}$, unless possibly if n is even and any of the following holds:*

(1) $G_0 = \mathrm{P}\Omega_n^+(q)$ and U is totally singular of dimension $n/2$;
(2) $G_0 = \mathrm{P}\Omega_n^{\pm}(q)$, q is odd and U is non-degenerate with $\dim U$ odd.

In any case $\mathcal{S}$ is the union of at most two G-orbits.

Proof. This follows from Witt's lemma (see Theorem 2.1) and Propositions 4.1.3, 4.1.4, 4.1.6, 4.1.18, 4.1.19, 4.1.20 of [5]. For the definition of c in the statements of these propositions see [5, Section 3.2]. □

Lemma 3.5. *If (1) of Lemma 3.4 holds, then* $\mathrm{diam}(X, G) = [k/2]$.

Proof. First we show that $\mathrm{diam}(X, G) \geq [k/2]$. It is sufficient to prove that G has an orbital graph whose diameter is at least $[k/2]$. Recall that n is even and $k = n/2$. Let Q be the quadratic form on $V_n(q)$ and let $\langle , \rangle$ be the associated bilinear form. Let $\{e_1, \ldots, e_k, f_1, \ldots, f_k\}$ be the standard basis of $V_n(q)$ with $Q(e_i) = Q(f_i) = 0$ and $(e_i, f_j) = \delta_{ij}$ for all i and j with $1 \leq i, j \leq k$. We claim that there are vector spaces W, W', W'' in $\mathcal{S}$ such that $\dim(W \cap W')$ is equal to 0 if k is even, is 1 if k is odd and $\dim(W \cap W'') = k - 2$. Let $W = \langle e_1, \ldots, e_k \rangle$. Take W' to be $\langle f_1, \ldots, f_k \rangle$ if k is even and $W' = \langle e_1, f_2, \ldots, f_k \rangle$ if k is odd. Put $W'' = \langle e_1, e_2, \ldots, f_{k-1}, f_k \rangle$.

The set $\mathcal{S}$ is the union of two $\Omega_n^+(q)$-orbits by Description 4 on page 30 of [5]. Let these be $\mathcal{U}_k^1$ and $\mathcal{U}_k^2$. For each $i \in \{1, 2\}$ vector spaces A and B are in $\mathcal{U}_k^i$ if and only if $k - \dim(A \cap B)$ is even by [5, Description 4]. It follows that W, W', W'' belong to the same $\Omega_n^+(q)$-orbit.

Let $\mathcal{O}'$ be the orbital graph containing the edge $\{W, W''\}$ for the permutation group $\mathrm{PSL}_n(q)$ acting on $\mathcal{S}$. By Lemma 3.1, the distance between W and W' in the graph $\mathcal{O}'$ and thus also the distance in the orbital graph for G containing the edge $\{W, W''\}$ is at least $[k/2]$.

Now we prove that $\mathrm{diam}(X, G) \leq [k/2]$. By [9, Lemma 6.4.2.(3)], the rank of this action is $[k/2] + 1$ so the upper bound on the orbital diameter follows immediately. □

Lemma 3.6. *If (2) of Lemma 3.4 holds, then the diameter of $\mathcal{O}$ is at least k.*

Proof. The set $\mathcal{S}$ of all subspaces isomorphic to U consists of at most two G_0-orbits, one of these being X. By inspecting the non-singular symmetric bilinear form on $V_n(q)$ with $n \geq 3$ (see [14, Section 3.4.6]), we see that there exist three subspaces A_1, A_2, A_3 in $\mathcal{S}$ such that $\dim(A_i \cap A_j) = k - 1$ for all i and j with $1 \leq i < j \leq 3$. It follows that there is $A' \in X$ with $\dim(U \cap A') = k - 1$. Similarly, one can show that there are subspaces B_1, B_2, B_3 in $\mathcal{S}$ such that $\dim(B_i \cap B_j) = 0$ for every i and j with $1 \leq i < j \leq 3$. It follows that there is $B' \in X$ such that $\dim(U \cap B') = 0$. Thus the diameter of $\mathcal{O}$ is at least k by Lemma 3.1. □

From now on we assume that $X = \mathcal{S}$.

Lemma 3.7. *If $X = \mathcal{S}$, then the diameter of $\mathcal{O}$ is at least k.*

Proof. It is sufficient to show by Lemma 3.1 that there are U', $U'' \in \mathcal{S}$ such that $\dim(U \cap U') = k-1$ and $\dim(U \cap U'') = 0$. This is clear in case $G_0 = \mathrm{PSL}_n(q)$.

