CONJUGACY CLASSES OF π-ELEMENTS AND NILPOTENT/ABELIAN HALL π-SUBGROUPS

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Abstract. Let G be a finite group and π be a set of primes. We study finite groups with a large number of conjugacy classes of π-elements. In particular, we obtain precise lower bounds for this number in terms of the π-part of the order of G to ensure the existence of a nilpotent or abelian Hall π-subgroup in G.

1. Introduction

Let G be a finite group. The number $k(G)$ of conjugacy classes of G is an important and much investigated invariant in group theory. It is equal to the number of complex irreducible representations of G. The probability $\mathrm{Pr}(G)$ that two uniformly and randomly chosen elements from G commute is given by $k(G)/|G|$ where $|G|$ denotes the order of G. This is called the commuting probability or the commutativity degree of G and it has a large literature, see [Gus73, Neu89, Les01, GR06, Ebe15] and references therein. The commuting probability has also been studied for infinite groups, see [Toi20].

A starting point of our work is a much cited theorem of Gustafson [Gus73] stating that $\mathrm{Pr}(G) > 5/8$ for a finite group G if and only if it is abelian. Let p be the smallest prime divisor of the order of a finite group G. It was observed by Guralnick and Robinson [GR06, Lemma 2] that if $\mathrm{Pr}(G) > 1/p$, then G is nilpotent. Moreover, Burness, Guralnick, Moretó and Navarro [BGMN22, Lemma 4.2] recently showed that if $\mathrm{Pr}(G) > \frac{p^2+p-1}{p^3}$, then G is abelian. An aim of this paper is to give a generalization of all three of these results.

Let π be a set of primes. A positive integer is called a π-number if it is not divisible by any prime outside π. The π-part n_π of a positive integer n is the largest π-number which divides n. An element of a finite group is called a π-element if its order is a π-number. The set of all π-elements in a finite group is a union of conjugacy classes of the group. Let $k_\pi(G)$ be the number of conjugacy classes of π-elements in a finite group G and let

$$d_\pi(G) := k_\pi(G)/|G|_\pi.$$

This invariant is always at most 1 by an old result of Robinson, see [MNR21, Lemma 3.5]. The main theorem of the paper [MN14] is that if $d_\pi(G) > 5/8$ for a finite group G and a

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set of primes π, then G possesses an abelian Hall π-subgroup. The following result is a far reaching generalization of this statement.

Theorem 1.1. *Let G be a finite group and let π be a set of primes. Let p be the smallest member of π. If $\mathrm{d}_\pi(G) > \frac{1}{p}$, then G has a nilpotent Hall π-subgroup, whose derived subgroup has size at most p. Moreover, if $\mathrm{d}_\pi(G) > \frac{p^2+p-1}{p^3}$, then G has an abelian Hall π-subgroup.*

A well-known theorem of Wielandt [Wie54] states that if a finite group G contains a nilpotent Hall π-subgroup for some set of primes π then all Hall π-subgroups of G are conjugate and every π-subgroup of G is contained in a Hall π-subgroup. Therefore, the π-subgroups of a group satisfying the hypothesis of Theorem 1.1 behave like Sylow subgroups.

There are several results in the literature on the existence of abelian or nilpotent Hall subgroups in finite groups. For example [BFMMNSST16, Theorem B] states that if G is a finite group and π a set of primes, then G has nilpotent Hall π-subgroups if and only if for every pair of distinct primes p, q in π the class sizes of the p-elements of G are not divisible by q.

For certain sets π, Tong-Viet [TV20] proved some nice results on the existence of normal π-complements in finite groups G under the condition that $\mathrm{d}_\pi(G)$ is large. For example, [TV20, Theorem E] states that if $p > 2$ is the smallest prime in π and $d_\pi(G) > (p+1)/2p$, then G contains not only an abelian Hall π-subgroup but also a normal π-complement. Another is [TV20, Theorem A], which states that if $d_2(G) > 1/2$ then G has a normal 2-complement. We in fact make use of this result to prove Theorem 1.1 in the case $2 \in \pi$. As a consequence, the proof for this case does not depend on the classification of finite simple groups. The other case $2 \notin \pi$, however, is more challenging and our proof has to rely on the classification.

The paper is organized as follows. In Section 2 we prove some preliminary results on the commuting probability $\mathrm{Pr}(G)$. In Section 3 we prove some basic properties of the π-class invariant $\mathrm{d}_\pi(G)$ and, in particular, we show in Theorem 3.4 that in order to prove the main result, it suffices to show the existence of a nilpotent Hall π-subgroup under the hypothesis $\mathrm{d}_\pi(G) > \frac{1}{p}$. We then establish this statement in Section 4, modulo a result on finite simple groups (Theorem 4.9) that will be proved in Section 5. Finally, in Section 6, we present examples showing that the converse of Theorem 1.1 is false and that the obtained bounds are sharp in general.

2. Commuting probability

In this section we recall and prove some results about the commuting probability $\mathrm{Pr}(G)$ that will be needed later.

The first lemma is a generalisation of Gustafson's result [Gus73] mentioned earlier. The inequality part is due to Burness, Guralnick, Moretó, and Navarro [BGMN22].

Lemma 2.1. *Let G be a finite group and p the smallest prime dividing $|G|$. If G is not abelian, then $\mathrm{Pr}(G) \leq \frac{p^2+p-1}{p^3}$ with equality if and only if $G/\mathbf{Z}(G) = C_p \times C_p$.*

Proof. The first part of the lemma is [BGMN22, Lemma 4.2]. Following its proof, we see that the equality $\mathrm{Pr}(G) = \frac{p^2+p-1}{p^3}$ holds if and only if $G/\mathbf{Z}(G) = C_p \times C_p$ and $|x^G| = p$ for every $x \in G \setminus \mathbf{Z}(G)$. It suffices to prove that if $G/\mathbf{Z}(G) = C_p \times C_p$, then $|x^G| = p$ for every $x \in G \setminus \mathbf{Z}(G)$.

Assume that $G/\mathbf{Z}(G) = C_p \times C_p$ and let $x \in G \setminus \mathbf{Z}(G)$. Since $x \in \mathbf{C}_G(x) \setminus Z(G)$, we have that $\mathbf{Z}(G) < \mathbf{C}_G(x)$. Therefore, $|x^G| = \frac{|G|}{|C_G(x)|}$ is a proper divisor of $\frac{|G|}{|\mathbf{Z}(G)|} = p^2$. On the other hand, since x is not central, $|x^G| > 1$. Thus, $|x^G| = p$, and the claim follows. □

Note that if G is an extraspecial p-group of order p^3 with p odd or if G is a dihedral group when $p = 2$, then $G/\mathbf{Z}(G) = C_p \times C_p$. Therefore, the bound in Lemma 2.1 is sharp for all p.

We next give a bound for $\mathrm{Pr}(G)$ in terms of the smallest prime factor of the order of G and the order of its derived subgroup G'.

Lemma 2.2. *If p is the smallest prime dividing the order of a finite group G, then*

$$\mathrm{Pr}(G) \leq \frac{1 + (p^2 - 1)/|G'|}{p^2}.$$

Proof. Let $\mathrm{Irr}(G)$ denote the set of all irreducible complex characters of G. We have

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \geq |G/G'| + p^2(k(G) - |G/G'|),$$

since $\chi(1)$ divides $|G|$ for every $\chi \in \mathrm{Irr}(G)$. After dividing both sides of the previous inequality by $|G|$, we obtain $1 \geq 1/|G'| + p^2(\mathrm{Pr}(G) - 1/|G'|)$. This yields $\mathrm{Pr}(G) \leq (1 + (p^2 - 1)/|G'|)/p^2$, as we claimed. □

Lemma 2.3. *Let G be a finite group and p the smallest prime dividing $|G|$. Suppose that $|G'| \leq p$. Then $G' \leq \mathbf{Z}(G)$, and thus $G/\mathbf{Z}(G)$ is abelian. In particular, G is nilpotent.*

Proof. The case $|G'| = 1$ is obvious, so we assume $|G'| = p$. Since G' is normal and its order is the smallest prime dividing $|G|$, we deduce that G' is central in G, and the result follows. □

Next we refine Lemma 2.1. It follows from [GR06, Lemma 2(xiii)] of Guralnick and Robinson that if $\mathrm{Pr}(G) > \frac{1}{p}$, where p is the smallest prime dividing $|G|$, then G is nilpotent.

