

MIXABILITY OF FINITE SOLVABLE GROUPS AND OF SOME SIMPLE GROUPS

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ABSTRACT. A finite group G is mixable if there exists a sequence $g_1, \dots, g_n$ of elements of G and independent Bernoulli random variables $\varepsilon_1, \dots, \varepsilon_n$ such that $\prod_{i=1}^n g_i^{\varepsilon_i}$ is the uniform distribution of G . This notion was introduced by Amir, Blachar, Ghosh, and Vishne. They proved that a finite group with a nontrivial quotient of odd order cannot be mixable and conjectured that if a finite group has no nontrivial quotient of odd order then it is mixable. We confirm this conjecture for solvable groups and we establish the mixability of several finite simple groups.

1. INTRODUCTION

A finite group G is said to be *mixable* if there exists a sequence $g_1, \dots, g_n$ of elements of G and independent Bernoulli random variables $\varepsilon_1, \dots, \varepsilon_n$ (with values 0 or 1) such that $\prod_{i=1}^n g_i^{\varepsilon_i}$ is the uniform distribution of G . The sequence g_i , together with the probabilities $p_i = P(\varepsilon_i = 1)$ (for $i \in \{1, \dots, n\}$), is called a *mixing sequence* for G and n is its *length*. The smallest length of a mixing sequence of G is called the *mixing length* of G and is denoted by $\ell(G) = \ell_{\text{mix}}(G)$. For example, the symmetric group $G = S_3$ on 3 letters is mixable since $g_1 = g_3 = (12)$, $g_2 = (23)$ together with $p_1 = p_3 = 1/2$, $p_2 = 2/3$ is a mixing sequence for G and $\ell(G) = 3$. Note that $\ell(G) \geq \log_2 |G|$ for every mixable group G , since a sequence of length n has image of size at most 2^n .

These notions were introduced by Amir, Blachar, Ghosh, and Vishne [1]. They ask which finite groups are mixable. All symmetric groups are mixable by [2] and [6, Remark 4.7], see [1, Example 2.5]. All finite 2-groups are mixable by [1, Corollary 3.1]. All alternating groups of degree at least 5 are mixable by [1, Corollary 4.5]. All finite irreducible Coxeter groups, with the possible exception of those of types E_6 , E_7 , and E_8 , are mixable by [1, Theorem 8.1]. In this paper we show that the latter three groups are also mixable.

Theorem 1.1. *Every finite irreducible Coxeter group is mixable.*

The groups $\text{SL}_2(q)$ are mixable for every power q of 2 by [1, Theorem 10.2]. The groups $\text{SL}_n(q)$, $\text{PSL}_n(q)$, and $\text{PGL}_n(q)$ are mixable if $q - 1$ is a power of 2 by [1, Corollary 11.3]. The group $\text{GL}_n(q)$ is mixable if and only if $q - 1$ is a power of 2

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by [1, Corollary 11.5]. In this paper we establish the mixability of further linear groups.

Theorem 1.2. *If $n \geq 2$ is an integer and q is a prime power such that $q \not\equiv 3 \pmod{4}$ or $q \in \{7, 11, 19\}$, then the groups $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$ and $\mathrm{ASL}_n(q)$ are mixable.*

The Mathieu groups M_{11} , M_{12} , M_{22} , M_{23} , M_{24} and the Higman-Sims group HS are mixable by [1, Corollary 11.7] and [1, Corollary 11.8]. We show that there are further sporadic simple groups which are mixable.

Theorem 1.3. *The sporadic simple groups J_2 , Suz, McL, Co_3 , Co_2 , Fi_{22} , Fi_{23} , Fi_{24} , J_1 are mixable.*

We also prove the mixability of the following groups (see Theorem 5.1): $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_6(2)$, $\mathrm{Sp}_6(2)$, $\mathrm{O}_8^+(2)$, $G_2(4)$.

Amir, Blachar, Ghosh, and Vishne [1, Theorem 3.2] proved that a finite group with a nontrivial quotient of odd order is not mixable. They conjectured [1, Conjecture 6.3] that a finite group is mixable if and only if it has no nontrivial quotient of odd order.

If G is a mixable group, then $\ell(G) \geq \log_2 |G|$. This bound is tight for 2-groups (see [1, Corollary 3.1]) and also for symmetric and alternating groups up to a factor of $3/2$ (see [1, Theorem 4.3] and [1, Corollary 4.5]). It is asked [1, Question 1.2] whether there exists a universal constant c such that $\ell(G) \leq c \cdot \log_2 |G|$ holds for every mixable group G . It is mentioned that a test case could be the dihedral group $G = D_{2n}$ for which the current best bound is $\ell(G) \leq \log_2(|G|_2) + |G|_{2'} - 1$.