Now consider the case when G_0 is not orthogonal. Assume that $V_n(q)$ admits a non-singular alternating bilinear form. Let

$$\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$$

be a standard symplectic basis with $n = 2m$. If $U = \langle e_1, \ldots, e_\ell, f_1, \ldots, f_\ell\rangle$ is a non-degenerate space with $k = 2\ell$, then put $U' = \langle e_1, \ldots, e_\ell, f_1, \ldots, f_\ell + f_{\ell+1}\rangle$ and $U'' = \langle e_{\ell+1}, \ldots, e_{2\ell}, f_{\ell+1}, \ldots, f_{2\ell}\rangle$. If $U = \langle e_1, \ldots, e_k\rangle$ is totally singular, then put $U' = \langle e_1, \ldots, e_{k-1}, f_k\rangle$ and $U'' = \langle f_1, \ldots, f_k\rangle$.

Let $V_n(q)$ admit a non-singular conjugate-symmetric sesquilinear form and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for this form. If $U = \langle e_1, \ldots, e_k\rangle$ is non-degenerate, then put $U' = \langle e_1, \ldots e_{k-1}, e_{k+1}\rangle$ and $U'' = \langle e_{k+1}, \ldots, e_{2k}\rangle$. If U is totally singular, then we may write $V_n(q)$ as a perpendicular direct sum of non-singular 2-spaces (together with a 1-space in case n is odd) such that each 2-space admits a symplectic basis (see [14, p. 67]). Now proceed as in the second part of the previous paragraph.

Now consider the case when G_0 is orthogonal.

Consider the case when U is non-degenerate. Recall the standard basis of $V_n(q)$. This is $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ or $\{e_1, \ldots, e_{m-1}, f_1, \ldots, f_{m-1}, x, y\}$ for $n = 2m$ of type O_n^+ or O_n^- respectively, and for $n = 2m+1$ it is $\{e_1, \ldots, e_m, f_1, \ldots, f_m\, x\}$. For all i, j we have $Q(e_i) = Q(f_i) = 0$, $(e_i, e_j) = (f_i, f_j) = 0$, $(e_i, f_j) = \delta_{i,j}$ and $(x, y) = 1$, $(e_i, x) = (f_i, x) = (e_i, y) = (f_i, y) = 0$, $Q(x) = 1$ and $Q(y) = \alpha$ as in [5, Proposition 2.5.3]. Now we will find U, U', U'' such that $\dim(U \cap U') = k-1$ and $\dim(U \cap U'') = 0$, and then the result will follow by Lemma 3.1. We may assume that $k \geq 3$ and so $n \geq 6$. We have several cases to consider. Assume that n is even.

Assume $k = 2l$.

Assume U is of type O_{2l}^+. For $k < n/2$ or $G_0 = \mathrm{P}\Omega_n^+(q)$, choose

$$U = \langle e_1, \ldots, e_l, f_1, \ldots, f_l\rangle,$$

$$U' = \langle e_1, \ldots, e_l, f_1, \ldots, f_{l-1}, f_l + f_{l+1}\rangle$$

and

$$U'' = \langle e_{l+1}, \ldots, e_{2l}, f_{l+1}, \ldots, f_{2l}\rangle.$$

For $k = n/2$ or $G_0 = \mathrm{P}\Omega_n^-(q)$, choose

$$U = \langle e_1, \ldots, e_l, f_1, \ldots, f_l\rangle,$$

$$U' = \langle e_1, \ldots, e_l, f_1, \ldots, f_{l-1}, e_l - f_l + x\rangle$$

and

$$U'' = \langle e_{l+1}, \ldots, e_{2l-1}, f_{l+1}, \ldots, f_{2l-1}, e_1 - f_1 + x, e_1 - \alpha f_1 + y\rangle.$$

Assume U is of type O_{2l}^-. For $G_0 = \mathrm{P}\Omega_n^-(q)$, choose

$$U = \langle e_1, \ldots, e_{l-1}, f_1, \ldots, f_{l-1}, f_l + \alpha e_l, f_{l+1} + x + e_l\rangle,$$

$$U' = \langle e_1, \ldots, e_{l-1}, f_1, \ldots, f_{l-1}, f_l + \alpha e_l, f_{l+1} + e_{l+1} + e_l\rangle$$

and

$$U'' = \langle e_{l+1}, \ldots, e_{2l-1}, f_{l+1}, \ldots, f_{2l-1}, y, e_1 + x\rangle.$$

For $G_0 = \mathrm{P}\Omega_n^+(q)$, choose

$$U = \langle e_1, \ldots, e_{l-1}, f_1, \ldots, f_{l-1}, f_l + \alpha e_l, f_{l+1} + e_{l+1} + e_l\rangle,$$
$$U' = \langle e_1, \ldots, e_{l-1}, f_1, \ldots, f_{l-1}, f_l + \alpha e_l, f_{l+2} + e_{l+2} + e_l\rangle$$

and

$$U'' = \langle e_{l+2}, \ldots, e_{2l}, f_{l+2}, \ldots, f_{2l}, e_1 + \alpha f_1 + f_l, e_2 + f_2 + f_{l+1} + f_1\rangle.$$