Theorem 2.4. *Let G be a finite group and p the smallest prime dividing $|G|$. Then $\frac{1}{p} < \mathrm{Pr}(G) \leq \frac{p^2+p-1}{p^3}$ if and only if $|G'| = p$. Moreover, in such case,*

$$\mathrm{Pr}(G) = \frac{1}{p} + \frac{p-1}{p|G : \mathbf{Z}(G)|}.$$

Proof. By Lemma 2.1 we may assume that G is non-abelian. Assume that $|G'| > p$. Then $|G'| \geq p + 1$ and hence, applying Lemma 2.2, we have $\mathrm{Pr}(G) \leq \frac{1}{p}$. The only if part is therefore done.

Conversely, assume that $|G'| = p$. Then $G' \leq \mathbf{Z}(G)$ by Lemma 2.3. By [Isa76, Problem 2.13], we have $\chi(1)^2 = |G : \mathbf{Z}(G)|$ for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$. We deduce that

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|/p + |G : \mathbf{Z}(G)|(k(G) - |G|/p),$$

and it follows that

$$\Pr(G) = \frac{1}{p} + \frac{p-1}{p|G:\mathbf{Z}(G)|} > \frac{1}{p},$$

as stated. □

Remark 2.5. It is worth noting that if $G/\mathbf{Z}(G) \cong C_p \times C_p$, then, by Lemma 2.1, we have $\Pr(G) = \frac{p^2+p-1}{p^3} > \frac{1}{p}$, and hence $|G'| = p$ by Theorem 2.4.

Let us denote

$$g_p(x) := \frac{1 + (p^2-1)/x}{p^2}.$$

We note that the function $g_p(x)$ is decreasing in terms of x. Also, $g_p(1) = 1$, $g_p(p) = \frac{p^2+p-1}{p^3}$, and $g_p(p+1) = \frac{1}{p}$. These values of g_p, that appear in our main result, explain the relevance of g_p.

The next theorem could be compared with a result of Lescot [Les01] stating that $\Pr(G) = \frac{1}{2}$ if and only if G is isoclinic to the symmetric group Σ_3.

Theorem 2.6. *Let G be a finite group and p the smallest prime dividing $|G|$. If $|G'| > p$, then*

$$\Pr(G) \leq \frac{n(p) + p^2 - 1}{p^2 n(p)} \leq \frac{1}{p},$$

where $n(p)$ denotes the smallest prime larger than p. Moreover, $\Pr(G) = \frac{1}{p}$ *if and only if $p = 2$ and $G/\mathbf{Z}(G) \cong \Sigma_3$.*

Proof. By Bertrand's postulate, we know that $n(p) < 2p \leq p^2$. Therefore, if $|G'| > p$ then $|G'| \geq n(p)$ and hence, applying Lemma 2.2, we have

$$\Pr(G) \leq g_p(n(p)) = \frac{n(p) + p^2 - 1}{p^2 n(p)}.$$

The second inequality holds because $g_p(n(p)) \leq g_p(p+1) = \frac{1}{p}$.

Suppose that $\Pr(G) = \frac{1}{p}$. This forces $n(p) = p+1$, which implies that $p = 2$ and $|G'| = 3$. We claim that $\Pr(G) = \frac{1}{2}$ if and only if $G/\mathbf{Z}(G) = \Sigma_3$. Assume first that $G/\mathbf{Z}(G) = \Sigma_3$. Let q be a prime dividing $|G|$ and let $Q \in \mathrm{Syl}_q(G)$. Since $G/\mathbf{Z}(G) = \Sigma_3$, we deduce that $|Q : \mathbf{Z}(Q)| \leq q$ and hence Q is abelian. It follows that G possesses an abelian Sylow q-subgroup for every prime q dividing $|G|$. Thus, by [GR06, Lemma 2(xiii)], we have

$$\Pr(G) = \Pr(G/\mathbf{Z}(G)) = \Pr(\Sigma_3) = \frac{1}{2}.$$

The other direction of the claim follows from the above-mentioned theorem of Lescot [Les01] since if G is isoclinic to Σ_3, then $G/\mathbf{Z}(G) = \Sigma_3$. □

3. Hall π-subgroups

In this section we prove that the second statement of Theorem 1.1 follows from the first.

Let $\mathcal{D}_\pi$ be the collection of all finite groups G such that G has a Hall π-subgroup, any two Hall π-subgroups of G are conjugate, and any π-subgroup of G is contained in a Hall π-subgroup. Of course $\mathcal{D}_\pi$ is everything when π is a single prime by Sylow's theorems. Also, $\mathcal{D}_\pi$ contains all π-separable groups. The following easy observation is useful to bound $\mathrm{d}_\pi(G)$ in the case $G \in \mathcal{D}_\pi$.

Lemma 3.1. *Let G be a finite group in $\mathcal{D}_\pi$. If H is a Hall π-subgroup of G, then*

$$\mathrm{d}_\pi(G) \leq \mathrm{Pr}(H).$$

Proof. Since $|H| = |G|_\pi$, it suffices to see that $k_\pi(G) \leq k(H)$. If $x, y \in H$ are not conjugate in G, then they cannot be conjugate in H. Since $G \in \mathcal{D}_\pi$, every G-class of π-elements has a representative in H. □

From this, we can easily prove Theorem 1.1 in case $G \in \mathcal{D}_\pi$.

Theorem 3.2. *Let π be a set of primes and G a finite group in $\mathcal{D}_\pi$. Then Theorem 1.1 holds for G.*

Proof. By hypothesis, G has a Hall π-subgroup H and all the Hall π-subgroups of G are G-conjugates of H. Thus, by Lemma 3.1, we have $\mathrm{d}_\pi(G) \leq \mathrm{Pr}(H)$. Let p be the smallest prime in π. Assume that $\mathrm{d}_\pi(G) > \frac{1}{p}$. We then have

$$\mathrm{Pr}(H) > \frac{1}{p}.$$

Theorem 2.4 and Lemma 2.3 then imply that $|H'| \leq p$ and H is nilpotent, as claimed. Moreover, if $\mathrm{d}_\pi(G) > \frac{p^2+p-1}{p^3}$ then H is abelian by Lemma 2.1. □

As a consequence of Theorem 3.2, we have that Theorem 1.1 holds if $\pi = \{p\}$ or if G is π-separable.

We also recall some facts on the groups in $\mathcal{D}_\pi$. The first one is a result of Wielandt [Wie54] mentioned in the Introduction and the second one is due to Hall [Hal56, Theorem D5].

Lemma 3.3. *Let G be a finite group and π a set of primes.*

(i) *If G possesses a nilpotent Hall π-subgroup, then $G \in \mathcal{D}_\pi$.*
(ii) *If N possesses nilpotent Hall π-subgroups, G/N possesses solvable Hall π-subgroups, and $G/N \in \mathcal{D}_\pi$, then $G \in \mathcal{D}_\pi$.*

Theorem 3.4. *The second statement of Theorem 1.1 follows from the first.*

Proof. Let G be a group with $\mathrm{d}_\pi(G) > \frac{p^2+p-1}{p^3} > \frac{1}{p}$. By hypothesis, G possesses a nilpotent Hall π-subgroup. It then follows that $G \in \mathcal{D}_\pi$ by Lemma 3.3. The result follows by Theorem 3.2. □

The rest of the paper is therefore devoted to prove that G has a nilpotent Hall π-subgroup under the condition $\mathrm{d}_\pi(G) > \frac{1}{p}$.