Our final result is a proof of [1, Conjecture 6.3] in the special case of solvable groups in which we generalize the previous mixing length bound for dihedral groups.

Theorem 1.4. *A finite solvable group is mixable if and only if it has no nontrivial quotient of odd order. Moreover, if G is a mixable solvable group, then*

$$\ell(G) \leq \log_2(|G|_2) + |G|_{2'} - 1.$$

2. GENERAL LEMMAS

In this section we collect a few general sufficient conditions for a group (or for a transitive action of a group) to be mixable.

We start with [1, Definition 2.1 (1)] and [1, Definition 2.6]. A *random subproduct* in a finite group G is a product $\prod_{i=1}^n g_i^{\varepsilon_i}$, where $g_1, \dots, g_n$ are fixed elements in G and $\varepsilon_1, \dots, \varepsilon_n$ are independent Bernoulli random variables. If the random subproduct is the uniform distribution of G , then it is called a *mixing sequence* for the group G . The mixing sequence is sometimes written in the form $(p_1 : g_1), \dots, (p_n : g_n)$ where $p_i = P(\varepsilon_i = 1)$ for $i \in \{1, \dots, n\}$.

Let G be a finite group acting transitively on a set Ω . The action of G on Ω is called *mixable* if there exists a random subproduct $\prod_{i=1}^n g_i^{\varepsilon_i}$ in G such that $\omega_0 \prod_{i=1}^n g_i^{\varepsilon_i}$ is distributed uniformly on Ω , where ω_0 is some fixed point in Ω . Such a random subproduct is called a *mixing sequence* for the action of G on Ω . If an

action of G on Ω is mixable with respect to some $\omega_0 \in \Omega$, then it is mixable with respect to every $\omega \in \Omega$. The smallest length of a mixing sequence for the mixable action of the group G on Ω is denoted by $\ell(G, \Omega)$.

If $S_1, \dots, S_n$ are independent random subproducts for a finite group G , then the concatenation of these sequences will be denoted by $S_1 \cdots S_n$. Let S be a random subproduct of a finite group G . The associated random variable will sometimes be denoted by $\mathbf{s}$, that is, written in lower case letter and in bold face. For the set of values of $\mathbf{s}$, we will write $\text{Im}(S)$ or $\text{Im}(\mathbf{s})$.

The first lemma will be used in the proof of Theorem 1.4.

Lemma 2.1. *Let H_0, H_1, H_2 be subgroups of a finite group G . Suppose that H_1 and H_2 are mixable. Let Ω be the set of right cosets of H_0 in G and let G act on Ω by right multiplication. Let Δ be the H_1 -orbit of $\omega_0 = H_0$ in Ω . Suppose that for any H_2 -orbit Σ the fraction $|\Delta \cap \Sigma|/|\Sigma|$ is a constant. Then $G = H_0 H_1 H_2$ and the action of G on Ω is mixable. Moreover, if H_0 is also mixable, then G is mixable and $\ell(G) \leq \sum_{i=0}^2 \ell(H_i)$.*

Proof. Let the H_2 -orbits of Ω be $\Sigma_1, \dots, \Sigma_k$. Let $|\Delta \cap \Sigma_i|/|\Sigma_i|$ be c for every i with $1 \leq i \leq k$. Since $\Omega = \bigcup_{i=1}^k \Sigma_i$, we have $c \neq 0$. Therefore

$$|\Omega| = \sum_{i=1}^k |\Sigma_i| = \sum_{i=1}^k (|\Delta \cap \Sigma_i|/c) = |\Delta|/c.$$

Let $\mathbf{h}_1, \mathbf{h}_2$ be independent random variables associated to mixing sequences of H_1, H_2 respectively, acting on Ω . Such exist because H_1 and H_2 are mixable. We prove that $\mathbf{h}_1 \mathbf{h}_2$ is a mixing sequence for the action of G on Ω . Fix $\tau \in \Omega$. There exists a Σ_i containing τ . We have

$$\begin{aligned} P(\omega_0 \mathbf{h}_1 \mathbf{h}_2 = \tau) &= \sum_{\omega \in \Delta \cap \Sigma_i} P(\omega_0 \mathbf{h}_1 = \omega \text{ and } \omega \mathbf{h}_2 = \tau) \\ &= \sum_{\omega \in \Delta \cap \Sigma_i} P(\omega_0 \mathbf{h}_1 = \omega) \cdot P(\omega \mathbf{h}_2 = \tau) \\ &= \sum_{\omega \in \Delta \cap \Sigma_i} \frac{1}{|\Delta|} \frac{1}{|\Sigma_i|} = \frac{|\Delta \cap \Sigma_i|}{|\Delta| |\Sigma_i|} = \frac{c}{|\Delta|} = \frac{1}{|\Omega|}. \end{aligned}$$

