ON THE ORDERS OF COMPOSITION FACTORS IN COMPLETELY REDUCIBLE GROUPS

ATTILA MARÓTI AND SAVELIY V. SKRESANOV

Abstract. We obtain an asymptotic upper bound for the product of the p-parts of the orders of certain composition factors of a finite group acting completely reducibly and faithfully on a finite vector space of order divisible by a prime p. An application is given for the diameter of a nondiagonal orbital graph of an affine primitive permutation group.

1. Introduction

The composition length $s(G)$ of a finite group G is the length of any composition series of G. Obtaining bounds for this invariant has been an important area of study in finite group theory. For instance, Glasby, Praeger, Rosa, Verret [5, Theorem 1.2] proved that if G is a permutation group of degree d with r orbits, then $s(G) \leq \frac{4}{3}(d-r)$. In the special case when G is primitive, they [5, Theorem 1.3] gave a logarithmic bound in d for $s(G)$, namely $s(G) \leq \frac{8}{3}\log_2 d - \frac{4}{3}$.

A finite primitive permutation group G is affine if it has an abelian minimal normal subgroup V. The group G decomposes into a semidirect product HV, where H is a point stabilizer in G, moreover, $H \cap V = 1$ and the vector space V may be viewed as an irreducible H-module. More generally, let V have dimension n over the finite field $\mathbb{F}_q$ of size q and let V be a completely reducible, faithful $\mathbb{F}_qH$-module for a finite group H. Glasby, Praeger, Rosa, Verret [5, Theorem 1.4] and Holt, Tracey [8, Theorem 4.1] gave sharp upper bounds for $s(H)$. These are at most $Cn\log q$ for explicit constants C. All logarithms in this paper are in base e unless specified otherwise.

The bound can be made more precise if one focuses on cyclic composition factors only. We continue to assume that V is a completely reducible, faithful $\mathbb{F}_qH$-module for a finite group H with $|V| = q^n$ for $q = p^f$ with p a prime and n, f are positive integers. Let r be the number of irreducible summands of V. Giudici, Glasby, Li, Verret [4, Theorem 1] proved that the number of composition factors of H of order p is at most $\frac{\epsilon_q n - r}{p-1}$ where ϵ_q is $\frac{4}{3}$ if $p = 2$ and f is even, is $\frac{p}{p-1}$ if p is a Fermat prime (a prime of the form $2^{2^k}+1$ for some integer $k \geq 0$), and is 1 otherwise.

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Some results bound products of orders of special kinds of composition factors in a composition series of a finite group. For example, Guralnick, the first author, and Pyber [6] investigate products of the orders of abelian or nonabelian composition factors of a finite group and use these results to classify primitive permutation groups A and G of degree d with G normal in A and $|A/G| \geq d$.

In this paper we will also present a bound on the product of the orders of certain composition factors. For a prime p and a positive integer N, let $v_p(N)$ denote the largest k such that p^k divides N. Given a finite group G with composition series $1 = G_0 < G_1 < \cdots < G_m = G$, let $s_p(G)$ be the sum of $v_p(|G_i/G_{i-1}|)$ for such $i \in \{1, \ldots, m\}$ that G_i/G_{i-1} is not isomorphic to a finite simple group of Lie type in characteristic p. By the Jordan–Hölder theorem $s_p(G)$ does not depend on the choice of the composition series, so it is an invariant of G. Notice that if the group does not contain composition factors isomorphic to finite simple groups of Lie type in characteristic p (if, for instance, the group is p-solvable), then $s_p(G)$ is equal to $v_p(|G|)$.

The following may be viewed as an asymptotic extension of the main theorem of Giudici, Glasby, Li, Verret [4].

Theorem 1.1. *There exists a universal constant C (independent of any parameter) such that the following holds. Let q be a power of a prime p and let V be a finite vector space of dimension n over the field of size q. If H is a subgroup of* $\mathrm{GL}(V)$ *acting completely reducibly with r irreducible summands, then*

$$s_p(H) \leq C \cdot \frac{n-r}{p-1}.$$

Note that one cannot hope to obtain a similar bound for $s_p(H)$ which is linear in n, for p fixed, unless one excludes composition factors isomorphic to finite simple groups of Lie type in characteristic p from the definition of $s_p(H)$. For instance, $v_p(|\mathrm{GL}(V)|) = n(n-1)/2$ if V has dimension n over a field of order p, and this is quadratic in n.

If the linear group H in Theorem 1.1 is p-solvable, then a good and explicit bound is known for $s_p(H) = v_p(|H|)$, namely, Schmid [17, p. 211] showed that $s_p(H) \leq np/(p-1)^2$. This is related to Brauer's $k(B)$ problem, to the $k(GV)$ theorem, and to the noncoprime $k(GV)$ problem. For example, Kovács and Robinson [10] (see also [17, Proposition 13.5]) proved that there exists a universal constant c such that whenever V is a finite, completely reducible, and faithful $\mathbb{F}_pH$-module of dimension n for a finite p-solvable group H with a prime p, then the number $k(HV)$ of conjugacy classes of the semidirect product HV is at most $c^n|V|$. It turned out after the proof of the $k(GV)$ theorem that $n \log_p c$ can be taken to be $v_p(|H|) = s_p(H) \leq np/(p-1)^2$.