4. Reducing to a problem on simple groups

In this section we prove Theorem 1.1, assuming a result on finite simple groups.

4.1. **Reducing to simple groups.** We begin by recalling two properties of $\mathrm{d}_\pi(G)$. The first one is [MN14, Proposition 5], essentially due to Robinson. The second is due to Fulman and Guralnick [FG12, Lemma 2.3].

Lemma 4.1. *Let G be a finite group and π a set of primes.*

(i) *Let $\mu \subseteq \pi$. Then $\mathrm{d}_\pi(G) \leq \mathrm{d}_\mu(G)$.*
(ii) *$\mathrm{d}_\pi(G) \leq \mathrm{d}_\pi(N)\,\mathrm{d}_\pi(G/N)$ for any normal subgroup N of G.*

Lemma 4.2. *Let G be a finite group, π a set of primes, and p the smallest prime in π. Let $q \in \pi$ and $Q \in \mathrm{Syl}_q(G)$. Suppose $\mathrm{d}_\pi(G) > \frac{1}{p}$. We have*

(i) *$Q/\mathbf{Z}(Q)$ is abelian and $|Q'| \leq q$.*
(ii) *If $q \in \pi \setminus \{p\}$, then Q is abelian.*

Proof. By Sylow's theorems and Lemma 3.1 we have $\mathrm{d}_q(G) \leq \mathrm{Pr}(Q)$. On the other hand, by Lemma 4.1(i), we have $\mathrm{d}_\pi(G) \leq \mathrm{d}_q(G)$. We deduce that

$$\frac{1}{q} \leq \frac{1}{p} < \mathrm{Pr}(Q).$$

Theorem 2.4 and Lemma 2.3 now imply that $Q/\mathbf{Z}(Q)$ is abelian and $|Q'| \leq q$.

Suppose $q > p$. Then $q \geq p+1$, and one can easily check that $\frac{q^2+q-1}{q^3} < \frac{1}{p}$. Now $\mathrm{Pr}(Q) > \frac{q^2+q-1}{q^3}$, and thus Q must be abelian by Lemma 2.1. □

The next lemma is [Mor13, Lemma 3.1], which allows us to work with a set of two primes instead of an arbitrary set.

Lemma 4.3 (Moretó). *Let G be a finite group and let π a set of primes. If G possesses a nilpotent Hall τ-subgroup for every $\tau \subseteq \pi$ with $|\tau| = 2$, then G possesses a nilpotent Hall π-subgroup.*

Proposition 4.4. *Suppose that Theorem 1.1 is false for a group G. Then there exists $\pi = \{p, q\}$, where $p < q$ are two primes, such that G does not possess nilpotent Hall π-subgroups and for all $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $P/\mathbf{Z}(P)$ is abelian, $|P'| \leq p$, and Q is abelian.*

Proof. By Theorem 3.4, we may assume that there exists π, a set of primes, such that $\mathrm{d}_\pi(G) > \frac{1}{p}$, but G does not possess nilpotent Hall π-subgroups, where p is the smallest member of π.

If G has a nilpotent Hall τ-subgroup for every $\tau \subseteq \pi$ with $|\tau| = 2$, then by Lemma 4.3, G has nilpotent Hall π-subgroups. Thus, there exists $\{q, r\} \subseteq \pi$ with $q < r$ such that G does not possess a nilpotent Hall $\{q, r\}$-subgroup. By Lemma 4.1(i), we also have $\mathrm{d}_\pi(G) \leq \mathrm{d}_{\{q,r\}}(G)$, and it follows that

$$\frac{1}{q} \leq \frac{1}{p} < \mathrm{d}_\pi(G) \leq \mathrm{d}_{\{q,r\}}(G).$$

Therefore, Theorem 1.1 fails for G and the set $\{q, r\}$, and hence we may assume that $|\pi| = 2$, that is $\pi = \{p, q\}$ with $p < q$.

Finally, the assertion on the Sylow subgroups follows from Lemma 4.2. □

Proposition 4.5. *Let G be a finite group with minimal order subject to the conditions that $\mathrm{d}_\pi(G) > \frac{1}{p}$ and G does not possess nilpotent Hall π-subgroups. Then G is non-abelian simple.*

Proof. We may assume that G is non-abelian and not simple. Let N be a nontrivial proper normal subgroup in G. By Lemma 4.1(ii), we have

$$\frac{1}{p} < \mathrm{d}_\pi(G) \leq \mathrm{d}_\pi(G/N)\,\mathrm{d}_\pi(N).$$

It follows that $\frac{1}{p} < \mathrm{d}_\pi(G/N)$ and $\frac{1}{p} < \mathrm{d}_\pi(N)$, as both $d_\pi(N)$ and $d_\pi(G/N)$ are at most one (see [MNR21, Lemma 3.5]). By the minimality of G, N and G/N possess nilpotent Hall π-subgroups. Applying Lemma 3.3, we then deduce that both N and G/N are members of $\mathcal{D}_\pi$. It follows that $G/N \in \mathcal{D}_\pi$, G/N possesses solvable Hall π-subgroups and N possesses nilpotent Hall π-subgroups. By Lemma 3.3(ii), we have $G \in \mathcal{D}_\pi$. Therefore, by Theorem 3.2, we have that G possesses nilpotent Hall π-subgroups, which is a contradiction. We conclude that G is non-abelian simple. □

4.2. **Reducing to a question on simple groups.** The following is a consequence of a result of Tong-Viet, which asserts that if $\mathrm{d}_2(G) > \frac{1}{2}$ then G possesses a normal 2-complement.

Lemma 4.6. *Let S be a non-abelian simple group and π be a set of primes containing 2. Then $\mathrm{d}_\pi(S) \leq \frac{1}{2}$.*

Proof. Suppose that $\mathrm{d}_\pi(S) > \frac{1}{2}$. Then $\frac{1}{2} < \mathrm{d}_\pi(S) \leq \mathrm{d}_2(S)$. By [TV20, Theorem A], S possesses a normal 2-complement, which is impossible. □

Proposition 4.7. *Let G be a group and π a set of primes such that $\mathrm{d}_\pi(G) > \frac{1}{p}$, where p is the smallest prime in π. Let $q \in \pi$ but $q \neq p$. Then q does not divide $|\mathbf{N}_G(P) : \mathbf{C}_G(P)|$ where $P \in \mathrm{Syl}_p(G)$.*

Proof. Assume by contradiction that q divides $|\mathbf{N}_G(P)/\mathbf{C}_G(P)|$. Let x be an element of order q in $\mathbf{N}_G(P)/\mathbf{C}_G(P)$ where $P \in \mathrm{Syl}_p(G)$. Consider the action of $X = \langle x\rangle$ on P. Let r be the number of elements of P fixed by X.

We claim that $r > \frac{|P|}{p^2}$. Assume to the contrary that $r \leq \frac{|P|}{p^2}$. We have $|P| = r + t \cdot q$, implying that $t = \frac{|P|-r}{q}$. Since each X-orbit on P is contained in a conjugacy class of p-elements it is easy to see that $k_p(G) \leq r + t$. Now we have

$$\frac{1}{p} < \mathrm{d}_\pi(G) \leq \mathrm{d}_p(G) = \frac{k_p(G)}{|P|} \leq \frac{r+t}{|P|} = \frac{1}{q}\left((q-1)\frac{r}{|P|} + 1\right) \leq \frac{1}{q}\left((q-1)\frac{1}{p^2} + 1\right).$$

It is not hard to see that this implies $q \leq p$, which is a contradiction. We have shown that $r > \frac{|P|}{p^2}$.