Since this probability is nonzero, for every $\tau \in \Omega$ there exist $h_1 \in H_1, h_2 \in H_2$ such that $\omega_0 h_1 h_2 = \tau$, therefore $G = H_0 H_1 H_2$. This also proves that the action of G on Ω is mixable and that $\ell(G, \Omega) \leq \ell(H_1) + \ell(H_2)$. Now let H_0 be mixable. The mixability of G and the claim about the mixing length follows from the above and [1, Proposition 2.12]. $\square$

Let H and K be subgroups of a finite group G . Let $a \in G$. The size of the set HaK is $|H||K|/|H \cap K^{a^{-1}}|$ and, for all $g \in HaK$, the set $\{(h, k) \in H \times K \mid hak = g\}$ has size $|H \cap K^{a^{-1}}|$.

Given two mixable subgroups H and K of a finite group G , we provide a sufficient condition for G to be mixable.

Lemma 2.2. *Let G be a finite group, let H, K be two mixable subgroups of G with mixing sequences $\mathbf{s}_H, \mathbf{s}_K$ respectively. Let $\mathbf{s}$ be a sequence of G such that*

$P(\mathbf{s} \in HgK) = |HgK|/|G|$ for all $g \in G$. Then $\tau = \mathbf{s}_H \cdot \mathbf{s} \cdot \mathbf{s}_K$ is a mixing sequence for G .

Proof. Fix $a \in G$ and fix $g \in HaK$.

$$\begin{aligned} P(\tau = g) &= \sum_{h \in H, k \in K} P(\mathbf{s} = h^{-1}gk^{-1})P(\mathbf{s}_H = h)P(\mathbf{s}_K = k) \\ &= \frac{1}{|H||K|} \sum_{h \in H, k \in K} P(\mathbf{s} = h^{-1}gk^{-1}) \\ &= \frac{1}{|H||K|} \sum_{v \in HaK} P(\mathbf{s} = v)|H \cap K^{a^{-1}}| = \frac{P(\mathbf{s} \in HaK)}{|HaK|} = \frac{1}{|G|}. \end{aligned}$$

This concludes the proof. $\square$

If H is a mixable subgroup of a finite group G and the action of G on the set of right cosets of H is 2-transitive, then G is mixable [1, Corollary 9.2]. This is precisely the situation when G has a partition of the form $H \sqcup HxH$ for some element $x \in G$. We can slightly extend [1, Corollary 9.2] in the following way. If G is a finite group, H and K are mixable subgroups in G and x is an element in G such that $G = HK \sqcup HxK$, then G is mixable. The next lemma is a further step in this direction.

Lemma 2.3. *Let G be a finite group and let H and K be two mixable subgroups in G . Let G have three double cosets with respect to H and K . Let these be HK , HxK and HyK where x and y are elements in G . If $HxK \cdot HyK$ is not contained in HK , then G is mixable.*

Proof. If $HxK \cdot HyK$ is not contained in HK , then there are $x' \in HxK$ and $y' \in HyK$ such that $x'y' \notin HK$. Without loss of generality, we may set $x' = x$ and $y' = y$. Define the following sequence $\mathbf{s}$ for G .

$$\mathbf{s} := \begin{cases} \left(\frac{|HxK|}{|G|} : x \right), \left(\frac{|HyK|}{|HK|+|HyK|} : y \right) & \text{if } xy \in HxK, \\ \left(\frac{|HxK|}{|HK|+|HxK|} : x \right), \left(\frac{|HyK|}{|G|} : y \right) & \text{if } xy \in HyK. \end{cases}$$

We obtain $P(\mathbf{s} \in HgK) = |HgK|/|G|$ for all $g \in G$. Apply Lemma 2.2. $\square$

Corollary 2.4. *Let G be a finite group and assume that G has a non-normal subgroup H such that the associated permutation group has rank 3. If H is mixable, then G is mixable.*

Proof. Let $\{H, HxH, HyH\}$ be the partition of G in double cosets of H where x and y are elements in G . We may suppose by Lemma 2.3 that $HxH \cdot HyH$ is contained in H . It follows that x and y normalize H . But H is assumed not to be normal in G . $\square$

The following lemma may be viewed as a further extension of [1, Corollary 9.2]. Note also that Corollary 2.4 can be seen as a direct application of Lemma 2.5.

