MINIMAL DEGREE, BASE SIZE, ORDER - SELECTED TOPICS ON PRIMITIVE PERMUTATION GROUPS

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ABSTRACT. In this survey article, we discuss the minimal degree, the base size, and the order of a finite primitive permutation group, along the lines of an article by Martin W. Liebeck.

1. INTRODUCTION

It has been forty years since Martin W. Liebeck's article "On minimal degrees and base sizes of primitive permutation groups" was published in Archiv der Mathematik. We take the opportunity to give a short survey of selected topics on the effects of this influential paper.

Let G be a finite permutation group acting on a set Ω. This is said to be *transitive* if for every α and β in Ω there is a permutation g in G such that $\alpha g = \beta$. A nonempty subset Δ of Ω is called a *block* for G if for every $g \in G$ the set $\Delta g \cap \Delta$ is either equal to Δ or is the empty set. A block is *trivial* if it has size 1 or if it is Ω. If there is no nontrivial block for a transitive permutation group G, then G is called a *primitive* permutation group. For an element $\alpha \in \Omega$, let G_α denote the *point-stabilizer* of α in G defined to be the subgroup of G consisting of those elements g such that $\alpha g = \alpha$. (A transitive permutation group G is primitive if and only if G_α is a maximal subgroup in G for some (any) $\alpha \in \Omega$.)

Let $\mu(G)$ be the smallest number of elements of Ω moved by any non-identity permutation in G. This invariant is called the *minimal degree* of G. A nonempty subset Δ of Ω is called a *base* for G if $\cap_{\alpha \in \Delta} G_\alpha = 1$. The minimal size of a base for G is denoted by $b(G)$. The third invariant which is discussed in this paper is the order $|G|$ of G.

The *degree* of G is defined to be $|\Omega|$, which we denote by n. Knowledge of the minimal size $b(G)$ of a base for G provides lower and upper bounds for $|G|$. It is easy to see that $2^{b(G)} \leq |G| \leq n(n-1)\cdots(n-b(G)+1) \leq n^{b(G)}$. The relationship between the minimal size of a base for G and the minimal degree of G is the following. If G is a transitive permutation group, then $n \leq b(G)\mu(G)$ (see [19, Exercise 3.3.7]).

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Martin Liebeck's article [28] contains three main results about a finite primitive permutation group G and these concern the invariants $|G|$, $\mu(G)$, and $b(G)$. We restate these groundbreaking theorems and discuss the three invariants in three separate sections. The survey could be considered as an account of existing results in the selected areas. Liebeck's paper [28] appeared soon after the classification of finite simple groups was announced. We give a brief history, with some relevant references for the interested reader, of work done on these topics before 1984. We then describe (in our point of view) the most direct influence of Liebeck's theorems. Many references are given on our selected topics. The reader is advised to consult these for more information and motivation. There is new progress (in different directions) and there are new questions and problems. These are not discussed here since we tried to keep the survey as coincise as possible, however we hope that the reader will become interested and can get acquainted with the most recent topics influenced by Martin Liebeck's paper [28].

2. Orders

Let G be a primitive permutation group of degree n different from the symmetric group S_n and the alternating group A_n. How large can the order $|G|$ of G be? This question was raised in the 19th century. For interesting historical accounts see [17, Section 4.10] and [1]. Apart from some early results of Jordan, probably the first successful estimate for the order of G was obtained by Bochert [6] (see also [19] or [41]): if G is a primitive permutation group of degree n and different from S_n and A_n, then $(S_n : G) \geq [\frac{1}{2}(n+1)]!$. This bound is good for very small degrees n.

Based on Wielandt's method [42] of bounding the orders of Sylow subgroups, Praeger and Saxl [36] obtained an exponential estimate, 4^n. Using entirely different combinatorial arguments, Babai [1] obtained an $e^{4\sqrt{n}\ln^2 n}$ estimate for uniprimitive (primitive but not doubly transitive) groups. (A permutation group G acting on a set Ω is *doubly transitive* if for any two tuples (α_1, β_1) and (α_2, β_2) from $\Omega \times \Omega$ such that $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$ there exists a permutation $g \in G$ with $\alpha_1 g = \alpha_2$ and $\beta_1 g = \beta_2$. Doubly transitive groups are primitive.) For the orders of doubly transitive groups not containing A_n, Babai [2] obtained the bound $\exp(\exp(c\sqrt{\ln n}))$ for some universal constant c. This was improved by Pyber [37] to an $n^{c \cdot \ln^2 n}$ bound by an elementary argument (using some ideas of [2]). Apart from $\mathrm{O}(\ln n)$ factors in the exponents, the estimates in [1] and [37] are asymptotically sharp.