Since r divides $|P|$, it follows that

$$r \in \{|P|, |P|/p\}.$$

If $r = |P|$ then X centralises P, which is impossible. Thus $r = |P|/p$ and hence there exists a subgroup H of order $|P|/p$ that is centralised by X. That is,

$$H = \mathbf{C}_P(X) = \{z \in P \mid z^x = z \text{ for all } x \in X\}.$$

Let $L := X \ltimes P$. Then $L/H \cong X \ltimes C$ for some $C \cong C_p$. Since H is maximal in P, the subgroup H is normal in P, and it is X-invariant, applying [Isa08, Corollary 3.28], we have

$$\mathbf{C}_{P/H}(X) = \mathbf{C}_P(X)H/H = H/H,$$

and hence X acts nontrivially on C. Let $\mathcal{O}$ be a nontrivial orbit of the action of X on C. We now have $q = |X| = |\mathcal{O}| \leq |C| = p$, which is a contradiction. □

Corollary 4.8. *Let G be a group and $\pi = \{p, q\}$ a set of primes with $p < q$ such that $\mathrm{d}_\pi(G) > \frac{1}{p}$. Let $P \in \mathrm{Syl}_p(G)$. Then q divides $|\mathrm{Syl}_p(G)| = |G : \mathbf{N}_G(P)|$ or G possesses a nilpotent Hall π-subgroup.*

Proof. We know that $|G|_q$ divides

$$|G| = |G : \mathbf{N}_G(P)||\mathbf{N}_G(P) : \mathbf{C}_G(P)||\mathbf{C}_G(P)|$$

but q cannot divide $|\mathbf{N}_G(P) : \mathbf{C}_G(P)|$ by Proposition 4.7. Assume that q does not divide $|G : \mathbf{N}_G(P)|$. Then $|G|_q$ divides $|\mathbf{C}_G(P)|$. Therefore, there exists $Q \in \mathrm{Syl}_q(G)$ with $Q \leq \mathbf{C}_G(P)$. Now PQ is a nilpotent Hall π-subgroup of G. □

Now we can prove Theorem 1.1, modulo the following statement about simple groups whose proof is deferred to the next section.

Theorem 4.9. *Let G be a non-abelian simple group and $\pi = \{p, q\}$ be a set of two odd primes with $p < q$. Assume that there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $P/\mathbf{Z}(P)$ is abelian, $|P'| \leq p$, Q is abelian, and q divides $|G : \mathbf{N}_G(P)|$. Then $\mathrm{d}_\pi(G) \leq \frac{1}{p}$.*

Observe that in Theorem 4.9 we may assume that both p and q divide the order of G.

Theorem 4.10. *Let G be a finite group, π be a set of primes, and p be the smallest prime in π. Assume Theorem 4.9. If $\mathrm{d}_\pi(G) > \frac{1}{p}$ then G has a nilpotent Hall π-subgroup.*

Proof. Assume that the theorem is false and let G be a minimal counterexample. In particular, $\mathrm{d}_\pi(G) > \frac{1}{p}$ but G has no nilpotent Hall π-subgroups. By Proposition 4.5, we know that G is non-abelian simple. Using Lemma 4.6, we know furthermore that $p \neq 2$.

By Proposition 4.4, there exists $\pi = \{p, q\}$ with (odd) $p < q$ such that $\mathrm{d}_\pi(G) > \frac{1}{p}$, $P/\mathbf{Z}(P)$ is abelian, $|P'| \leq p$, and Q is abelian, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. We also have that q divides $|G : \mathbf{N}_G(P)|$, by Corollary 4.8. We now have all the hypotheses of Theorem 4.9, and therefore deduce that $\mathrm{d}_\pi(G) \leq \frac{1}{p}$. This is a contradiction. □

Remark that we have indeed proved Theorem 1.1 when the set π contains the prime 2, and this result does not rely on the classification of finite simple groups.

5. Simple groups

In this section we prove Theorem 4.9, by using the classification. We begin with the alternating groups.

Lemma 5.1. *Let p be an odd prime, $n \geq 5$ be an integer and $P \in \mathrm{Syl}_p(\mathsf{A}_n)$.*

(i) *If $n \geq p^2$, then $P/\mathbf{Z}(P)$ is not abelian.*
(ii) *If $n < p^2$, then P is elementary abelian.*

Proof. For (i) it is sufficient to exhibit a subgroup H of P such that $H/\mathbf{Z}(H)$ is not abelian. If $n \geq p^2$, then $H = C_p \wr C_p$ is such a subgroup of P. Statement (ii) follows from the description of the Sylow p-subgroups of A_n found in [Hup67, Satz III.15.3]. $\square$

Theorem 5.2. *Let $n \geq 5$, $\pi = \{p, q\}$ be a set of two odd primes with $p < q$, and $P \in \mathrm{Syl}_p(\mathsf{A}_n)$. Assume that both p and q divide the order of A_n. If $P/\mathbf{Z}(P)$ is abelian, then $\mathrm{d}_\pi(\mathsf{A}_n) \leq \frac{1}{p}$. In particular, Theorem 4.9 holds for alternating groups.*

Proof. Let $P \in \mathrm{Syl}_p(\mathsf{A}_n)$ and $Q \in \mathrm{Syl}_q(\mathsf{A}_n)$. Since $P/\mathbf{Z}(P)$ is abelian, $n < p^2$ by Lemma 5.1. Let $n = rp + s = lq + t$, where $r, s \in \{0, 1, \ldots, p-1\}$ and $l, t \in \{0, 1, \ldots, q-1\}$. Then $P = (C_p)^r$ and $Q = (C_q)^l$ with both r and l at least 1.

It is easy to see that every π-element of A_n can be expressed as a product of the form $xy = yx$, where x is a product of cycles of length p and y is a product of cycles of length q. Since $n < p^2$, the supports of x and y are disjoint.

Assume first that $n \geq p + q + 2$. In this case we have that $k_p(\mathsf{A}_n) = 1 + r \leq p$, $k_q(\mathsf{A}_n) = 1 + l \leq q$ and $|\mathsf{A}_n|_\pi = p^r q^l$. Thus we have

$$\mathrm{d}_\pi(\mathsf{A}_n) = \frac{k_\pi(\mathsf{A}_n)}{|\mathsf{A}_n|_\pi} \leq \frac{pq}{p^r q^l}.$$

If $(r, l) \neq (1, 1)$, then $\mathrm{d}_\pi(\mathsf{A}_n) \leq \frac{1}{p}$. Assume now that $r = l = 1$. Then $k_\pi(\mathsf{A}_n) \leq k_p(\mathsf{A}_n)k_q(\mathsf{A}_n) = 4$ and hence $\mathrm{d}_\pi(\mathsf{A}_n) \leq \frac{4}{q}\frac{1}{p} < \frac{1}{p}$, where the last inequality holds because $q \geq 5$.

Assume now that $n \leq p + q + 1$ and so $l = 1$. In this case it may happen that a Σ_n-conjugacy class of π-elements splits in two different A_n-conjugacy classes. We thus have $k_\pi(\mathsf{A}_n) \leq (1+r)(1+l) + 1 = 2(1+r) + 1 = 2r + 3$. It follows that $\mathrm{d}_\pi(\mathsf{A}_n) \leq \frac{2r+1}{p^r q}$. If $r \geq 2$, then $\frac{2r+3}{p^r q} < \frac{1}{q} < \frac{1}{p}$. If $r = 1$, then $2r + 3 = 5 \leq q$ and so once again $\mathrm{d}_\pi(\mathsf{A}_n) \leq \frac{1}{p}$. $\square$

For convenience, we will consider the Tits group ${}^2F_4(2)'$ as a sporadic simple group.