Assume $k = 2l + 1$. For $G_0 = \mathrm{P}\Omega_n^-(q)$, choose

$$U = \langle e_1, \ldots, e_l, f_1, \ldots, f_l, x\rangle,$$
$$U = \langle e_1, \ldots, e_l, f_1, \ldots, f_l, f_{l+1} + e_{l+1}\rangle,$$

and

$$U'' = \langle e_{l+1}, \ldots, e_{2l}, f_{l+1}, \ldots, f_{2l}, y\rangle.$$

For $G_0 = \mathrm{P}\Omega_n^+(q)$, choose

$$U = \langle e_1, \ldots, e_l, f_1, \ldots, f_l, f_{2l+1} + e_{2l+1} + e_{l+1}\rangle,$$
$$U' = \langle e_1, \ldots, e_l, f_1, \ldots, f_l, f_{l+2} + e_{l+2}\rangle$$

and

$$U'' = \langle e_{l+2}, \ldots, e_{2l+1}, f_{l+2}, \ldots, f_{2l+1}, e_1 + f_1 + f_{l+1}\rangle.$$

Now assume n is odd. Now $k \leq (n-1)/2$. We can choose the same U, U' and U'' as we did for $G_0 = \mathrm{P}\Omega_{n-1}^+(q)$.

Now consider the case when U is totally singular.

Recall that there is a basis as in [5, Proposition 2.5.3]. In the various cases the maximum possible value of $k = \dim(U)$ is determined according to [5, Proposition 2.5.4]. Using this information one can see that there are subspaces U' and U'' in X with $\dim(U \cap U') = k - 1$ and $\dim(U \cap U'') = 0$. It follows by Lemma 3.1 that in this case the diameter of $\mathcal{O}$ is at least k. □

3.4. **Case (c).** Let (G, X) satisfy (c) of Definition 1.1. Recall that in this case $G_0 = \mathrm{PSL}_n(q)$, G contains a graph automorphism of G_0, and X is an orbit of pairs of subspaces $\{U, W\}$ of $V = V_n(q)$, where either $U \subseteq W$ or $V = U \oplus W$, and $\dim U = t$, $\dim W = n - t$. We may suppose that $k = t \leq n/2$. Let $\{e_1, \ldots, e_n\}$ be a basis for $V = V_n(q)$ and let σ be the inverse transpose automorphism with respect to this basis. We may view σ also as an automorphism of G_0 and thus an element of G. Since we aim to establish a lower bound for $\mathrm{diam}(X, G)$, there is no harm in assuming that G is as large as possible, $\mathrm{P\Gamma L}(V) \leq G$ and $\sigma \in G$. Put $U = \langle e_1, \ldots, e_t\rangle$ and define the subspaces W_1 and W_2 to be $\langle e_{t+1}, \ldots, e_n\rangle$ and $\langle e_1, \ldots, e_{n-t}\rangle$ respectively. Let p_1 be the permutation matrix swapping e_t and e_{t+1} and fixing all other basis vectors, and let p_2 be the permutation matrix interchanging e_t and e_{n-t+1} and fixing all other basis vectors. Let i be 1 or 2. Observe that if $\{U, W_i\} \in X$ then $\{Up_i, W_ip_i\} \in X$. Let us denote $\{Up_i, W_ip_i\}$ by $\{U, W_i\}p_i$. Let $\mathcal{O}_i$ be the orbital graph of G containing the edge $E_i = \{\{U, W_i\}, \{U, W_i\}p_i\}$. Let the image of $\{U, W_i\}$ under σ be denoted by $\{U, W_i\}\sigma$. This is $\{U\sigma, W_i\sigma\}$ where $U\sigma = W_1$, $W_1\sigma = U$ and $W_2\sigma = \langle e_{n-t+1}, \ldots, e_n\rangle$. Observe that σ centralizes p_i. It follows that the edge E_1 is fixed by σ. Let h be the permutation matrix swapping e_j with

e_{n+1-j} for every j in $\{1,\dots,t\}$ and fixing the vectors $e_{t+1},\dots,e_{n-t}$. Observe that $\sigma h \in G$ and $\{U, W_2\}\sigma h = \{U\sigma, W_2\sigma\}h = \{W_1, \langle e_{n-t+1},\dots,e_n\rangle\}h = \{W_2, U\}$. Observe also that h centralizes p_2. It follows that σh fixes the edge E_2. To summarize, $\mathcal{O}_i$ is an orbital graph also of $\mathrm{P\Gamma L}(V)$. From now on assume that $G = \mathrm{P\Gamma L}(V)$.