Another motivation to establish Theorem 1.1 was a recent work of Robinson [16] in which, answering a question of Etingof, he proved similar upper bounds for the index of an abelian normal subgroup of a p'-group contained in $\mathrm{GL}(n, \mathbb{C})$ for any given prime p.

For our final motivation, let G be a permutation group acting on a finite set X. Following [3, Section 1.11], an orbital graph of G is a directed graph with vertex set

X whose arc set is an orbit of G on $X \times X$. An orbital graph whose arcs are a subset of the diagonal $\{(x,x) \mid x \in X\}$ is called a diagonal orbital graph. A criterion of Higman [7] states that a transitive permutation group is primitive if and only if all its nondiagonal orbital graphs are (strongly) connected. Liebeck, Macpherson and Tent [12] described finite primitive permutation groups whose nondiagonal orbital graphs have bounded diameter (we note that in [12] orbital graphs are considered to be undirected). See also the papers of Sheikh [18] and Rekvényi [15].

The paper [14] contains upper bounds for the diameters of nondiagonal orbital graphs of affine primitive permutation groups. Improving on a bound in [14], the second author [19] proved that there exists a universal constant C such that the diameter of a nondiagonal orbital graph for an affine primitive permutation group G of degree p^n, for a prime p and an integer n, is at most Cn^3, provided that a point-stabilizer of G has order divisible by p.

As an application of Theorem 1.1, we obtain a strong upper bound for the orbital diameter of an affine primitive permutation group G with point-stabilizer H, under the condition that $s_p(H) \geq 1$, where p is the prime dividing the degree of G.

Corollary 1.2. *There exists a universal constant C such that whenever G is an affine primitive permutation group of degree p^n, where p is a prime and n is an integer, with a point-stabilizer H satisfying $s_p(H) \geq 1$, then the diameter of any nondiagonal orbital graph of G is less than $Cn(n-1)/s_p(H)$.*

Note that if the composition factors of H belong to a list of known finite simple groups, then Corollary 1.2 is independent from the classification of finite simple groups.

2. Bounds on prime divisors of the orders of finite simple groups

The purpose of this section is to establish Theorem 1.1 in the special case when H is a quasisimple group acting irreducibly on V. The main result of the section is Proposition 2.5.

The proof relies on bounds for prime divisors of the orders of finite simple groups of Lie type. Similar results have been obtained in [1, 2, 13], but we will require finer bounds in terms of the dimensions of irreducible projective modules of groups of Lie type.

We need the following corollary of a result of Artin [1].

Lemma 2.1. *Let $r > 1$ be an integer, and let p be a prime. If $a = \pm r$ or $a = r^2$ then*

$$v_p\left(\prod_{i=1}^{m}(a^i - 1)\right) \leq 2 \cdot \frac{\log\left((r+1)^m\right)}{\log p}.$$

Proof. In [1, p. 464], cf. [2, Lemma 4.2], the following bound for the largest power of p dividing $\prod_{i=1}^{m}(a^i-1)$ was given

$$p^{v_p\left(\prod_{i=1}^{m}(a^i-1)\right)} \leq \begin{cases} 3^{m/2}(r+1)^m, & \text{if } r \text{ is even, } a = \pm r \text{ or } a = r^2, \\ 2^m(r+1)^m, & \text{if } r \text{ is odd, } a = \pm r, \\ 4^m(r+1)^m, & \text{if } r \text{ is odd, } a = r^2. \end{cases}$$

The right-hand side can be bounded above by $(r+1)^{2m}$. The claim follows by taking base p logarithms. □

Our notation for finite simple groups of Lie type follows [9].

Lemma 2.2. *Let G be $\mathrm{L}_m(r)$, $\mathrm{U}_m(r)$, $\mathrm{PSp}_{2m}(r)$, $\Omega_{2m+1}(r)$, or $\mathrm{P}\Omega^{\pm}_{2m}(r)$. If p is a prime not dividing r, then*

$$v_p(|G|) \leq 2 \cdot \frac{\log\left((r+1)^m\right)}{\log p}.$$

Proof. It follows from the order formulae for the classical simple groups, see [9, Table 5.1.A], that the p-part of $|G|$ divides

$$\prod_{i=1}^{m}(r^{2i}-1).$$

By applying Lemma 2.1 with $a = r^2$ we obtain the required bound. □

Lemma 2.3. *Let G be an exceptional finite simple group of Lie type defined over the field of order r. If p is a prime not dividing r, then*

$$v_p(|G|) \leq 30 \cdot \frac{\log(r+1)}{\log p}.$$

Proof. We use the order formulae for the exceptional simple groups of Lie type, see [9, Table 5.1.B] and Lemma 2.1. For example, if $G = {}^3D_4(r)$, then the p-part of G divides $(r^8+r^4+1)(r^6-1)(r^2-1)$. This expression divides $(r^{12}-1)(r^6-1)(r^2-1)$, hence to estimate $v_p(|G|)$ we may apply Lemma 2.1 with $a = r^2$ and $m = 6$. The rest of the groups are processed similarly, and we collect the values of a and m in the following table.