Theorem 5.3. *Let S be a sporadic simple group and $\pi = \{p, q\}$ where $p < q$ are odd primes dividing $|S|$. If $(S, \pi) \neq (J_1, \{3, 5\})$ then $\mathrm{d}_\pi(S) \leq \frac{1}{p}$. In particular, Theorem 4.9 holds for S.*

Proof. In what follows we use information in [Atl] without further notice. We may assume that π is a set of primes such that $k_\pi(S) \geq 6$, for otherwise

$$\mathrm{d}_\pi(S) = \frac{k_\pi(S)}{|S|_\pi} \leq \frac{5}{pq} \leq \frac{1}{p}$$

There is no such π for the four smallest Mathieu groups. For each of the groups M_{24}, HS, J_2 there are two possibilities for π. In each of the six cases $k_\pi(S)$ is at most $|S|_p$ or $|S|_q$ and this is sufficient to obtain the bound $\mathrm{d}_\pi(S) \leq \frac{1}{p}$.

So we assume that S is not one of the groups already analyzed. If S is different from Fi_{23}, Fi'_{24} and J_1, then we count the total number of conjugacy classes of S of elements of odd order. These numbers are usually less that $|S|_r$ for a given prime divisor r of $|S|$. If this is the case for a prime r, then we can assume that r does not lie in π (otherwise we would be done). These give strong restrictions on the set π. In fact, given that $k_\pi(S) \geq 6$, we find this way that S must be J_4 and π is either $\{3, 7\}$ or $\{5, 7\}$. In each of these two cases we count the number of π-classes in S to obtain our bound of $\frac{1}{p}$ for $\mathrm{d}_\pi(S)$.

If S is Fi_{23} or Fi'_{24}, then we again count the number of conjugacy classes of S of elements of odd order. This allows us to conclude that 3 cannot lie in π. We then count the number of conjugacy classes of S whose elements have orders divisible neither by 2 nor 3. This number is 8 in the first case and 14 in the second. By looking at the prime factorization of $|S|$, the only case to consider is $S = Fi'_{24}$ and $\pi = \{11, 13\}$. But it turns out that $k_\pi(S) = 3$ in this case.

The only group remaining is $S = J_1$. The number of conjugacy classes of S of elements of odd order is 11 forcing π to be a subset of $\{3, 5, 7\}$. Then $k_\pi(S) = 3$ or $\pi = \{3, 5\}$ and $k_\pi(S) = 6$, giving $\mathrm{d}_\pi(S) = \frac{2}{5}$.

The last assertion follows from the fact that if $P \in \mathrm{Syl}_3(J_1)$, then 5 does not divide $|J_1 : \mathbf{N}_{J_1}(P)|$. ∎

We are left with the case of simple groups of Lie type $S \neq {}^2F_4(2)'$. For the sake of convenience, we rename the prime q in Theorem 4.9 to s in order to reserve q for the size of the underlying field of S.

The proof of Theorem 4.9 for groups of Lie type is divided into two fundamentally different cases: π contains the defining characteristic of S and π does not. The former case is fairly straightforward.

Theorem 5.4. *Let S be a finite simple group of Lie type in characteristic $p > 2$ and $\pi = \{p, s\}$, where s is an odd prime dividing $|S|$. Then,*

$$\mathrm{d}_\pi(S) \leq \frac{1}{s}.$$

In particular, Theorem 4.9 holds for simple groups of Lie type when π contains the defining characteristic of S.

Proof. First we observe that the desired inequality is satisfied if $k_\pi(S) \leq |S|_p$. We shall make use of well-known bounds of Fulman and Guralnick [FG12] for the numbers of conjugacy classes of finite Chevalley groups to show that, when S has high enough rank, even the stronger inequality $k(S) \leq |S|_p$ holds true.

Let $S = \mathrm{PSL}(n, q)$. Then $k(S) \leq \min\{2.5q^{n-1}, q^{n-1} + 3q^{n-2}\}$ by [FG12, Proposition 3.6]. This is certainly smaller than $|S|_p = q^{n(n-1)/2}$ if $n \geq 4$. Therefore, we just need to verify the theorem for $n = 2$ or 3. The theorem is in fact straightforward to verify for these low rank cases, using the known information on conjugacy classes of the group (see [Dor71, Chapter 38] for $n = 2$ and [SF73] for $n = 3$). The case $S = \mathrm{PSU}(n, q)$ is entirely similar.

Next, we consider $\mathrm{PSp}(2n,q)$ with $n \geq 3$. Then $k(S) \leq 10.8q^n$ for odd q, and it easily follows that $k(S) \leq |S|_p = q^{n^2}$. The case of orthogonal groups is similar, with a remark that $k(\Omega(2n+1,q)) \leq 7.3q^n$ for $n \geq 2$ and $k(\mathrm{P}\Omega^{\pm}(2n,q)) \leq 6.8q^n$ for $n \geq 4$.

Now we turn to exceptional groups. Recall that the defining characteristic p of S is odd, so we will exclude the types ${}^2B_2$ and ${}^2F_4$. By [FG12, Table 1] (or [Lüb] for more details), we observe that $k(S)$ is bounded above by a polynomial with positive coefficients, say g_S, evaluated at q. Suppose S is one of ${}^3D_4(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$. We then have

$$k(S) \leq g_S(1)q^{\deg(g_S)} \leq 252q^{\deg(g_S)} \text{ and } \frac{q^{\deg(g_S)}}{|S|_p} \leq \frac{1}{q^8}.$$

Therefore,

$$\mathrm{d}_\pi(S) \leq \frac{k(S)}{|S|_p|S|_s} \leq \frac{252}{sq^8} < \frac{1}{s},$$

as wanted. The remaining cases of the types G_2 and ${}^2G_2$ are even easier, using the more refined bounds $k(G_2(q)) \leq q^2+2q+9$ and $k({}^2G_2(q)) \leq q+8$. □

Lemma 5.5. *Let G be a finite group and let π be a set of primes such that $|\mathbf{Z}(G)|_\pi = 1$. Then, $k_\pi(G) = k_\pi(G/\mathbf{Z}(G))$.*

Proof. Let $Z := \mathbf{Z}(G)$. Every coset gZ of Z in G contains at most one π-element of G since $|Z|_\pi = 1$. The π-elements of G/Z are gZ where g runs through the π-elements of G. If g is a π-element, then the conjugacy class of gZ in G/Z consists of hZ where $h \in g^G$. Thus, there is a bijection between the π-conjugacy classes of G and the π-conjugacy classes of G/Z. □

In the case when π does not contain the defining characteristic of S, the conjugacy classes of π-elements of S will be semisimple classes, which can be conveniently described via an ambient algebraic group of S and its Weyl group.

It is well-known that every simple group of Lie type $S \neq {}^2F_4(2)'$ is of the form $S = \mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ for some simple algebraic group $\mathbf{G}$ of simply connected type and a suitable Steinberg endomorphism F on $\mathbf{G}$, see [MT11, Theorem 24.17] for instance.

Theorem 5.6. *Let S be a finite simple group of Lie type and $\mathbf{G}, F$ as above. Let $\pi = \{p, s\}$ with $p < s$ be a set of primes not containing the defining characteristic of S. Suppose that s divides $|\mathrm{Syl}_p(S)|$. Then*

$$\mathrm{d}_\pi(\boldsymbol{G}^F) \leq \frac{1}{p}.$$

In particular, if $|\mathbf{Z}(\boldsymbol{G}^F)|_\pi = 1$, then $\mathrm{d}_\pi(S) \leq \frac{1}{p}$.