Let $t < n/2$. For each pair $\{U', W\}$ in X one of $\dim(U')$ and $\dim(W)$ is smaller, and so, by disregarding the vector space of larger dimension from every pair in X, we may repeat the argument in Lemma 3.1 and obtain that the diameter of $\mathcal{O}_i$ is at least k. Thus $\mathrm{diam}(X, G) \geq k$.

Let $t = n/2$. The case when the vector spaces are equal in any pair in X was treated in (b). Assume that this is not the case and that $V = U \oplus W$ whenever $\{U, W\} \in X$. Since G_0 is contained in G, the orbit X is precisely the set of all pairs $\{U, W\}$ such that $V = U \oplus W$ and $\dim(U) = \dim(W) = t$. Consider the orbital graph $\mathcal{O}_1$. If $\{\{U_1, W_1\}, \{U_2, W_2\}\}$ is an edge in $\mathcal{O}_1$, then $\dim(U_1 \cap U_2) = t - 1$, $\dim(W_1 \cap W_2) = t - 1$, and $\dim(U_1 \cap W_2) = \dim(U_2 \cap W_1) = 1$ (with possibly U_1 and W_1 interchanged).

For arbitrary pairs $A = \{U_1, W_1\}$ and $B = \{U_2, W_2\}$ in X, let $d(A, B)$ denote their distance in $\mathcal{O}_1$ and define $\dim(A, B)$ to be

$$\max\{\dim(U_1 \cap W_2), \dim(U_1 \cap U_2), \dim(U_2 \cap W_1), \dim(W_1 \cap W_2)\}.$$

Lemma 3.8. *For pairs A and B in X we have $d(A, B) \geq t - \dim(A, B)$.*

Proof. Let $A = \{U_1, W_1\}$ and $B = \{U_2, W_2\}$. We proceed by induction on $d(A, B)$. If $d(A, B) = 1$, then $\dim(A, B) = t - 1$. Assume that $d(A, B) = d \geq 2$. If $\dim(A, B) \geq t - d$, then the claim holds. Assume that $\dim(A, B) \leq t - d - 1$. There exists $C = \{U_3, W_3\} \in X$ such that $d(A, C) = 1$ and $d(C, B) \leq d - 1$. There exist an element of A and an element of C whose intersection has dimension $t - 1$, by the definition of $\mathcal{O}_1$. Let these be U_1 and U_3. There exists, by induction, an element, say U_2, of B whose intersection with U_3 has dimension at least $t - d + 1$. Since $\dim(A, B) \leq t - d - 1$, we have $\dim(U_1 \cap U_2) \leq t - d - 1$. Since $(U_1 \cap U_3) + (U_2 \cap U_3)$ is contained in U_3, it follows that $\dim((U_1 \cap U_3) + (U_2 \cap U_3)) \leq \dim(U_3) = t$. But this leads to a contradiction since the left-hand side of this inequality is at least $t + 1$ since

$$t+1 = (t-1)+(t-d+1)-(t-d-1) \leq \dim(U_1 \cap U_3)+\dim(U_2 \cap U_3)-\dim(U_1 \cap U_2 \cap U_3).$$

□

The invariant $\dim(A, B)$ may take the value 0 as shown by the following example. Let $A = \{U_1, W_1\}$ be a pair in X. Let α be an isomorphism between U_1 and W_1 and let β be such an isomorphism of W_1 which fixes no non-zero vector in W_1. Define $U_2 = \{u_1 + u_1\alpha \mid u_1 \in U_1\}$ and $W_2 = \{u_1 + u_1\alpha\beta \mid u_1 \in U_1\}$. By transitivity, the pair $B = \{U_2, W_2\}$ is also in X, and, by construction, $\dim(A, B) = 0$.

We now have $\mathrm{diam}(X, G) \geq t = k$ by Lemma 3.8.

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4.

Let G be a primitive permutation group acting on a finite set X such that G has a standard t-action for some positive integer t.

4.1. **Case (b).** Let (X, G) be as in (b) of Definition 1.1. By the beginning of Section 3.2, we may assume that $\dim(U) = k$ where U is a member of X. Let $X \neq \mathcal{S}_t$. In particular, $G_0 \neq \mathrm{PSL}_n(q)$.

We may assume that U is a non-degenerate vector space. If $\dim(U) \geq 3$, then $\mathrm{diam}(X, G) \geq 3$ by Theorem 1.3. Thus, in order to prove Theorem 1.4, we assume that $k = 1$ or $k = 2$.