G	$G_2(r)$	$F_4(r)$	$E_6(r)$	$E_7(r)$	$E_8(r)$	${}^2B_2(r)$	${}^2G_2(r)$	${}^2F_4(r)$	${}^3D_4(r)$	${}^2E_6(r)$
a	r^2	r^2	r	r^2	r^2	r	r	r	r^2	$-r$
m	3	6	12	9	15	4	6	12	6	12

Clearly the $E_8(r)$ case dominates the rest, which gives us the claimed bound. □

The next lemma shows that the dimensions of cross-characteristic modules for a group of Lie type are large in comparison to the prime divisors of the order of the group.

Lemma 2.4. *There exists a universal constant C such that the following is true. Let G be a nonabelian finite simple group of Lie type defined over a field of order r having a faithful irreducible projective representation of dimension n over a field of characteristic p. If p divides $|G|$ and does not divide r, then $p \leq C \cdot n$. Moreover, the following are true:*

(1) *If G is $\mathrm{L}_m(r)$, $\mathrm{PSp}_{2m}(r)$, $\mathrm{U}_m(r)$, $\mathrm{P}\Omega^{\pm}_{2m}(r)$, or $\Omega_{2m+1}(r)$, then $r^{m-1} \leq C \cdot n$.*
(2) *If G is an exceptional simple group of Lie type, then $r \leq C \cdot n$.*

Proof. Assume first that G is $\mathrm{L}_m(r)$, $\mathrm{PSp}_{2m}(r)$, $\mathrm{U}_m(r)$, $\mathrm{P}\Omega^{\pm}_{2m}(r)$, or $\Omega_{2m+1}(r)$. We claim that for every type of the group (linear, symplectic, unitary or orthogonal) the dimension n is bounded from below by $C_1 \cdot r^{\alpha m+\beta}$ where C_1 is some universal constant and α, β depend only on the type of the group. For example, if $G = \mathrm{U}_m(r)$ and m is even, then by [9, Table 5.3.A], we have $n \geq (r^m - 1)/(r+1)$. Therefore $n \geq \frac{1}{2}r^{m-1}$, so $\alpha m + \beta$ is $m-1$ in this case.

The lower bounds on n extracted from [9, Table 5.3.A] are collected in the third column of Table 1, below. In the next table we list the expressions $\alpha m + \beta$ such that $n \geq C_1 \cdot r^{\alpha m+\beta}$ for classical simple groups:

G	$\mathrm{L}_m(r)$	$\mathrm{PSp}_{2m}(r)$	$\mathrm{U}_m(r)$	$\mathrm{P}\Omega^{\pm}_{2m}(r)$	$\Omega_{2m+1}(r)$
$\alpha m+\beta$	$m-1$	m	$m-1$	$2m-3$	$2m-2$

Clearly, for some constant C, we have $r^{m-1} \leq C \cdot n$, proving (1).

Since p divides $|G|$, it divides at least one of the factors from the order formula for $|G|$, see [9, Table 5.1.A]. In the second column of Table 1 we list the largest such factors, that is, only those which are not dominated by the lower bound on the dimension n. For instance, if $G = \Omega_{2m+1}(r)$, then p divides one of $r^{2i} - 1$, $i = 1, \dots, m$. We know that $n \geq C_1 \cdot r^{2m-2}$ from the table above, so $r^{2i} - 1 \leq C_1' \cdot n$ for $i = 1, \dots, m-1$ and some universal constant C_1'. Hence we put the factor $r^{2m} - 1$ in Table 1.

Note that $r^{2m} - 1$ factorizes as $(r^m - 1)(r^m + 1)$, so p divides one of the factors, and therefore one has $p \leq C_1' \cdot n$. Similar factorizations can be used for other classical simple groups, so we derive that $p \leq C \cdot n$ for some universal constant C.

Assume now that G is an exceptional simple group of Lie type. The dimension n can be bounded from below by $C_2 \cdot r^{\gamma}$ for some universal constants C_2 and γ depending only on the type of the group by [9, Table 5.3.A]. We list the corresponding γ for the exceptional simple groups of Lie type in the following table:

G	$E_6(r)$	$E_7(r)$	$E_8(r)$	$F_4(r)$	${}^2E_6(r)$	$G_2(r)$	${}^3D_4(r)$	${}^2F_4(r)$	${}^2B_2(r)$	${}^2G_2(r)$
γ	11	17	29	8	11	3	5	5	1	2

It immediately follows that $r \leq C \cdot n$ for some constant C, proving (2).

The prime p divides the order of the group and, hence, divides some factor in its order formula, see [9, Table 5.1.B]. As in the previous case, in the second column of Table 1 we list the largest such factor. Note that for the group ${}^3D_4(r)$ there are two factors not dominated by the lower bound for n.