Proof. Let $G := \mathbf{G}^F$. We first claim that a Hall π-subgroup of G, if exists, cannot be abelian. Assume by contradiction that G does have such subgroup, say H. Then $\overline{H} := H\mathbf{Z}(G)/\mathbf{Z}(G)$ would be an abelian Hall π-subgroup of S, implying that $\mathbf{N}_S(P)$ contains $\overline{H}$, where P is a Sylow p-subgroup of S that is contained in $\overline{H}$. It follows that s does not divide $|S : \mathbf{N}_S(P)|$, violating the hypothesis.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, and let $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the Weyl group of $\mathbf{G}$. Since π does not contain the defining characteristic of S, the conjugacy classes of

π-elements of G are semisimple classes. According to [Car85, Proposition 3.7.3] and its proof, there is a well-defined bijection

$$\tau : \mathrm{Cl}_{ss}(G) \to (\mathbf{T}/W)^F$$

between the set $\mathrm{Cl}_{ss}(G)$ of semisimple conjugacy classes of G and the set $(\mathbf{T}/W)^F$ of F-stable orbits of W on $\mathbf{T}$. Malle, Navarro, and Robinson showed in [MNR21, Theorem 3.15] that this bijection τ preserves element orders, and therefore the counting formula (and its proof) for the number of F-stable orbits of W on $\mathbf{T}$ in [Car85, Proposition 3.7.4] implies that

$$k_\pi(G) = \frac{1}{|W|} \sum_{w \in W} |\mathbf{T}^{w^{-1}F}|_\pi.$$

It follows that

$$\mathrm{d}_\pi(G) = \frac{1}{|W|} \sum_{w \in W} \frac{|\mathbf{T}^{w^{-1}F}|_\pi}{|G|_\pi}.$$

Now, if $|\mathbf{T}^{w^{-1}F}|_\pi = |G|_\pi$ for some $w \in W$ then a Hall π-subgroup of $\mathbf{T}^{w^{-1}F}$, which is abelian, would be a Hall π-subgroup of G, and this contradicts the above claim. Thus

$$\frac{|\mathbf{T}^{w^{-1}F}|_\pi}{|G|_\pi} \leq 1/p$$

for every $w \in W$. It then follows that

$$\mathrm{d}_\pi(G) \leq \frac{1}{p},$$

proving the first part of the theorem.

For the second part, assume that $|\mathbf{Z}(G)|_\pi = 1$. By Lemma 5.5, we then have

$$\mathrm{d}_\pi(S) = \mathrm{d}_\pi(G/\mathbf{Z}(G)) = \mathrm{d}_\pi(G) \leq \frac{1}{p},$$

as stated. □

Theorem 5.6 already proves Theorem 4.9 in several cases, as seen in the next result. In what follows, to unify the notation, we use GL^ϵ, SL^ϵ and PSL^ϵ for linear groups when $\epsilon = +$ and for unitary groups when $\epsilon = -$. We also use E_6^+ for E_6 and E_6^- for ${}^2E_6$.

Theorem 5.7. *Let S be a simple group of Lie type, π be a set of two odd primes not containing the defining characteristic of S, and p be the smaller prime in π. Assume that we are not in one of the following situations:*

(i) *$S = E_6^\epsilon(q)$ and $3 \in \pi$.*
(ii) *$S = \mathrm{PSL}^\epsilon(n, q)$ with $n \geq 3$ and $\gcd(n, q - \epsilon)_\pi \neq 1$.*

Then $\mathrm{d}_\pi(S) \leq \frac{1}{p}$.

Proof. Let $\mathbf{G}$ and F be as in Theorem 5.6. According to [MT11, Table 24.12], if we are not in one of the stated situations, then $|\mathbf{Z}(\mathbf{G}^F)|_\pi = 1$. The result then follows from Theorem 5.6. □

Next we prove Theorem 4.9 for case (i) in Theorem 5.7.

Proposition 5.8. *Let $S = E_6^\epsilon(q)$ with $(3,q) = 1$ and $P \in \mathrm{Syl}_3(S)$. Then $|P'| > 3$. In particular, Theorem 4.9 holds in the case $S = E_6^\epsilon(q)$ and $3 \in \pi$.*

Proof. Let $\mathbf{G}$ be a simple algebraic group of simply connected type and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius map such that $S = \mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$. By [MT11, Theorem 25.17], we know that every Sylow 3-subgroup of $\mathbf{G}^F$ lies in $\mathbf{N}_{\mathbf{G}^F}(\mathbf{T})$ for some maximal F-stable torus $\mathbf{T}$ of $\mathbf{G}$. Therefore Sylow 3-subgroups of $\mathbf{N}_{\mathbf{G}^F}(\mathbf{T})/\mathbf{T}^F = \mathrm{SO}(5,3)$ (the Weyl group of E_6) are homomorphic images of Sylow 3-subgroups of $S = \mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$. Since the size of the derived subgroup of Sylow 3-subgroups of $\mathrm{SO}(5,3)$ is 9, we deduce that $|P'| > 3$. □

For the rest of this section, we will prove Theorem 4.9 for case (ii) in Theorem 5.7.

Lemma 5.9. *Let p be an odd prime and $S = \mathrm{PSL}^\epsilon(n,q)$. Assume that p divides $\gcd(n, q-\epsilon)$ and Sylow p-subgroups of S are abelian. Then $n = p = 3$. Furthermore, $q - \epsilon$ is divisible by 3 but not 9.*

Proof. It is argued in Lemma 2.8 of [KS21] that if Sylow p-subgroups of S are abelian and $p \geq 5$ then p cannot divide $|\mathbf{Z}(\mathrm{SL}^\epsilon(n,q))|$. Therefore our hypotheses imply that $p = 3$.

We first prove that $n = 3$. The condition $p = 3$ divides $\gcd(n, q-\epsilon)$, implies that $n \geq 3$. Assume by contradiction that $n > 3$. Let w be the (unique) element of order 3 of $\mathbb{F}_{q^2}^\times$, and consider the element $g := \mathrm{diag}(I_{n-2}, w, w^{-1})$. We have

$$\mathbf{C}_{\mathrm{GL}^\epsilon(n,q)}(g) = \mathrm{GL}^\epsilon(n-2,q) \times \mathrm{GL}^\epsilon(1,q)^2,$$

and so

$$|\,\mathrm{GL}^\epsilon(n,q) : \mathbf{C}_{\mathrm{GL}^\epsilon(n,q)}(g)| = q^{2n-1}\frac{(q^n - \epsilon^n)(q^{n-1} - \epsilon^{n-1})}{(q-\epsilon)^2}.$$

Since 3 divides $\gcd(n, q-\epsilon)$, we have that 3 must divide $|\,\mathrm{GL}^\epsilon(n,q) : \mathbf{C}_{\mathrm{GL}^\epsilon(n,q)}(g)|$. In fact, we also have 3 divides $|\,\mathrm{SL}^\epsilon(n,q) : \mathbf{C}_{\mathrm{SL}^\epsilon(n,q)}(g)|$. On the other hand, as 1 is the only eigenvalue of g with multiplicity larger than 1 (recall that $n > 3$), it is easy to see that $\mathbf{C}_{\mathrm{SL}^\epsilon(n,q)}(g)$ is the full pre-image of $\mathbf{C}_{\mathrm{PSL}^\epsilon(n,q)}(\overline{g})$ under the natural projection from SL^ϵ to PSL^ϵ, where $\overline{g}$ is the image of g in $\mathrm{PSL}^\epsilon(n,q)$. In particular, $|\,\mathrm{SL}^\epsilon(n,q) : \mathbf{C}_{\mathrm{SL}^\epsilon(n,q)}(g)| = |\,\mathrm{PSL}^\epsilon(n,q) : \mathbf{C}_{\mathrm{PSL}^\epsilon(n,q)}(\overline{g})|$, and hence 3 divides $|\,\mathrm{PSL}^\epsilon(n,q) : \mathbf{C}_{\mathrm{PSL}^\epsilon(n,q)}(\overline{g})|$, implying that Sylow 3-subgroups of $S = \mathrm{PSL}^\epsilon(n,q)$ are not abelian. We have shown that $n = 3$.