To do better, one may want to use the O'Nan-Scott theorem and the classification of finite simple groups. If H is a permutation group acting on a set Γ, then the wreath product $H \wr S_r$ acts in a natural way on the set Γ^r. This action is a product action. In 1981 Cameron [15] proved the following theorem.

Theorem 2.1 (Cameron [15])**.** *There is a (computable) constant c with the property that, if G is a primitive permutation group of degree n, then at least one of the following holds.*

(1) G has an elementary abelian regular normal subgroup.
(2) G is a subgroup of $\mathrm{Aut}(T) \wr S_r$, containing T^r, where T is either an alternating group acting on k-element subsets, or a classical simple group acting on an orbit of subspaces or (in the case $T = \mathrm{PSL}_d(q)$ where d is an integer

and q is a prime power) pairs of subspaces of complementary dimension, and the wreath product has the product action.

(3) $|G| \leq n^{c \ln \ln n}$.

Groups in (1) of Theorem 2.1 have size at most $n^{1+\log n}$. (The bases of our logarithms will be 2.) It follows from Liebeck's paper [28] that if G is a group satisfying (2) with T a classical simple group, then $|G| < 9 \log n$. In this context see also [17, Theorem 4.13] and its proof.

A primitive permutation group is said to be *almost simple* if it has a unique minimal normal subgroup and that is nonabelian and simple. The class of almost simple primitive permutation groups plays a fundamental role in bounding the orders of primitive groups. The previously mentioned general upper bound $9 \log n$ of Liebeck, with known exceptions, for the order of a primitive permutation group of degree n was obtained from the following theorem.

Theorem 2.2 (Liebeck [28])**.** *Let G be an almost simple primitive permutation group of degree n with minimal normal subgroup T. At least one of the following holds.*

(1) $T = A_m$ *acting on k-subsets of* $\{1, \ldots, m\}$ *or on partitions of* $\{1, \ldots, m\}$ *into a subsets of size b, where* $ab = m$, $a > 1$, $b > 1$, $n = \binom{m}{k}$ *or* $m!/(b!)^a a!$, *respectively.*

(2) *T is a classical simple group acting on an orbit of subspaces of the natural module, or (in the case* $T = \mathrm{PSL}_d(q)$ *where d is an integer and q is a prime power) pairs of subspaces of complementary dimensions.*

(3) $|G| < n^9$.

The bound $|G| < n^9$ in part (3) of Theorem 2.2 may be replaced by $|G| \leq n^c$ where $c = 6.077948094$ is adjusted to the Mathieu group M_{24} (Martin Liebeck, unpublished). For more on this comment, see [17, p. 116]. See also Theorem 4.2.

Let us close this section by recording the following theorem.

Theorem 2.3 (M [34])**.** *Let G be a primitive permutation group of degree n. Then at least one of the following holds.*

(1) *G is a subgroup of* $S_m \wr S_r$ *containing* $(A_m)^r$, *where the action of* S_m *is on k-element subsets of* $\{1, \ldots, m\}$ *and the wreath product has the product action of degree* $n = \binom{m}{k}^r$.

(2) $G = \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{23}$ *or* M_{24} *with their 4-transitive actions.*

(3) $|G| \leq n \cdot \prod_{i=0}^{[\log_2 n]-1} (n - 2^i) < n^{1+[\log_2 n]}$.

3. Minimal degree

One of the classical problems in the theory of permutation groups was to classify the permutation groups whose minimal degree is small. Let G be a primitive permutation group of degree n. If G contains a transposition or a 3-cycle, then G must be S_n or A_n. This means that if G does not contain A_n, then its minimal degree, $\mu(G)$, is at least 4. There are many classical results on minimal degrees of primitive permutation groups. For example, see Bochert [7], Jordan [26], and Manning [33]. A summary can be found in Wielandt's book [41]. See also [19].