We factorize the polynomials from the order formulae in order to obtain a bound of the form $p \leq C \cdot n$ for some universal constant C. For example, if $G = E_6(r)$ and p divides $r^{12} - 1$, we deduce that p divides one of $r^6 - 1$ or $r^6 + 1$ which is smaller than r^{11}. Different reasoning is applied to ${}^3D_4(r)$, ${}^2F_4(r)$, ${}^2B_2(r)$ and ${}^2G_2(r)$. If $G = {}^3D_4(r)$ and p divides $r^8 + r^4 + 1$, we use the factorization

$$r^8 + r^4 + 1 = (r^4 + r^2 + 1)(r^4 - r^2 + 1),$$

hence $p \leq 3 \cdot r^5$. If $G = {}^2F_4(r)$ and p divides $r^6 + 1$, then we use $r^6 + 1 = (r^2 + 1)(r^4 - r^2 + 1)$, so $p \leq 3 \cdot r^5$. If $G = {}^2B_2(r)$ and p divides $r^2 + 1$, then recall that $r = 2^{2e+1}$ for some integer $e \geq 1$ so that $\sqrt{2r}$ is an integer and we have

$$r^2 + 1 = (r + 1 - \sqrt{2r})(r + 1 + \sqrt{2r}).$$

Therefore $p \leq 3 \cdot r$ in this case. Finally, if $G = {}^2G_2(r)$ and p divides $r^3 + 1$, then recall that $r^3 + 1 = (r + 1)(r^2 - r + 1)$, hence $p \leq r^2$, finishing the proof of the lemma. □

Notice that in the setting of the lemma we also have bounds of the form $p - 1 \leq C'(n - 1)$, $r^{m-1} - 1 \leq C'(m - 1)$ in case (1), and $r - 1 \leq C'(n - 1)$ in case (2) for some universal constant C'.

Group	Largest factors	Lower bounds
$\mathrm{L}_2(r)$	$r^2 - 1$	$(r-1)/\gcd(2, r-1)$
$\mathrm{L}_m(r)$, $m \geq 3$	$r^m - 1$	$r^{m-1} - 1$
$\mathrm{PSp}_{2m}(r)$, $m \geq 2$	$r^{2i} - 1$, $m < 2i \leq 2m$	$(r^m - 1)/2$, r odd
		$r^{m-1}(r^{m-1} - 1)(r - 1)/2$, r even
$\mathrm{U}_m(r)$, $m \geq 3$	$r^m - (-1)^m$	$r(r^{m-1} - 1)/(r + 1)$, m odd
		$(r^m - 1)/(r + 1)$, m even
$\mathrm{P}\Omega^+_{2m}(r)$, $m \geq 4$	$r^{2m-2} - 1$	$(r^{m-1} - 1)(r^{m-2} + 1)$, $r \neq 2, 3, 5$
		$r^{m-2}(r^{m-1} - 1)$, $r = 2, 3, 5$
$\mathrm{P}\Omega^-_{2m}(r)$, $m \geq 4$	$r^{2m-2} - 1$	$(r^{m-1} + 1)(r^{m-2} - 1)$
$\Omega_{2m+1}(r)$, $m \geq 3$, r odd	$r^{2m} - 1$	$r^{2m-2} - 1$, $r > 5$
		$r^{m-1}(r^{m-1} - 1)$, $r = 3, 5$
$E_6(r)$	$r^{12} - 1$	$r^9(r^2 - 1)$
$E_7(r)$	$r^{18} - 1$	$r^{15}(r^2 - 1)$
$E_8(r)$	$r^{30} - 1$	$r^{27}(r^2 - 1)$
$F_4(r)$	$r^{12} - 1$	$r^6(r^2 - 1)$, r odd
		$r^7(r^3 - 1)(r - 1)/2$, r even
${}^2E_6(r)$	$r^{12} - 1$	$r^9(r^2 - 1)$
$G_2(r)$	$r^6 - 1$	$r(r^2 - 1)$
${}^3D_4(r)$	$r^8 + r^4 + 1$, $r^6 - 1$	$r^3(r^2 - 1)$
${}^2F_4(r)$	$r^6 + 1$	$r^4\sqrt{r/2}(r - 1)$
${}^2B_2(r)$	$r^2 + 1$	$\sqrt{r/2}(r - 1)$
${}^2G_2(r)$	$r^3 + 1$	$r(r - 1)$

Table 1. Largest factors in order formulae and lower bounds of dimensions of representations for groups of Lie type

The following result will be used in the main proof. Recall that G is quasisimple if it is perfect and $G/Z(G)$ is nonabelian simple.

Proposition 2.5. *There exists a universal constant C such that the following is true. Let G be a quasisimple group such that $G/Z(G)$ is not isomorphic to a group of Lie type in characteristic p. If G has a faithful irreducible projective representation of dimension n over a field of characteristic p, then*

$$v_p(|G|) \leq C \cdot \frac{n - 1}{p - 1}.$$

Proof. By [9, Corollary 5.3.3], the degree of a minimal faithful projective p-modular representation of G is bounded below by the corresponding number for $G/Z(G)$. We may thus replace G by $G/Z(G)$ and assume that G is simple.

Let C be a large fixed constant (how to specify C will be clear from the proof). Notice that by choosing C large enough we may assume that G is not a sporadic simple group.

If G is isomorphic to $\mathrm{Alt}(m)$, $m \geq 5$, then by [9, Proposition 5.3.7] one has $n \geq m-4$. Thus by Legendre's formula

$$v_p(|\mathrm{Alt}(m)|) \leq \frac{m-1}{p-1} \leq 5 \cdot \frac{n-1}{p-1},$$

where the last inequality uses the fact that $m-1 \leq n+3 \leq 5(n-1)$ for $n \geq 2$. Now the claimed inequality follows for $C \geq 5$.