Finally, assume that 9 divides $q - \epsilon$. Let λ be the element of order 9 in $\mathbb{F}_{q^2}^\times$ and consider $h := \mathrm{diag}(\lambda, \lambda^3, \lambda^5) \in \mathrm{SL}^\epsilon(3,q)$, also of order 9. We then have $\mathbf{C}_{\mathrm{GL}^\epsilon(3,q)}(h) = \mathrm{GL}^\epsilon(1,q)^3$, so that $|\mathbf{C}_{\mathrm{SL}^\epsilon(3,q)}(h)| = (q-\epsilon)^2$. Moreover, as $\{\lambda, \lambda^3, \lambda^5\} = \{a\lambda, a\lambda^3, a\lambda^5\}$ if and only if $a = 1$, $\mathbf{C}_{\mathrm{SL}^\epsilon(3,q)}(h)$ is the full pre-image of $\mathbf{C}_{\mathrm{PSL}^\epsilon(3,q)}(\overline{h})$. We deduce that $|\mathbf{C}_{\mathrm{PSL}^\epsilon(3,q)}(\overline{h})| = (q-\epsilon)^2/3$. This is smaller than the 3-part of $|\,\mathrm{PSL}^\epsilon(3,q)|$, and thus Sylow 3-subgroups of $\mathrm{PSL}^\epsilon(3,q)$ are not abelian, violating the hypothesis. So 9 cannot divide $q-\epsilon$, as stated. □

Theorem 5.10. *Let p be an odd prime, $n \geq 4$, and $(n,p) \neq (6,3)$. Let $G := \mathrm{SL}^\epsilon(n,q)$ defined in characteristic not equal to p, $S := G/\mathbf{Z}(G) = \mathrm{PSL}^\epsilon(n,q)$, and $P \in \mathrm{Syl}_p(S)$. Suppose that $P/\mathbf{Z}(P)$ is abelian. Then p does not divide $|\mathbf{Z}(G)|$.*

Proof. Assume by contradiction that $p \mid |\mathbf{Z}(G)| = \gcd(n, q-\epsilon)$. Lemma 5.9 already shows that P is non-abelian, but we need to work harder to achieve that $P/\mathbf{Z}(P)$ is non-abelian.

Let $\lambda \in \mathbb{F}_{q^2}^\times$ be an element of order p and consider the p-element

$$x := \operatorname{diag}(\lambda, \lambda^{-1}, I_{n-2}) \in G.$$

Let $V = \mathbb{F}_q^n$, respectively $\mathbb{F}_{q^2}^n$, denote the natural G-module for $\epsilon = +$, respectively $\epsilon = -$. Fix a basis $B = \{v_1, v_2, ..., v_n\}$ of V, and consider the permutation y on B defined by

$$y := \{v_1 \mapsto v_2, v_2 \mapsto v_3, ..., v_{p-1} \mapsto v_p, v_p \mapsto v_1, v_i \mapsto v_i \text{ for } p < i \leq n\},$$

which is well-defined as $p \leq n$. Note that, as $p > 2$, we have $y \in G$ and $\operatorname{ord}(y) = p$. Direct calculation shows that

$$[x, y] = \operatorname{diag}(\lambda^{-1}, \lambda^2, \lambda^{-1}, I_{n-3}) =: s.$$

Suppose that the p-part of $q - \epsilon$ is p^a and let C be the (unique) cyclic subgroup of order p^a of $\mathbb{F}_{q^2}^\times$. As y permutes the diagonal matrices in G with diagonal entries in C, one can form the corresponding semidirect product that is then a p-group. It follows that x and y both belong to a Sylow p-subgroup, say $\widehat{P}$, of G. We deduce that $s = [x, y] \in \widehat{P}'$, which implies that $s\mathbf{Z}(G) \in P'$, where $P \in \operatorname{Syl}_p(S)$ is the image of $\widehat{P}$ under the natural projection $\mathrm{SL}^\epsilon \to \mathrm{PSL}^\epsilon$.

We will show that $s\mathbf{Z}(G)$ does not belong to $\mathbf{Z}(P)$, which is enough to conclude that $P/\mathbf{Z}(P)$ is not abelian.

Let $\widetilde{G} := \mathrm{GL}^\epsilon(n, q)$. We have

$$\mathbf{C}_{\widetilde{G}}(s) = \begin{cases} \mathrm{GL}^\epsilon(3, q) \times \mathrm{GL}^\epsilon(n-3, q) & \text{if } p = 3 \\ \mathrm{GL}^\epsilon(1, q) \times \mathrm{GL}^\epsilon(2, q) \times \mathrm{GL}^\epsilon(n-3, q) & \text{if } p > 3. \end{cases}$$

It is easy to see that $|S : \mathbf{C}_S(s\mathbf{Z}(G))| = |G : \mathbf{C}_G(s)| = |\widetilde{G} : \mathbf{C}_{\widetilde{G}}(s)|$. Hence,

$$|S : \mathbf{C}_S(s\mathbf{Z}(G))| = \begin{cases} \dfrac{|\mathrm{GL}^\epsilon(n, q)|}{|\mathrm{GL}^\epsilon(3, q)||\mathrm{GL}^\epsilon(n-3, q)|} & \text{if } p = 3 \\ \dfrac{|\mathrm{GL}^\epsilon(n, q)|}{|\mathrm{GL}^\epsilon(1, q)||\mathrm{GL}^\epsilon(2, q)||\mathrm{GL}^\epsilon(n-3, q)|} & \text{if } p > 3. \end{cases}$$

It follows that, if ℓ is the defining characteristic of S, then

$$|S : \mathbf{C}_S(s\mathbf{Z}(G))|_{\ell'} = \begin{cases} \dfrac{(q^n - \epsilon^n)(q^{n-1} - \epsilon^{n-1})(q^{n-2} - \epsilon^{n-2})}{(q - \epsilon)(q^2 - 1)(q^3 - \epsilon^3)} & \text{if } p = 3 \\ \dfrac{(q^n - \epsilon^n)(q^{n-1} - \epsilon^{n-1})(q^{n-2} - \epsilon^{n-2})}{(q - \epsilon)^2(q^2 - 1)} & \text{if } p > 3. \end{cases}$$

Using the condition $p \mid \gcd(n, q - \epsilon)$ and the assumption $(n, p) \neq (6, 3)$, we see that this is divisible by p. It follows that $s\mathbf{Z}(G)$ does not belong to $\mathbf{Z}(P)$, and this finishes the proof. □

Lemma 5.11. *Let* $S = \mathrm{PSL}^\epsilon(n, q)$ *with* $n \geq 4$. *If* 3 *divides* $q - \epsilon$, *then* $d_3(S) \leq \frac{1}{3}$. *In particular, if* 3 *divides* $q - \epsilon$ *and* $3 \in \pi$, *then* $d_\pi(S) \leq \frac{1}{3}$.