The best result on the minimal degree of a primitive permutation group obtained prior to the classification of finite simple groups is due to Babai [1, Theorem 6.14]. He claims that this is the central result of his paper [1].

Theorem 3.1 (Babai [1])**.** *If G is a primitive permutation group of degree n and not containing A_n, then $\mu(G) > (\sqrt{n}-1)/2$.*

The following stronger bound was proved using the classification of finite simple groups.

Theorem 3.2 (Liebeck [28])**.** *Let G be a primitive permutation group of degree n. At least one of the following holds.*

(1) *G is a subgroup of $S_m \wr S_r$ containing $(A_m)^r$, where the action of S_m is on k-element subsets of $\{1,\dots,m\}$ and the wreath product has the product action of degree $n = \binom{m}{k}^r$.*
(2) *$\mu(G) > n/(9\log n)$.*

Liebeck and Saxl [32] improved the bound $\mu(G) > n/(9\log n)$ in part (2) of Theorem 3.2 to $\mu(G) \geq n/3$. This result is deduced from the stronger result [32, Theorem 6.1] where the $n/3$ is replaced by $n/2$ at the cost of further exceptions in part (1) of Theorem 3.2. It is noted in [32, p. 268] that in all cases the minimal degrees of the groups $S_m \wr S_r$ in Theorem 3.2 are realized by a transposition in one of the factors S_m of the base group. A useful consequence [32, Corollary 3] of these results is an improvement of Babai's general bound (Theorem 3.1) to $\mu(G) > 2(\sqrt{n}-1)$. Later Guralnick and Magaard [22] classified all primitive permutation groups of degree n with $\mu(G) \leq n/2$. All examples are essentially variants on alternating or symmetric groups acting on the set of subsets of some cardinality k or from orthogonal groups over the field of two elements acting on some collection of 1-spaces or hyperplanes.

The strongest theorem to date on the minimal degree of a primitive permutation group is due to Burness and Guralnick [9, Theorem 4].

Theorem 3.3 (Burness, Guralnick [9])**.** *Let G be a primitive permutation group acting on a finite set Ω of size n. Let a point-stabilizer be H. Either $\mu(G) \geq 2n/3$ or one of the following holds (up to permutation isomorphism).*

(1) *$G = S_m$ or A_m acting on k-element subsets of $\{1,\dots,m\}$ with $1 \leq k < m/2$.*
(2) *$G = S_m$, $H = S_{m/2} \wr S_2$, and $\mu(G) = (1+1/(m-1))(n/2)$.*
(3) *$G = \mathrm{M}_{22}:2$ or $G = \mathrm{L}_3(4).2_2$ with $n = 22$ and $\mu(G) = 14$.*
(4) *G is an almost simple classical group in a subspace action and the few possibilities are listed in* [9, Table 2].
(5) *$G = V : H$ is an affine group with unique minimal normal subgroup $V = (C_2)^d$ and $H \leq \mathrm{GL}_d(2)$ contains a transvection and $\mu(G) = 2^{d-1} = n/2$.*
(6) *$G \leq L \wr S_r$ is a product type primitive group with its product action on $\Omega = \Gamma^r$ where $r \geq 2$ and $L \leq \mathrm{Sym}(\Gamma)$ is one of the almost simple groups in (1)-(4).*

4. Base size

The minimal size of a base of a primitive permutation group has been much investigated. Already in the nineteenth century Bochert [6] showed that $b(G) \leq n/2$ for a primitive permutation group G of degree n not containing A_n. This bound was substantially improved by Babai to $b(G) < 4\sqrt{n}\ln n$, for uniprimitive groups G, in [1], and to the estimate $b(G) < 2^{c\sqrt{\log n}}$ for a universal constant c, for doubly transitive groups G not containing A_n, in [2]. The latter bound was improved by Pyber [37] to $b(G) < c(\log n)^2$ where c is a universal constant. These estimates are elementary in the sense that their proofs do not require the classification of finite simple groups.

Using the classification, Liebeck [28] classified all primitive permutation groups G of degree n with $b(G) \geq 9\log n$.

Theorem 4.1 (Liebeck [28])**.** *Let G be a primitive permutation group of degree n. At least one of the following holds.*

(1) G is a subgroup of $S_m \wr S_r$ containing $(A_m)^r$, where the action of S_m is on k-element subsets of $\{1,\dots,m\}$ and the wreath product has the product action of degree $n = \binom{m}{k}^r$.
(2) $b(G) < 9\log n$.