Now we assume that G is a group of Lie type not in characteristic p. We first consider classical simple groups. Fix r and m as in Lemma 2.2, and notice that for $r \geq 2$ and $m \geq 2$ we have $(r+1)^m \leq 9(r^{m-1}-1)^2$. Lemma 2.2 implies

$$v_p(|G|) \leq 3 \cdot \frac{\log(9(r^{m-1}-1)^2)}{\log p}.$$

If $p \geq \sqrt{3(r^{m-1}-1)}$, then by Lemma 2.4 for C large enough

$$v_p(|G|) \leq 3 \cdot \frac{\log(9(r^{m-1}-1)^2)}{\log\sqrt{3(r^{m-1}-1)}} = 12 \leq C \cdot \frac{n-1}{p-1}.$$

If $p < \sqrt{3(r^{m-1}-1)}$, then

$$v_p(|G|) \leq 3 \cdot \frac{\log(9(r^{m-1}-1)^2)}{\log 2} < C_1\sqrt{r^{m-1}-1} < C_2 \cdot \frac{r^{m-1}-1}{p-1},$$

for some constants C_1, C_2. By Lemma 2.4 (1), we have $r^{m-1}-1 \leq C_3 \cdot (n-1)$ for some constant C_3. Therefore

$$v_p(|G|) \leq C_2 \cdot C_3 \cdot \frac{n-1}{p-1} \leq C \cdot \frac{n-1}{p-1},$$

whenever $C \geq C_2 \cdot C_3$.

We turn to the exceptional simple groups of Lie type. Now, $r+1 \leq 3(r-1)$ and Lemma 2.3 implies that

$$v_p(|G|) \leq 30 \cdot \frac{\log(r+1)}{\log p} \leq 30 \cdot \frac{\log(3(r-1))}{\log p}.$$

If $p \geq \sqrt{3(r-1)}$, then

$$v_p(|G|) \leq 30 \cdot \frac{\log(3(r-1))}{\log\sqrt{3(r-1)}} = 60 \leq C \cdot \frac{n-1}{p-1},$$

where the last inequality uses Lemma 2.4. If $p < \sqrt{3(r-1)}$, then

$$v_p(|G|) \leq 30 \cdot \frac{\log(3(r-1))}{\log 2} < C_1'\sqrt{r-1} < C_2' \cdot \frac{r-1}{p-1},$$

for some constants C_1', C_2'. By Lemma 2.4 (2), we have $r-1 \leq C_3' \cdot (n-1)$, hence

$$v_p(|G|) \leq C_2' \cdot C_3' \cdot \frac{n-1}{p-1} \leq C \cdot \frac{n-1}{p-1},$$

for $C \geq C_2' \cdot C_3'$. $\square$

3. Nonabelian composition factors

For a finite group G with composition series $1 = G_0 < \cdots < G_m = G$, let $\overline{s}_p(G)$ be the sum of $v_p(|G_i/G_{i-1}|)$ over such $i \in \{1, \ldots, m\}$ that G_i/G_{i-1} is *nonabelian* and not isomorphic to a finite simple group of Lie type in characteristic p. The main result of [4] bounds the number $c_p(G)$ of composition factors isomorphic to cyclic groups of order p, so in order to bound $s_p(G) = c_p(G) + \overline{s}_p(G)$ we may focus on bounding $\overline{s}_p(G)$ first.

Proposition 3.1. *There exists a universal constant C such that the following holds. Let q be a power of a prime p and let V be a finite vector space of dimension n over the field of size q. If H is a subgroup of $\mathrm{GL}(V)$ acting completely reducibly with r irreducible summands, then*

$$\overline{s}_p(H) \leq C \cdot \frac{n-r}{p-1}.$$

Proof. Let $H \leq \mathrm{GL}(V)$ be a counterexample to the statement of the theorem with $n \geq 2$ minimal. Under this condition, assume that $|H|$ is as small as possible. The proof proceeds in several steps; we choose the constant $C = \max\{20/3, C_1\}$, where C_1 is the constant C from Proposition 2.5.

Step 1: H acts irreducibly on V. Assume that W is a nonzero proper irreducible submodule of V. Let K be the centralizer of W in H. The factor group H/K acts irreducibly and faithfully on W. Thus $\overline{s}_p(H/K) \leq C \cdot \frac{m-1}{p-1}$ where m is the dimension of W over the field of size q. Since H acts completely reducibly on V, there exists a submodule U of V such that $V = W \oplus U$. The group K acts faithfully on U. Since K is normal in H, it acts completely reducibly on U by Clifford's theorem. By the minimality of n again, we have $\overline{s}_p(K) \leq C \cdot \frac{(n-m)-(r-1)}{p-1}$. These give

$$\overline{s}_p(H) = \overline{s}_p(H/K) + \overline{s}_p(K) \leq C \cdot \frac{m-1}{p-1} + C \cdot \frac{(n-m)-(r-1)}{p-1} = C \cdot \frac{n-r}{p-1},$$

a contradiction.

Step 2: H is perfect. Since H acts irreducibly on V, its derived subgroup $[H,H]$ acts completely reducibly. Now, $\overline{s}_p(H) = \overline{s}_p([H,H])$ and we may assume that $H = [H,H]$ by the minimality of $|H|$.