Lemma 2.5. *Let G be a group and let H and K be two subgroups of G such that $G = HK \sqcup Hx_2K \sqcup \dots \sqcup Hx_rK$ for some elements $x_1, \dots, x_r$ in G . Assume that*

- (1) H and K are mixable, and
- (2) $x_{i_1}x_{i_2}\dots x_{i_t} \in Hx_{i_1}K$ for all $2 \leq i_1 < i_2 < \dots < i_t \leq r$.

Then G is mixable with mixing length at most $\ell(H) + \ell(K) + r - 1$.

Proof. For all i with $2 \leq i \leq r$ we set $\delta_i = |Hx_iK|/|G|$. We set $\mathbf{s} = \mathbf{x}_2 \dots \mathbf{x}_r$ where the $\mathbf{x}_i$ are independent Bernoulli random variables with probabilities $p_i = P(\mathbf{x}_i = x_i)$ defined as follows.

$$p_i = \begin{cases} \delta_2 & \text{if } i = 2, \\ \frac{\delta_i}{1 - (\delta_2 + \dots + \delta_{i-1})} & \text{if } 3 \leq i \leq r. \end{cases}$$

Note that the image of $\mathbf{s}$ is the set

$$\text{Im}(\mathbf{s}) = \{x_{i_1}x_{i_2}\dots x_{i_t} \mid t \geq 0, 2 \leq i_1 < i_2 < \dots < i_t \leq r\}.$$

In order to prove the lemma, we apply Lemma 2.2 and check that $P(\mathbf{s} \in HgK) = |HgK|/|G|$ for all $g \in G$. For all i with $2 \leq i \leq r$, by assumption (2) we have that

$$Hx_iK \cap \text{Im}(\mathbf{s}) = \{x_{i_1}x_{i_2}\dots x_{i_t} \mid t \geq 0, i < i_2 < \dots < i_t \leq r\}.$$

In particular to reach an element inside Hx_iK the exponents of the random elements $\mathbf{x}_j$ in the sequence $\mathbf{s}$ must be all 0 for $j < i$, precisely 1 for $j = i$ and arbitrary for $j > i$. Therefore the value of $P(\mathbf{s} \in Hx_iK)$ is given by

$$\prod_{j < i} P(\mathbf{x}_j = 1) \cdot P(\mathbf{x}_i = x_i) \cdot \sum_{S \subseteq \{x_{i+1}, \dots, x_r\}} \left(\prod_{j \in S} P(\mathbf{x}_j = x_j) \prod_{j \notin S} P(\mathbf{x}_j = 1) \right),$$

that is

$$P(\mathbf{s} \in Hx_iK) = (1 - p_2)(1 - p_3) \dots (1 - p_{i-1})p_i.$$

For all $i \geq 3$ we have that

$$\begin{aligned} \prod_{j=2}^{i-1} (1 - p_j) &= (1 - \delta_2) \left(1 - \frac{\delta_3}{1 - \delta_2} \right) \dots \left(1 - \frac{\delta_{i-1}}{1 - (\delta_2 + \dots + \delta_{i-2})} \right) \\ &= 1 - (\delta_2 + \dots + \delta_{i-1}) = \delta_i/p_i \end{aligned}$$

and therefore $P(\mathbf{s} \in Hx_iK) = \delta_i$. The claim follows. $\square$

We end this section with another condition for a transitive action of a finite group to be mixable.

Lemma 2.6. *Let $\mathbf{s}$ be a sequence for G and assume it mixes its image $A \subseteq G$ (which is not necessarily a subgroup of G). If H is a subgroup of G such that $HA = G$ and $|Ha \cap A| = |H \cap A|$ for all $a \in A$, then $\mathbf{s}$ mixes the right multiplication action of G on $\Omega = \{Hx \mid x \in G\}$.*

Proof. Consider the map $f : H \times A \rightarrow G$, $(h, a) \mapsto ha$, which is surjective by assumption. Fix $g = ha \in G$ and let $F = \{(x, y) \in H \times A \mid xy = g\}$. The map $Ha \cap A \rightarrow F$, $y \mapsto (gy^{-1}, y)$ is a bijection, hence $|F| = |Ha \cap A| = |H \cap A|$. Therefore $|G| = |f(H \times A)| = |H||A|/|H \cap A|$.