We first consider the case when $\dim(U) = 1$. Since U is non-degenerate and the case $G_0 = \mathrm{PSL}_n(q)$ was treated earlier, we can assume that G_0 is unitary or orthogonal.

Lemma 4.1. *If G_0 is a unitary group with $n \geq 5$ and X is the set of all non-degenerate 1-spaces, then* $\mathrm{diam}(X, G) = 2$.

Proof. The natural module $V = V_n(q)$ is a unitary space with a non-degenerate unitary form f. Let U be a member of X. This is a non-degenerate subspace of dimension 1. In this case $X = \{\langle v\rangle | f(v, v) = 1\}$. Let

$$S_{\langle v\rangle} = \{\alpha v | f(\alpha v, \alpha v) = 1\} = \{\alpha v | \alpha^{q+1} = 1\}$$

where $\alpha \in \mathbb{F}_{q^2}$. This set has cardinality larger than 1. Let $D = \{\alpha \in \mathbb{F}_{q^2} | \alpha^{q+1} = 1\}$. This is a cyclic group of order $q + 1$. For a scalar $\lambda \in \mathbb{F}_{q^2}$, we define

$$\{\lambda, \overline{\lambda}\}D = \{\lambda\alpha | \alpha \in D\} \cup \{\overline{\lambda}\alpha | \alpha \in D\}$$

and let

$$\Delta_{\lambda D} = \{\{\langle v\rangle, \langle w\rangle\} | f(v, v) = f(w, w) = 1, f(v, w) \in \{\lambda, \overline{\lambda}\}D\}.$$

The set $\Delta_{\lambda D}$ is non-empty for every $\lambda \in \mathbb{F}_{q^2}$ provided that $n \geq 3$. To see this, let $\mathcal{B}$ be a unitary basis for V and let $\chi \in \mathbb{F}_{q^2}$ be such that $\chi + \overline{\chi} = 1$ (χ exists by the surjectivity of the trace map). Take $v = v_\lambda = e_1 + \chi f_1$ where $e_1,\ f_1 \in \mathcal{B}$. If $n \geq 3$ is odd, then $x \in \mathcal{B}$ and we may take $w = w_\lambda = x + \overline{\lambda} f_1$. On the other hand, if $n \geq 4$ is even, then $e_2, f_2 \in \mathcal{B}$ and we take $w = w_\lambda = e_2 + \chi f_2 + \overline{\lambda} f_1$.

Observe that $\Delta_{\lambda D}$ is invariant under G. By applying Theorem 2.1 on 2-spaces of the form $\langle v, w\rangle$ such that $f(v, v) = f(w, w) = 1, f(v, w) \in \{\lambda, \overline{\lambda}\}D$, it follows that $\Delta_{\lambda D}$ is an orbital. For any $u, v \in V$ such that $f(u, u) = f(v, v) = 1$ we have that $f(u, v)$ is either 0 or is in the union of one or two cosets of D, hence all non-diagonal orbitals are of the form $\Delta_{\lambda D}$ for some λ. In particular, the rank of the permutation group G on X is at least 3 implying that $\mathrm{diam}(X, G) \neq 1$.

Let $\lambda,\ \mu \in \mathbb{F}_{q^2}$ such that $\lambda \neq \mu$. Let $\mathcal{B}$ be a unitary basis for V. In this case e_1, f_1, e_2, f_2 are in $\mathcal{B}$. Let $v = e_1 + \chi f_1$ and $w = \mu f_1 + e_2 + \chi f_2$ for some $\chi \in \mathbb{F}_{q^2}$ such that $\chi + \overline{\chi} = 1$. If $n = 5$, then $x \in \mathcal{B}$ and we may take u to be $x + \lambda f_1 + \lambda f_2$. If $n \geq 6$, then $e_3,\ f_3 \in \mathcal{B}$ and we take u to be $f_3 + \chi e_3 + \lambda f_1 + \lambda f_2$. We have $f(v, v) = f(w, w) = f(u, u) = 1$ and $f(v, w) = \overline{\mu}$, $f(u, v) = \lambda$, $f(u, w) = \lambda$. □

Lemma 4.2. *If G_0 is an orthogonal group with $n \geq 5$, q odd and X is an orbit of non-degenerate 1-spaces, then* $\mathrm{diam}(X, G) = 2$.

Proof. The natural module $V = V_n(q)$ is an orthogonal space with a non-degenerate quadratic form Q and associated bilinear form f. For a non-zero vector $v \in V$ such that $Q(v)$ is a square in $\mathbb{F}_q$ there are two vectors u in $\langle v \rangle$ such that $Q(u) = 1$. Let us denote these vectors by v_0 and $-v_0$.