Step 3: H acts primitively on V. Assume that H acts imprimitively on V, that is, H preserves a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_t$ of the vector space V to nonzero subspaces V_i of the same size where $1 \leq i \leq t$ for some integer $t > 1$. Let the kernel of the action of H on $\{V_1, \ldots, V_t\}$ be B. We have $\overline{s}_p(H/B) \leq (t-1)/(p-1)$ by considering the p-part of $t!$. Since B is a normal subgroup of H, it acts completely reducibly on V. The B-module V has at least t irreducible direct summands. Since

$|B| < |H|$, we have $\overline{s}_p(B) \leq C \cdot \frac{n-t}{p-1}$ by the assumption that $|H|$ is as small as possible. These give

$$\overline{s}_p(H) = \overline{s}_p(H/B) + \overline{s}_p(B) \leq \frac{t-1}{p-1} + C \cdot \frac{n-t}{p-1} \leq C \cdot \frac{n-1}{p-1},$$

where $C \geq 1$ is used.

Step 4: For every normal subgroup R of H, every irreducible constituent of the R-module V is absolutely irreducible. Let R be a normal subgroup of H. Since H acts primitively on V, the group R acts homogeneously on V by Clifford's theorem. In particular, if R is abelian then R is cyclic by Schur's lemma.

Assume that an irreducible constituent of the R-module V is not absolutely irreducible. The center of $\mathrm{End}_R(V)$ is the field E of size q^e for some $e > 1$. This is normalized by H. Since H is perfect, E is centralized by H. The group H may be viewed as a subgroup of $\mathrm{GL}(V)$ where V is the vector space of dimension n/e over the field E. The EH-module V remains irreducible. The minimality of n gives

$$\overline{s}_p(H) \leq C \cdot \frac{n/e - 1}{p-1} < C \cdot \frac{n-1}{p-1}.$$

A contradiction.

Notice that, in particular, H itself acts absolutely irreducibly on V.

Step 5: The group H is quasisimple. Let R be a normal subgroup of H minimal subject to being noncentral. The center $Z(R)$ of R is contained in $Z(H)$ and $R/Z(R)$ is characteristically simple. As in the proof of [6, Theorem 4.1], the group R is either a central product of say t quasisimple groups Q_i (with the $Q_i/Z(Q_i)$ all isomorphic) or $R/Z(R)$ is an elementary abelian s-group for some prime s. In the second case R is an s-group with s different from p and it may be proved that R is of symplectic type with $|R/Z(R)| = s^{2a}$ for some integer $a \geq 1$, and $|Z(R)| = s$ or 4.

We follow the proof of [6, Theorem 4.1] and introduce some notation. Let $J_1, \ldots, J_k$ denote the distinct normal subgroups of H that are minimal with respect to being noncentral in H. Let $J = J_1 \cdots J_k$ be the central product of these subgroups. Let W be an irreducible constituent of the J-module V. This is absolutely irreducible by Step 4. We thus have by [9, Lemmas 5.5.5 and 2.10.1] that $W = U_1 \otimes \cdots \otimes U_k$ where U_i is an irreducible J_i-module. If J_i is the central product of t_i copies of a quasisimple group, then $\dim(U_i) \geq 2^{t_i}$ and if J_i is of symplectic type with $J_i/Z(J_i)$ of order $s_i^{2a_i}$ for some prime s_i and positive integer a_i, then $\dim(U_i) = s_i^{a_i}$.

We have

$$(Z(H)J)/Z(H) \cong J/Z(J) \cong \mathrm{Inn}(J) \cong \mathrm{Inn}(J_1) \times \cdots \times \mathrm{Inn}(J_k)$$

so that $H/(Z(H)J) \leq \mathrm{Out}(J_1) \times \cdots \times \mathrm{Out}(J_k)$, that is, $H/(Z(H)J)$ embeds into the direct product of the outer automorphism groups of the J_i. We claim that

$$\overline{s}_p(H/(Z(H)J)) \leq \frac{5}{3} \cdot \frac{n-1}{p-1}.$$

We inductively define subgroups $T_i \leq \mathrm{Out}(J_i)$ and $K_i \leq H/(Z(H)J)$, $i = 1, \ldots, k$. Let T_1 be the projection of $H/(Z(H)J)$ to $\mathrm{Out}(J_1)$ with kernel K_1. Now define

T_{i+1} to be the projection of K_i to $\mathrm{Out}(J_{i+1})$ with kernel K_{i+1}, for $i = 1, \dots k-1$. Since $\overline{s}_p(H/(Z(H)J)) = \overline{s}_p(T_1) + \overline{s}_p(K_1)$ and $\overline{s}_p(K_i) = \overline{s}_p(T_{i+1}) + \overline{s}_p(K_{i+1})$, $i = 1, \dots, k-1$, and $K_k = 1$, we have $\overline{s}_p(H/(Z(H)J)) = \sum_{i=1}^{k} \overline{s}_p(T_i)$. Therefore