In 1981 Babai conjectured (see [38, p. 207]) that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that any primitive group that has no alternating or classical composition factor of degree or dimension greater than d has base size less than $f(d)$. An important special case was a famous theorem of Seress [39]: if G is a (finite) solvable primitive permutation group then $b(G) \leq 4$. Babai's conjecture was proved by Gluck, Seress, Shalev [21] with f a quadratic function and improved to a linear function f by Liebeck and Shalev [29].

In order to state a second conjecture on base sizes of primitive permutation groups, we return to Theorem 2.2 (and to the comment that follows it). If G is a primitive permutation group satisfying part (1) or part (2) of Theorem 2.2, then (the action of) G is called *standard*. Otherwise the action is said to be *nonstandard*. A well-known conjecture of Cameron and Kantor (see [16] and [18]) asserted that there exists an absolute constant c such that $b(G) \leq c$ for all almost simple primitive permutation groups G in nonstandard actions. Referring to c, Cameron wrote (see [17, p. 122]), 'Probably this constant is 7, and the extreme case is the Mathieu group M_{24}'. The Cameron-Kantor conjecture was proved by Liebeck and Shalev [29] and in the strong form with $c = 7$ by Burness [8], Burness, Guralnick, Saxl [10], Burness, Liebeck, Shalev [12], and Burness, O'Brien, Wilson [13].

Theorem 4.2 (Burness, Liebeck, Shalev [12]; Burness, O'Brien, Wilson [13])**.** *If G is an almost simple primitive permutation group in nonstandard action, then $b(G) \leq 7$, with equality if and only if G is the Mathieu group M_{24} in its natural action of degree 24.*

For primitive permutation groups G in standard actions, the minimal base size $b(G)$ cannot be bounded from above by an absolute constant. This is because, in general, the orders of the permutation groups in standard actions are not bounded from above by a fixed polynomial function of their degree and thus $b(G)$ is unbounded by the observation that $b(H) > \log|H|/\log n$ for any finite permutation group H of degree n. Bounds (and formulas) for $b(G)$ for groups G satisfying part (1) of Theorem 2.2 were obtained by Halasi [23], Bailey, Cameron [3], [14], [5] and James [25]. For bounds and calculations for $b(G)$ in case G satisfies part (2) of Theorem 2.2, see [35], [24], and [11, Section 4].

A well-known conjecture of Pyber [38, Page 207] stated that there exists a universal constant c such that $b(G) < c(\log|G|/\log n)$ for all primitive permutation groups G of degree n. For a collection of results towards this conjecture, see the survey article [31]. See also the important paper [30] of Liebeck and Shalev. The proof of Pyber's conjecture was completed by Duyan, Halasi and the author [20]. It was shown in [20] that there exists a universal constant c such that, for every primitive permutation group G of degree n, we have $b(G) < 45(\log|G|/\log n) + c$. The theorem was made effective by Halasi, Liebeck and the author [24] (where the multiplicative constant 2 is best possible).

Theorem 4.3 (Halasi, Liebeck, M [24]). *If G is a primitive permutation group of degree n, then $b(G) \leq 2(\log|G|/\log n) + 24$.*

We continue this section with a beautiful improvement of Theorem 4.1 due to Moscatiello and Roney-Dougal [35]. The next theorem should also be compared with Theorem 2.3.

Theorem 4.4 (Moscatiello, Roney-Dougal [35]). *Let G be a primitive permutation group of degree n and not satisfying part (1) of Theorem 4.1. Then $b(G) \geq 1 + \log n$ if and only if G is one of the following.*

(1) A subgroup of $\mathrm{AGL}_d(2)$ for some integer d and $b(G) = 1 + d = 1 + \log n$.
(2) The group $\mathrm{Sp}_d(2)$, acting on the cosets of $\mathrm{GO}_d^-(2)$ with $d \geq 4$, in which case $1 + \log n < b(G) = 1 + \lceil \log n \rceil$.
(3) A Mathieu group M_n in its natural permutation representation with n in the set $\{12, 23, 24\}$. If $n = 12$ or 23 then $b(G) = 1 + \lceil \log n \rceil$, while if $n = 24$ then $b(G) = 7 > 1 + \lceil \log n \rceil$.