Since $HA = G$, the cosets of H in G have the form Ha for $a \in A$. We evaluate

$$P(\mathbf{s} \in Ha) = \sum_{h \in H} P(\mathbf{s} = ha) = \frac{|Ha \cap A|}{|A|} = \frac{|H \cap A|}{|A|} = \frac{|H|}{|G|} = \frac{1}{|G : H|},$$

in other words $\mathbf{s}$ mixes the action of G on Ω . $\square$

3. PROOF OF THEOREM 1.4

One direction of Theorem 1.4 follows from [1, Theorem 3.2] which states that any finite group with a nontrivial quotient of odd order is not mixable. For the other direction, let G be a finite solvable group with the property that whenever N is a normal subgroup in G with G/N of odd order then $N = G$. We claim that G is mixable and that $\ell(G) \leq \log_2(|G|_2) + |G|_{2'} - 1$, where ℓ is the mixing length.

Assume that G is a counterexample to the claim with $|G|$ minimal. If N is a nontrivial normal subgroup in G , then G/N is mixable with

$$\ell(G/N) \leq \log_2(|G/N|_2) + |G/N|_{2'} - 1,$$

by the fact that G is a minimal counterexample. Observe that this implies that N cannot be mixable, since otherwise G would be mixable by [1, Corollary 2.14] and with mixing length

$$\begin{aligned} \ell(G) &\leq \ell(G/N) + \ell(N) \\ &\leq \log_2(|G/N|_2) + |G/N|_{2'} - 1 + \log_2(|N|_2) + |N|_{2'} - 1 \\ &\leq \log_2(|G|_2) + |G|_{2'} - 1. \end{aligned}$$

Every proper nontrivial normal subgroup of G is therefore not mixable. The group G is not a 2-group by [1, Corollary 3.1] (note that when G is a 2-group then G is mixable of mixing length precisely $\log_2 |G|$). Let $N = O^2(G)$ be the intersection of all normal subgroups A of G such that G/A is a 2-group. This is a characteristic subgroup of G with the property that G/N is the largest possible quotient of G being a 2-group by [7, 6.3.2(a)]. The group N is proper in G since G is solvable. Let $L = O^{2'}(N)$ be the intersection of all normal subgroups A of N such that N/A is a $2'$ -group. This is a characteristic subgroup of N with the property that N/L is the largest possible quotient of N being a $2'$ -group, again by [7, 6.3.2(a)]. It follows that $O^{2'}(L) = L$ and since $|L| < |G|$, the group L is mixable. If $L \neq 1$, this is a contradiction. Thus $L = 1$ and G has the form $N \rtimes A$ where N is a normal subgroup of G of odd order and A is a Sylow 2-subgroup of G .

We claim that $|A| = 2$. Let Q be a maximal subgroup of A . Consider $T = \langle Q^x \mid x \in N \rangle$. Since Q is a Sylow 2-subgroup of NQ , we have $T = O^{2'}(NQ)$ and therefore $O^{2'}(T) = T$. Since $|T| < |G|$, the group T is mixable. Since T is a characteristic subgroup of NQ and NQ is normal in $G = NA$, the group T must be normal in G . If $T \neq 1$, this contradicts our assumptions on G , so $T = 1$, implying that $Q = 1$ so $|A| = 2$. This proves the claim.

We are in the following situation: $G = N \rtimes A$ with $|N|$ odd and $|A| = 2$. Let a be the generator of A . Let C be the group $C_N(A)$ of fixed points of A acting on N . The group $[N, A] = \langle [x, a] \mid x \in N, a \in A \rangle$ is a normal A -invariant subgroup of N . We have $N = [N, A]C$ by [7, 8.2.7]. It follows that the group $[N, A]A$ contains

all G -conjugates of A forcing that $[N, A]A = G$, since $O^{2'}(G) = G$. Therefore $[N, A] = N$.

We claim that $C \leq N'$. Consider the canonical projection $G \rightarrow G/N'$ and use the 'bar' notation to denote the image of $g \in G$ by $\bar{g}$ and the image of a subgroup $X \leq G$ by $\bar{X}$. We have $\bar{C} = C_{\bar{N}}(\bar{A})$ by [7, 8.2.2 (a)] and, since $[N, A] = N$, $[\bar{N}, \bar{A}] = \overline{[N, A]} = \bar{N} = [\bar{N}, \bar{A}] \times \bar{C}$ by [7, 8.4.2], so $\bar{C} = \bar{1}$.