Observe that for each $U \in X$ there exists $u \in U$ such that $Q(u) = 1$. Let U and U' be two distinct vertices in X. Let u be a generator of U and u' a generator of U'. Consider the orbital graph $\mathcal{O} = \{U, U'\}^G$ and the subspace $W = \langle u, u' \rangle = \langle u_0, u_0' \rangle$. Denote $f(u_0, u_0')$ by α. Observe that the conditions $Q(u_0) = Q(u_0') = 1$, $f(u_0, u_0') = \alpha$ uniquely determine the isomorphism type of the space W. It follows by Theorem 2.1 that the graph $\mathcal{O}$ is uniquely determined by the isomorphism type of W. On the other hand, the 2-space W is uniquely determined by $\pm\alpha$. Hence all orbitals are of the form

$$\Delta_{\pm\alpha} = \{\{\langle v \rangle, \langle w \rangle\} | \langle v \rangle \neq \langle w \rangle, Q(v) = Q(w) = 1, f(v, w) = \pm\alpha\}.$$

To prove that the diameter of each associated orbital graph is at most two, we need to show that for any two non-degenerate 1-space, there is an element of x that neighbours them both in $\Delta_{\pm\lambda}$. As all pairs $\{\langle v \rangle, \langle w \rangle\}$ such that $Q(v) = Q(w) = 1$ and $f(v, w) = \alpha$ are in the same orbit, it is sufficient to show that for two specific vectors with $Q(v) = Q(w) = 1$ and $f(v, w) = \alpha$ there is x such that $Q(x) = 1$ and $f(x, w) = \pm\lambda$ and $f(v, x) = \pm\lambda$. Let $v = e_1 + f_1$ and $w = \alpha f_1 + e_2 + f_2$. For $n = 5$ and $(n, \epsilon) = (6, -)$ choosing $x = \lambda f_1 + \lambda f_2 + x$ works and for $n \geq 7$ or $(n, \epsilon) = (6, +)$ choosing $x = \lambda f_1 + \lambda f_2 + e_3 + f_3$ works. $\square$

The following result concerns the case when G_0 is orthogonal, q is even and X is the set of non-singular 1-spaces.

Lemma 4.3. *Let G_0 be orthogonal, with $n \geq 8$ even, q even and X the set of non-singular 1-spaces. Then* $\mathrm{diam}(X, G) = 2$.

Proof. The action of G on X is equivalent to on the set X' of non-singular vectors v such that $Q(v) = 1$. We can see using Witt's Lemma (see Theorem 2.1) that the orbitals are of the form

$$\Delta_\alpha = \{\{a, b\} | a, b \in X', (a, b) = \alpha\}$$

with $\alpha \in \mathbb{F}_q$. Note that $Q(a + b) = (a, b)$. Consider the orbital graph Δ_λ, $\lambda \in \mathbb{F}_q$.

Fix $x_0, v_0 \in X'$. Let $Q(v_0 + x_0) = \sigma \in \mathbf{F}_q$. Assume there is $w_0 \in X'$ such that $Q(v_0 + w_0) = Q(w_0 + x_0) = \lambda$. Let $x, v \in X'$ arbitrary such that $Q(x + v) = \sigma$. Now there exists $g \in G$ such that $(v_0, x_0)^g = (v, x)$, so $Q(v + w_0^g) = Q(w_0^g + x) = \lambda$, so w_0^g is a common neighbour of v and x.

For $\lambda = 0$ choose $v_0 = f_2 + e_2 + e_1$ and $x_0 = f_2 + e_2 + \sigma f_1$. Then for $w_0 = e_2 + f_2$ we have $Q(v_0 + w_0) = Q(w_0 + x_0) = \lambda$, as required. For $\lambda \neq 0$, choose $v_0 = e_1 + e_2 + e_3 + f_3$ and $x_0 = (\sigma - \lambda) f_1 + e_3 + f_3 + \lambda f_2$. Then for $w_0 = e_2 + e_3 + f_3 + \lambda f_1$ we have $Q(v_0 + w_0) = Q(w_0 + x_0) = \lambda$, as required. $\square$

Now consider the case when $\dim(U) = 2$ where $U \in X$.

Lemma 4.4. *Let $n \geq 4$. Let G_0 be $\mathrm{PSp}_n(q)$, $\mathrm{PSU}_n(q)$ or $\mathrm{P\Omega}_n^\epsilon(q)$ provided that, in the latter case, $(n, \epsilon) \neq (4, -)$. Let X be the set of those non-degenerate subspace* *of V which have dimension 2. Furthermore, assume that each subspace in X is of type O_2^+ when $G_0 = \mathrm{P}\Omega_n^\epsilon(q)$. Then* $\mathrm{diam}(X, G) \geq 3$.