Concerning the groups satisfying part (1) of Theorem 4.1, we remark that if G is a primitive permutation group of degree n not containing A_n then $b(G) \leq \max\{\sqrt{n},\ 25\}$ by [24, Corollary 1.3].

The base size has important applications in computational group theory. See for example [40]. Blaha [4] proved that the problem of computing a minimal base for a permutation group is NP-hard. However, an irredundant base (see the next paragraph) can be computed in polynomial time (see [40]).

Let G be a permutation group acting on a finite set Ω. A nonempty subset Δ of Ω is called an *irredundant base* for G if it is a base for G but no proper subset of Δ is a base for G. The maximal size of an irredundant base for G (acting on Ω) is denoted by $I(G)$. Another nice improvement of Theorem 4.1 was obtained by Kelsey and Roney-Dougal [27]. Let G be a primitive permutation group of degree n not satisfying part (1) of Theorem 4.1. Then $I(G) < 5\log n$. It follows from this and the previous paragraph that a base for G of size at most $5\log n$ can be constructed in polynomial time (see [27]).

References

[1] Babai, László. On the orders of uniprimitive permutation groups. *Ann. of Math.* **113**, (1981), 553–568.

[2] Babai, László. On the order of doubly transitive permutation groups. *Invent. Math.* **65**, (1981/82), no. 3, 473–484.

[3] Bailey, Robert F.; Cameron, Peter J. Base size, metric dimension and other invariants of groups and graphs. *Bull. Lond. Math. Soc.* **43** (2011), no. 2, 209–242.

[4] Blaha, Kenneth D. Minimum bases for permutation groups: the greedy approximation. *J. Algorithms* **13** (1992), no. 2, 297–306.

[5] Benbenishty, Carmit; Cohen, Jonathan A.; Niemeyer, Alice C. The minimum length of a base for the symmetric group acting on partitions. *European J. Combin.* **28** (2007), no. 6, 1575–1581.

[6] Bochert, Alfred. Über die Transitivitätsgrenze der Substitutionengruppen, welche die Alternierende ihres Grades nicht einhalten. *Math. Ann.* **33**, (1889), 572–583.

[7] Bochert, Alfred. Über die Zahl der verschiedenen Werthe, die eine Function gegebener Buchstaben durch Vertauschung derselben erlangen kann. *Math. Ann.* **33** (1889), no. 4, 584–590.

[8] Burness, Timothy C. On base sizes for actions of finite classical groups. *J. Lond. Math. Soc.* (2) **75** (2007), 545–562.

[9] Burness, Timothy C.; Guralnick, Robert M. Fixed point ratios for finite primitive groups and applications. *Adv. Math.* **411** (2022), part A, Paper No. 108778, 90 pp.

[10] Burness, Timothy C.; Guralnick, Robert M.; Saxl, Jan. On base sizes for symmetric groups. *Bull. Lond. Math. Soc.* **43** (2011), no. 2, 386–391.

[11] Burness, Timothy C.; Guralnick, Robert M.; Saxl, Jan. On base sizes for algebraic groups. *J. Eur. Math. Soc. (JEMS)* **19** (2017), no. 8, 2269–2341.

[12] Burness, Timothy C.; Liebeck, Martin W.; Shalev, Aner. Base sizes for simple groups and a conjecture of Cameron. *Proc. Lond. Math. Soc.* (3) **98** (2009), 116–162.

[13] Burness, Timothy C.; O'Brien, E. A.; Wilson, Robert A. Base sizes for sporadic simple groups. *Israel J. Math.* **177** (2010), 307–333.

[14] Cáceres, José; Garijo, Delia; González, Antonio; Márquez, Alberto; Puertas, María Luz. The determining number of Kneser graphs. *Discrete Math. Theor. Comput. Sci.* **15** (1) (2013) 1–14.

[15] Cameron, Peter J. Finite permutation groups and finite simple groups. *Bull. London Math. Soc.* **13**, (1981), 1–22.

[16] Cameron, Peter J. Some open problems on permutation groups. Groups, combinatorics, geometry (Durham, 1990), 340–350, London Math. Soc. Lecture Note Ser., 165, Cambridge Univ. Press, Cambridge, 1992.

[17] Cameron, Peter J. Permutation groups. London Mathematical Society Student Texts, 45. Cambridge University Press, Cambridge, 1999.