Proof. Assume, to the contrary, that $d_3(S) > 1/3$. Then $d_3(P) > 1/3$, and thus $|P'| \leq 3$ by Theorem 2.4. The proof of Theorem 5.10 shows that P' contains two elements $s\mathbf{Z}(G)$ and

$t\mathbf{Z}(G)$, where $s = \mathrm{diag}(\lambda^{-1}, \lambda^2, \lambda^{-1}, I_{n-3})$ and $t = \mathrm{diag}(1, \lambda^{-1}, \lambda^2, \lambda^{-1}, I_{n-4})$. Obviously these elements generate a group of order greater than 3, a contradiction. □

Lemma 5.12. *Let $S = \mathrm{PSL}^\epsilon(3, q)$ and π a set of odd primes with $3 \in \pi$. Then $\mathrm{d}_\pi(S) \leq \frac{1}{3}$.*

Proof. If 3 does not divide $q - \epsilon$, then the result follows by Proposition 5.6. We therefore assume that 3 divides $q - \epsilon$. In particular, 3 divides $q^2 + \epsilon q + 1$. Denote $t := \frac{(q-\epsilon)_3}{3}$. We have

$$|S|_3 = \frac{((q-\epsilon)^2(q+\epsilon)(q^2+\epsilon q+1))_3}{3} \geq (q-\epsilon)_3^2 = 9t^2.$$

On the other hand, counting the number of conjugacy classes of 3-elements in $\mathrm{PSL}^\epsilon(3, q)$ (see for example [SF73]) we have $k_3(S) = (t^2 + t + 2)/2 \leq 2t^2$. Therefore,

$$\mathrm{d}_\pi(S) \leq \mathrm{d}_3(S) = \frac{k_3(S)}{|S|_3} \leq \frac{2t^2}{9t^2} < \frac{1}{3},$$

as wanted. □

Proposition 5.13. *Theorem 4.9 holds for $S = \mathrm{PSL}^\epsilon(n, q)$ with $n \geq 3$ and $\pi = \{p, s\}$ with $p < s$ be odd primes such that q is not divisible by neither p nor s.*

Proof. The result follows by Theorem 5.6 in the case $\gcd(n, q-\epsilon)_\pi = 1$. So assume that $\gcd(n, q-\epsilon)_\pi > 1$, so that there exists $r \in \pi$ such that r divides $\gcd(n, q-\epsilon)$. The case $n = 3$ is then done by Lemma 5.12. So we assume furthermore that $n \geq 4$.

Let $R \in \mathrm{Syl}_r(S)$. We have that $R/\mathbf{Z}(R)$ is abelian by hypothesis. This and the condition r divides $\gcd(n, q-\epsilon)$ contradict Theorem 5.10 if $r \geq 5$. The remaining case $r = 3$ is handled by Lemma 5.11. □

We have completed the proof of Theorem 4.9, by combining Theorems 5.2, 5.3, 5.4, 5.7 and Propositions 5.8 and 5.13.

As mentioned before, Theorem 1.1 follows from Theorem 4.9 and Theorem 4.10 together with Theorem 3.4.

6. Examples and further discussion

In this section, we present examples showing that the converses of both statements of Theorem 1.1 are false and the bounds are generically sharp.

Consider the converse of the first sentence of Theorem 1.1. Assume first that $2 \in \pi$ and $3 \notin \pi$. If G is the direct product of Σ_4 and an abelian group, then $\mathrm{d}_\pi(G) = \frac{1}{6}$. Now, let π have size at least 2 and $p > 2$. Let P be a finite p-group with $|P'| = p$. Let C be the cyclic group which is the direct product of the groups C_q where q runs over all primes in π except for p. Let T be the elementary abelian 2-group of rank $|\pi| - 1$. Let $G = P \times (C : T)$ where

$C : T = \prod_{p\neq q\in\pi}(C_q : C_2)$. In this case

$$\begin{aligned}\mathrm{d}_\pi(G) &\leq \Big(\frac{p^2+p-1}{p^3}\Big)\Big(\prod_{p\neq q\in\pi}\frac{q+1}{2q}\Big)\\ &\leq \Big(\frac{p^2+p-1}{p^3}\Big)\cdot\Big(\frac{p+1}{2p}\Big)^{|\pi|-1}\\ &\leq \Big(\frac{p^2+p-1}{p^3}\Big)\cdot\Big(\frac{p+1}{2p}\Big).\end{aligned}$$

Since $p \geq 3$, this is less than $\frac{5}{6p}$, so the converse of the first statement is false.

Consider the converse of the second statement. Assume first that $2 \in \pi$ and $3 \notin \pi$. If G is the direct product of A_4 and an abelian group, then $\mathrm{d}_\pi(G) = \frac{1}{6}$. Now, let $p \neq 2$ and let $|\pi| \geq 3$. Let $C = \prod_{q\in\pi} C_q$. Let $T = C_{p-1} \times (C_2)^{|\pi|-1}$ and set $G = C : T$. Then

$$\mathrm{d}_\pi(G) = \frac{2}{p}\cdot\prod_{p\neq q\in\pi}\frac{q+1}{2q}.$$

Since $|\pi| \geq 3$, $q \geq p+2$ and all primes q in π are odd, we get

$$\mathrm{d}_\pi(G) \leq \Big(\frac{2}{p}\Big)\cdot\Big(\frac{(p+2)+1}{2(p+2)}\Big)\cdot\Big(\frac{(p+4)+1}{2(p+4)}\Big) \leq \frac{24}{35p}.$$

Thus the converse of the second statement of Theorem 1.1 is also false.

The inequality $\mathrm{d}_\pi(G) > \frac{p^2+p-1}{p^3}$ in the second statement of Theorem 1.1 is sharp for every set of primes π. Take G to be the direct product of a finite non-abelian p-group P such that $P/\mathbf{Z}(P)$ is isomorphic to $C_p \times C_p$ with an abelian group. In this case $\mathrm{d}_\pi(G) = \frac{p^2+p-1}{p^3}$ and G does not contain an abelian Hall π-subgroup.

Let us consider now the inequality $\mathrm{d}_\pi(G) > 1/p$ of the first sentence. This condition is best possible when $p = 2$ and $3 \in \pi$, for if G is a direct product of Σ_3 and an abelian group, then $\mathrm{d}_\pi(G) = 1/2$ and G does not contain a nilpotent Hall π-subgroup. However the bound is certainly not best possible when p is odd. In fact, following our proofs closely, it can be seen that in such case, the group G still possesses a nilpotent Hall π-subgroup even when $d_\pi(G) = 1/p$.

Now let p be odd. We will show that in certain cases the inequality $\mathrm{d}_\pi(G) > 1/2p$ does not imply that G has a nilpotent Hall π-subgroup. To see this let $\pi = \{p, q\}$ where $q = 2p+1$; that is, p is a Sophie Germain prime. Let G be the direct product of $C_q : C_p$ and an abelian group. Elementary character theory gives $k_\pi(C_q : C_p) = p+(q-1)/p$. Thus

$$\mathrm{d}_\pi(G) = \mathrm{d}_\pi(C_q : C_p) = \frac{1}{2p+1}\left(1+\frac{2}{p}\right),$$

which is strictly larger than $1/2p$.

The last example naturally raises the following question: for π a set of *odd* primes, what is the exact (lower) bound for $d_\pi(G)$ to ensure the existence of a nilpotent Hall π-subgroup in G? This seems nontrivial to us at the time of this writing.

Let G be a finite group and let p be the smallest prime dividing $|G|$. If $n(p)$ denotes the smallest prime larger than p and

$$\Pr(G) > \frac{n(p)+p^2-1}{p^2n(p)} =: f(p),$$

then $|G'| \leq p$ and thus G is nilpotent by Theorem 2.6 and Lemma 2.3. Note that $f(p) \leq 1/p$ and equality occurs if and only if $p = 2$.

Now let π be a set of primes and p be the smallest member in π. It is perhaps true that if $d_\pi(G) > f(p)$ then G possesses a nilpotent Hall π-subgroup, but this would require significant more effort, especially on the part of simple groups of Lie type in characteristic not belong to π. We have decided to work with the bound $1/p$ instead in order to make our arguments flowing smoothly. We certainly do not claim that $f(p)$ is the (conjectural) best possible bound for $d_\pi(G)$ to ensure the existence of a nilpotent Hall π-subgroup in G, and thus the question we just raised above remains open.

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