Proof. It is sufficient to exhibit an orbital graph with diameter at least 3. Let $\mathcal{B}$ be a symplectic, a unitary or a standard basis and let e_1, e_2, f_1, f_2 as defined in Section 2. Let $U = \langle e_1, f_1 \rangle$ and $U' = \langle e_1 + e_2, f_1 \rangle$. These are vertices in X with intersection $\langle f_1 \rangle$, a singular space of dimension 1. Consider the orbital graph $\mathcal{O} = \{U, U'\}^G$. Let $U'' = \langle e_2, f_2 \rangle$. We will show that $d(U, U'') \geq 3$. Clearly, U and U'' are not adjacent in $\mathcal{O}$. Assume for a contradiction that there is a vertex W in X such that W is adjacent to both U and U''. In this case $U \cap W = \langle u \rangle$ and $U'' \cap W = \langle v \rangle$ are singular 1-spaces where u and v are non-zero vectors in V. Since $U \cap U'' = 0$, the 1-spaces $\langle u \rangle$ and $\langle v \rangle$ are distinct and so $W = \langle u, v \rangle$. On the other hand, since U and U'' are perpendicular, u is orthogonal to v, and so W is totally singular. This is a contradiction to the fact that W is non-degenerate. □

We continue this section by proving that for $G_0 = \mathrm{P}\Omega_n^\epsilon(q)$ such that $q \equiv 1 \mod 4$ and X is of type O_2^-, the orbital diameter is at least 3.

Lemma 4.5. *Let $n \geq 7$. Let G_0 be $\mathrm{P}\Omega_n^\epsilon(q)$ such that $q \equiv 1 \mod 4$ and X be the set of subspaces of type O_2^-. Then* $\mathrm{diam}(X, G) \geq 3$.

Proof. It is sufficient to exhibit an orbital graph with diameter at least 3. Let e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4 and x as defined in Section 2, and let $\zeta \in \mathbb{F}_q$ such that $x^2 + x + 1$ is irreducible. Let $U = \langle e_1 + f_1, e_1 + \zeta e_2 + f_2 \rangle$ and $U' = \langle e_1 + f_1, e_1 + \zeta e_3 + f_3 \rangle$. Since both U and U' are spanned by two vectors whose quadratic forms are 1 and ζ, respectively, and $(e_1 + f_1, e_1 + \zeta e_2 + f_2) = (e_1 + f_1, e_1 + \zeta e_3 + f_3) = 1$, they are of type O_2^-. These are vertices in X with intersection $\langle e_1 + f_1 \rangle$, a 1-space containing a vector with quadratic form equal to 1. Consider the orbital graph $\mathcal{O} = \{U, U'\}^G$. Let $U'' = \langle \zeta e_3 + f_3, e_3 + v \rangle$, where $v = x$ for $n = 7$ and $\mathrm{P}\Omega_8^-(q)$ and $v = e_4 + f_4$ otherwise. We will show that $d(U, U'') \geq 3$. Clearly, U and U'' are not adjacent in $\mathcal{O}$. Assume for a contradiction that there is a vertex W in X such that W is adjacent to both U and U''. In this case, without loss of generality, $U \cap W = \langle u \rangle$ and $U'' \cap W = \langle w \rangle$ where $Q(u) = 1$ and $Q(w) = 1$. Since $U \cap U'' = 0$, the 1-spaces $\langle u \rangle$ and $\langle w \rangle$ are distinct and so $W = \langle u, w \rangle$. Let $\sigma \in \mathbb{F}_q$ such that $\sigma^2 = -1$. Since $q \equiv 1 \mod 4$, we know that that -1 is a square, so such σ exists. We have that $u + \sigma w \in W$ and since since U and U'' are perpendicular, $f(u, w) = 0$ and so we have $Q(u + \sigma w) = 1 + \sigma^2 = 0$, so W is of type O_2^+, as a 2-space of type O_2^- does not contain a totally singular 1-space. This is a contradiction. □

We finish the proof of Theorem 1.4 by considering pairs (X, G) as in (c) of Definition 1.1. We show that in this case the orbital diameter is at least 3.