$$\begin{aligned} \overline{s}_p(H/(Z(H)J)) = \sum_{i=1}^{k} \overline{s}_p(T_i) = \sum_{i=1}^{k} \overline{s}_p(T_i/(\mathrm{Sol}(\mathrm{Out}(J_i)) \cap T_i)) \leq \\ \leq \sum_{i=1}^{k} v_p(|T_i/(\mathrm{Sol}(\mathrm{Out}(J_i)) \cap T_i)|) \leq \sum_{i=1}^{k} v_p(|\mathrm{Out}(J_i)/\mathrm{Sol}(\mathrm{Out}(J_i))|), \end{aligned} \tag{1}$$

where $\mathrm{Sol}(X)$ denotes the largest solvable normal subgroup of a finite group X. Fix an index i. Assume that J_i is a central product of t_i quasisimple groups Q_i. The outer automorphism group $\mathrm{Out}(J_i)$ in this case may be viewed as a subgroup of $\mathrm{Out}(Q_i/Z(Q_i)) \wr \mathrm{Sym}(t_i)$. Since $\mathrm{Out}(Q_i/Z(Q_i))$ is solvable by Schreier's conjecture, the factor group $\mathrm{Out}(J_i)/\mathrm{Sol}(\mathrm{Out}(J_i))$ is a section of $\mathrm{Sym}(t_i)$. It follows that

$$v_p(|\mathrm{Out}(J_i)/\mathrm{Sol}(\mathrm{Out}(J_i))|) \leq v_p(|\mathrm{Sym}(t_i)|) \leq \frac{t_i - 1}{p-1}.$$

Now let J_i be a group of symplectic type with $|J_i/Z(J_i)| = s_i^{2a_i}$. In this case $\mathrm{Out}(J_i)$ may be viewed as a subgroup of $\mathrm{Sp}_{2a_i}(s_i)$ and so

$$v_p(|\mathrm{Out}(J_i)/\mathrm{Sol}(\mathrm{Out}(J_i))|) \leq v_p(|\mathrm{Out}(J_i)|) \leq v_p(|\mathrm{Sp}_{2a_i}(s_i)|),$$

which is at most $\frac{(4/3)s_i^{a_i}-1}{p-1}$ by [4, (3)]. In any case, for any i, we have that

$$v_p(|\mathrm{Out}(J_i)/\mathrm{Sol}(\mathrm{Out}(J_i))|) \leq \frac{(4/3)\dim(U_i) - 1}{p-1}. \tag{2}$$

From the above, this gives

$$\overline{s}_p(H/(Z(H)J)) \leq \sum_{i=1}^{k} \frac{(4/3)\dim(U_i) - 1}{p-1} \leq \frac{(4/3)(\sum_{i=1}^{k} \dim(U_i)) - 1}{p-1}.$$

Since $n = \dim(V) \geq \dim(W) = \prod_i \dim(U_i) \geq \sum_i \dim(U_i)$, we have

$$\overline{s}_p(H/(Z(H)J)) \leq \frac{(4/3)n - 1}{p-1} \leq \frac{5}{3} \cdot \frac{n-1}{p-1}. \tag{3}$$

We claim that exactly one of the J_i has a nonabelian composition factor of order divisible by p but different from a group of Lie type in characteristic p. Suppose otherwise. If there is no such J_i, then $\overline{s}_p(Z(H)J) = 0$ and so

$$\overline{s}_p(H) \leq \overline{s}_p(H/(Z(H)J)) + \overline{s}_p(Z(H)J) \leq \frac{5}{3} \cdot \frac{n-1}{p-1} < C \cdot \frac{n-1}{p-1},$$

by (3) and the fact that $C > 5/3$, a contradiction. Let the number of such J_i be $m > 1$. Without loss of generality, let these be $J_1, \dots, J_m$. We have $\overline{s}_p(Z(H)J) = \sum_{i=1}^{m} \overline{s}_p(J_i)$. For each i with $1 \leq i \leq k$, let $\dim(U_i) = n_i$. By the minimality of n, we find that

$$\overline{s}_p(Z(H)J) = \sum_{i=1}^{m} \overline{s}_p(J_i) \leq C \cdot \frac{(\sum_{i=1}^{m} n_i) - m}{p-1}. \tag{4}$$

Let $m \geq 3$, or $m = 2$ and $\max\{n_1, n_2\} \geq 4$. In this case $\sum_{i=1}^{m} n_i \leq \frac{3}{4} \prod_{i=1}^{m} n_i \leq \frac{3}{4} n$. Inequalities (3) and (4) give

$$\overline{s}_p(H) = \overline{s}_p(Z(H)J) + \overline{s}_p(H/(Z(H)J)) \leq \frac{3C}{4} \cdot \frac{n-1}{p-1} + \frac{5}{3} \cdot \frac{n-1}{p-1} \leq C \cdot \frac{n-1}{p-1},$$

where the last inequality holds since $C \geq 20/3$. A contradiction.

Let $m = 2$ and $\max\{n_1, n_2\} \leq 3$. In this case $\mathrm{Out}(J_1)$ and $\mathrm{Out}(J_2)$ are solvable and so $\overline{s}_p(T_i) = 0$ for $i = 1, 2$. Inequalities (1) and (2) give

$$\overline{s}_p(H/(Z(H)J)) \leq \frac{(4/3)\sum_{i=3}^{k} n_i - 1}{p-1},$$

so

$$\overline{s}_p(H) \leq C \cdot \frac{(n_1 + n_2) - 1}{p-1} + \frac{(4/3)\sum_{i=3}^{k} n_i - 1}{p-1} \leq C \cdot \frac{n-1}{p-1},$$

by the minimality of n, where the last inequality holds since $C \geq 4/3$. A contradiction. We thus have $m = 1$.