[18] Cameron, Peter J.; Kantor, William M. Random permutations: some group-theoretic aspects. *Combin. Probab. Comput.* **2** (1993), no. 3, 257–262.

[19] Dixon, John D.; Mortimer, Brian Permutation groups. Graduate Texts in Mathematics, 163. Springer-Verlag, New York, 1996.

[20] Duyan, Hülya; Halasi, Zoltán; Maróti, Attila. A proof of Pyber's base size conjecture. *Adv. Math.* **331** (2018), 720–747.

[21] Gluck, David; Seress, Ákos; Shalev, Aner. Bases for primitive permutation groups and a conjecture of Babai. *J. Algebra* **199** (1998), 367–378.

[22] Guralnick, Robert; Magaard, Kay. On the minimal degree of a primitive permutation group. *J. Algebra* **207** (1998), no. 1, 127–145.

[23] Halasi, Zoltán. On the base size for the symmetric group acting on subsets. *Studia Sci. Math. Hungar.* **49** (2012), no. 4, 492–500.

[24] Halasi, Zoltán; Liebeck, Martin W.; Maróti, Attila. Base sizes of primitive groups: bounds with explicit constants. *J. Algebra* **521** (2019), 16–43.

[25] James, Jonathan P. Partition actions of symmetric groups and regular bipartite graphs. *Bull. London Math. Soc.* **38** (2006), no. 2, 224–232.

[26] Jordan, Camille. Theoremes sur les groupes primitifs. *J. Math. Pures Appl.* **16** (1871) 383–408

[27] Kelsey, Veronica; Roney-Dougal, Colva M. On relational complexity and base size of finite primitive groups. *Pacific J. Math.* **318** (2022), no. 1, 89–108.

[28] Liebeck, Martin W. On minimal degrees and base sizes of primitive permutation groups. *Arch. Math. (Basel)* **43** (1984), no. 1, 11–15.
[29] Liebeck, Martin W.; Shalev, Aner. Simple groups, permutation groups, and probability. *J. Amer. Math. Soc.* **12** (1999), 497–520.
[30] Liebeck, Martin W.; Shalev, Aner. Bases of primitive linear groups II. *J. Algebra* **403** (2014), 223–228.
[31] Liebeck, Martin W.; Shalev, Aner. Bases of primitive permutation groups. Groups, combinatorics, geometry (Durham, 2001), 147–154, World Sci. Publ., River Edge, NJ, 2003.
[32] Liebeck, Martin W.; Saxl, Jan. Minimal degrees of primitive permutation groups, with an application to monodromy groups of covers of Riemann surfaces. *Proc. London Math. Soc.* (3) **63** (1991), no. 2, 266–314.
[33] Manning, William A. The degree and class of multiply transitive groups. III. *Trans. Amer. Math. Soc.* **35** (1933), no. 3, 585–599.
[34] Maróti, Attila. On the orders of primitive groups. *J. Algebra* **258** (2002), no. 2, 631–640.
[35] Moscatiello, Mariapia; Roney-Dougal, Colva M. Base sizes of primitive permutation groups. *Monatsh. Math.* **198** (2022), no. 2, 411–443.
[36] Praeger, Cheryl; Saxl, Jan. On the order of primitive permutation groups. *Bull. London Math. Soc.* **12**, (1980), 303-308.
[37] Pyber, László. On the orders of doubly transitive permutation groups, elementary estimates. *J. Comb. Theory (A)* **62**, (1993), 361–366.
[38] Pyber, László. Asymptotic results for permutation groups. Groups and computation, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 11 (ed. L. Finkelstein and W. M. Kantor) Amer. Math. Soc. Providence RI 1993, pp. 197–219.
[39] Seress, Ákos. The minimal base size of primitive solvable permutation groups. *J. London Math. Soc.* (2) **53** (1996), no. 2, 243–255.
[40] Sims, Charles C. Computational methods in the study of permutation groups. 1970 Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967) pp. 169–183 Pergamon, Oxford.
[41] Wielandt, Helmut. Finite permutation groups. *Acad. Press, New York*, 1964.
[42] Wielandt, Helmut. Permutation groups through invariant relations and invariant functions. *Lecture Notes, Ohio State University, Columbus, Ohio*, 1969.

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