4.2. **Case (c).** Let (X, G) be as in (c) of Definition 1.1. In this case $G_0 = \mathrm{PSL}_n(q)$, the group G contains a graph automorphism σ of G_0, and X is an orbit of pairs $\{U, W\}$ of subspaces of $V = V_n(q)$, where either $U \subseteq W$ or $V = U \oplus W$, and $\dim U = t$, $\dim W = n - t$. We may suppose that $k = t$. The possibilities are $t = 1$ or $t = 2$ by Theorem 1.3. Let $n \geq 8$. The automorphism σ defines a bijection between the set of t-dimensional subspaces of V to the set of $(n - t)$-dimensional subspaces of V. Let $\{e_1, \ldots, e_n\}$ be a basis for V. Let $U = \langle e_1, \ldots, e_t \rangle$. Let W_1 be the subspace $\langle e_1, \ldots, e_{n-t}\rangle$ in the first case and let W_1 be the subspace $\langle e_{t+1}, \ldots, e_n\rangle$ in the second case. Let W_2 be the subspace $\langle e_1, \ldots, e_{n-t-1}, e_{n-t+1}\rangle$ in the first case and let W_2 be the subspace $\langle e_{t+1}+e_1, e_{t+2}, \ldots, e_n\rangle$ in the second case. Consider the orbital graph $\Gamma = \{\{U, W_1\}, \{U, W_2\}\}^G$. Observe that if $\{\{U_1, W_3\}, \{U_2, W_4\}\}$ is an edge in Γ, then $\{U_1, W_3\} \cap \{U_2, W_4\} \neq \emptyset$. Let $\pi \in G_0$ be the element which permutes the basis vectors in such a way that e_1 is taken to e_2, e_2 is taken to e_n, e_n is taken to e_1 and all other basis vectors are fixed. It is easy to see that $\{U^\pi, W_1^\pi\} \in X$. The pair $\{\{U, W_1\}, \{U^\pi, W_1^\pi\}\}$ is not an edge in Γ since $\{U, W_1\} \cap \{U^\pi, W_1^\pi\} = \emptyset$. Assume for a contradiction that the diameter of Γ is 2. Let $\{U', W'\} \in X$ be a vertex adjacent to both $\{U, W_1\}$ and $\{U^\pi, W_1^\pi\}$ with $\dim U' = t$ and $\dim W' = n-t$. There are two cases: (i) $U' = U$ and $W' = W_1^\pi$ or (ii) $W' = W_1$ and $U' = U^\pi$. In both cases we arrive to the contradiction that $\{U', W'\} \notin X$.

References

[1] Cameron, P. J. Finite permutation groups and finite simple groups. *Bull. London Math. Soc.* **13** (1981), no. 1, 1–22.

[2] Cameron, P. J. Permutation groups. London Math. Soc. Stud. Texts, 45 Cambridge University Press, Cambridge, 1999.

[3] Higman, D. G. Intersection matrices for finite permutation groups. *J. Algebra* **6** (1967) 22–42.

[4] Inglis, N. F. J. The embedding $\mathrm{O}(2m, 2^k) \leq \mathrm{Sp}(2m, 2^k)$. *Arch. Math. (Basel)* **54** (1990), no. 4, 327–330.

[5] Kleidman, P. B.; Liebeck, M. W. The subgroup structure of the finite classical groups, LMS Lecture Note Ser. 129, Cambridge Univ. Press, Cambridge 1990.

[6] Liebeck, M. W.; Macpherson, D.; Tent, K. Primitive permutation groups of bounded orbital diameter. *Proc. Lond. Math. Soc.* (3) **100** (2010), no. 1, 216–248.

[7] Maróti, A., Skresanov, S. Bounds for the diameters of orbital graphs of affine groups. *Vietnam J. Math* **51** (2023), no. 3, 617–631.

[8] Maróti, A., Skresanov, S. On the orders of composition factors in completely reducible groups. *J. Algebra* Vol. **645** (2024), 183–196.

[9] Sheikh, A. Orbital diameters of primitive permutation groups. PhD thesis, Imperial College London, 2017.

[10] Sheikh, A. Orbital diameters of the symmetric and alternating groups. *J. Algebraic Combin.* **45** (2017), no. 1, 1–32.

[11] Skresanov, S. On a polynomial bound for the orbital diameter of primitive affine groups. *Comm. Algebra* **53** (2025), no. 4, 1412–1415.

[12] Rekvényi, K. On the orbital diameter of groups of diagonal type. *J. Combin. Theory Ser. A* **190** (2022), no. 105636, 26 pp.

[13] Rekvényi, K. On the orbital diameter of primitive affine groups. *Comm. Algebra* **53** (2025), no. 9, 3662–3706.

[14] Wilson, R. A. The finite simple groups. Graduate Texts in Mathematics, 251. Springer-Verlag London, Ltd., London, 2009.

Hun-Ren Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, H-1053, Budapest, Hungary

Email address: maroti.attila@renyi.hu

Department of Mathematics, University of Manchester, Manchester, M13 9PL, United Kingdom. Also affiliated with: Heilbronn Institute for Mathematical Research, Bristol, BS8 1UG, United Kingdom.

Email address: kamilla.rekvenyi@manchester.ac.uk