We claim that $k = 1$. Assume that $k \geq 2$. By the previous paragraph and without loss of generality, $\overline{s}_p(J_1) \geq 1$ and $\overline{s}_p(J_i) = 0$ for every i with $2 \leq i \leq k$. By the minimality of n and the fact that $k \geq 2$ and $n_2 \geq 2$, we have

$$\overline{s}_p(Z(H)J) = \overline{s}_p(J_1) \leq C \cdot \frac{n_1 - 1}{p-1} \leq \frac{C}{2} \cdot \frac{n-1}{p-1}.$$

This together with the bound (3) and $C \geq 10/3$ give $\overline{s}_p(H) < C \cdot \frac{n-1}{p-1}$, a contradiction.

The group $J = J_1$ is a central product of say $t = t_1$ quasisimple groups Q_i (with the $Q_i/Z(Q_i)$ all isomorphic). We claim that $t = 1$. Assume for a contradiction that $t \geq 2$. Let W be an irreducible constituent of the J-module V. Since J is nontrivial and normal in H and V is an irreducible and primitive H-module, W is nontrivial by Clifford's theorem. Moreover, W is absolutely irreducible by Step 4. Then $W = W_1 \otimes \cdots \otimes W_t$ where W_i is an irreducible Q_i-module for every i with $1 \leq i \leq t$ by [9, Lemmas 5.5.5 and 2.10.1]. From the same reference, for each i with $1 \leq i \leq t$, the integer $m_i = \dim(W_i)$ is at least 2 (otherwise J_i acts trivially on W and so also on V). Thus $n \geq \prod_{i=1}^{t} m_i \geq \sum_{i=1}^{t} m_i$. It follows that $\overline{s}_p(Z(H)J) = \overline{s}_p(J) \leq C \cdot \frac{n-t}{p-1}$ by the minimality of n. The factor group $H/(Z(H)J)$ modulo its solvable radical may be viewed as a section of $\mathrm{Sym}(t)$, by Schreier's conjecture. It follows that

$$\overline{s}_p(H/(Z(H)J)) = \overline{s}_p((H/(Z(H)J))/\mathrm{Sol}(H/(Z(H)J))) \leq v_p(|\mathrm{Sym}(t)|) \leq \frac{t-1}{p-1}.$$

Thus

$$\overline{s}_p(H) = \overline{s}_p(Z(H)J) + \overline{s}_p(H/(Z(H)J)) \leq C \cdot \frac{n-t}{p-1} + \frac{t-1}{p-1} \leq C \cdot \frac{n-1}{p-1},$$

since $C \geq 1$. This is a contradiction. We conclude that $t = 1$.

Since H is perfect, $H = JZ(H)$ and so $H = J$ is quasisimple. Since H acts absolutely irreducibly on V and is quasisimple, the final contradiction follows from Proposition 2.5. □

4. Proofs of the main results

Proof of Theorem 1.1. Let V be a finite vector space of dimension n over the field of size q. Let H be a subgroup of $\mathrm{GL}(V)$ acting completely reducibly on V. Let r be the number of irreducible summands of the H-module V. We claim that $s_p(H) \leq C \cdot \frac{n-r}{p-1}$ for some universal constant C.

We prove the bound by induction on n. If $n = 1$, then the size of H is not divisible by p and so $s_p(H) = 0$. Assume that $n \geq 2$ and that the claim is true for $n-1$. If the H-module V contains an irreducible summand W of dimension 1 and K denotes the centralizer of W in H, then $s_p(H) = s_p(K) \leq C \cdot \frac{(n-1)-(r-1)}{p-1}$ by the induction hypothesis. We may assume that every submodule of V has dimension at least 2. In particular, $r \leq n/2$. The number of composition factors of H isomorphic to the cyclic group of order p is $c_p(H) \leq ((4/3)n - r)/(p-1)$ by [4, Theorem 1]. This is at most $\frac{8}{3} \cdot \frac{n-r}{p-1}$ since $r \leq n/2$. Thus

$$s_p(H) = c_p(H) + \overline{s}_p(H) \leq \frac{8}{3} \cdot \frac{n-r}{p-1} + \overline{s}_p(H) \leq C \cdot \frac{n-r}{p-1},$$

where C is $8/3$ plus a constant whose existence is assured by Proposition 3.1. □

Proof of Corollary 1.2. Let C be a constant whose existence is assured by Theorem 1.1. Let G be an affine primitive permutation group of degree p^n where p is a prime and n is an integer. Let H be a point-stabilizer in G satisfying $s_p(H) \geq 1$. The diameter of any nondiagonal orbital graph of G is at most $(p-1)n$ by [14, Proposition 3.2]. On the other hand, $p - 1 \leq C(n-1)/s_p(H)$ by Theorem 1.1. □

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Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, H-1053, Budapest, Hungary

Email address: maroti.attila@renyi.hu

Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, H-1053, Budapest, Hungary

Email address: skresanov.savelii@renyi.hu