Let M be a maximal subgroup of N containing N' . We claim that M is A -invariant. As $\bar{N} = N/N'$ is abelian and $\bar{A}$ acts on $\bar{N}$ fixed-point-freely by the previous paragraph, every element of $\bar{N}$ is inverted by the generator of $\bar{A}$ by [7, 8.1.8]. Thus for every $m \in M$ we have that $m^a \in N'm^{-1}$ and since $N' \leq M$ we deduce that M is A -invariant.

Let $\mathcal{S} = \{[x, a] \mid x \in N\}$. Then $\mathcal{S} = \{x \in N \mid x^a = x^{-1}\}$ by [7, 8.1.8]. Let H be the subgroup of M generated by $\mathcal{S} \cap M$. Since A inverts each element of $\mathcal{S} \cap M$, the subgroup H is A -invariant, and so HA is a subgroup of NA (and also of MA). Since $C \leq N' \leq M$, we have $C = C_M(A)$ and every coset of C in M contains an element of $\mathcal{S}$ by [7, 8.1.8], thus $M = CH$.

We claim that $H \trianglelefteq M$. Since $M = CH$ by the previous paragraph, it is sufficient to show that H is C -invariant. The subgroup C leaves invariant the set $\mathcal{S} = \{x \in N \mid x^a = x^{-1}\}$ and so also the set $\mathcal{S} \cap M$ which is a generating set for H . This proves the claim.

Recall that HA is a subgroup of G . Since $M = CH$, all MA -conjugates of A lie in HA , therefore $HA \trianglelefteq MA$ by the previous paragraph, moreover $O^{2'}(MA) \leq HA$. The group HA is generated by the set $(\mathcal{S} \cap M) \cup \{a\}$. For every element s in $\mathcal{S} \cap M$, the order of s is odd and every power of s also lies inside $\mathcal{S} \cap M$. Thus $\mathcal{S} \cap M = \{s^2 \mid s \in \mathcal{S} \cap M\}$. It follows that $\langle \mathcal{S} \cap M, a \rangle = \langle a^s \mid s \in \mathcal{S} \cap M \rangle$, since $a^s = s^{-1}as = as^2$ and $1 \in \mathcal{S} \cap M$. Thus $HA = O^{2'}(MA) = O^{2'}(HA)$. Therefore the group HA is a proper mixable subgroup of G and $\ell(HA) \leq |H|$.

Since $M \neq N$ and $N = \langle \mathcal{S} \rangle$, there exists an element $t \in \mathcal{S} \setminus M$. Setting $T = \langle t \rangle$ we have that $N = MT = HCT$. Since H is normal in MA , $K = H^T$ is normal in $G = NA$ and hence K is the normal closure of H in G . Every conjugate of H by any element in T is a normal subgroup in M . Thus $K \leq M$. We claim that $K = M$. For this it is sufficient to show that $C \leq K$. Consider the canonical projection $G \rightarrow G/K$ and for a subgroup X of G , let X^* be XK/K . The group $A^* \cong A$ acts on N^* and $N^* = [N^*, A^*]$, since $N = [N, A]$. We have that $C_{N^*}(A^*) = C^*$ by [7, 8.2.2 (a)]. Also, $C^* = CK/K = M/K = M^*$. Thus C^* is normal in N^* of prime index $p = |N : M|$. Since $[C, A] = 1$, $[A^*, C^*, N^*] = 1$ and $[C^*, N^*, A^*] \leq [C^*, A^*] = 1$, thus by the Three Subgroups Lemma ([7, 1.5.6]), $[N^*, A^*, C^*] = 1$. Since $N^* = [N^*, A^*]$, we deduce that C^* is central in N^* . Since $C^* = M^*$ has prime index p , it follows that N^* is abelian, so $N' \leq K$. But since $C \leq N'$, we conclude that $C \leq K$ and therefore $M = K = H^T$.

We claim that the core $(HA)_G$ of HA in G is trivial. Assume that $(HA)_G \neq 1$. Observe that $|(HA)_G| \geq 3$. The group $G/(HA)_G$ is mixable with mixing length $\ell(G/(HA)_G) \leq |G/(HA)_G|_{2'}$. Consider the action of G on the set X of right cosets of HA in G . This action is mixable if and only if the action of $G/(HA)_G$ on X is mixable. But the latter action is indeed mixable since $G/(HA)_G$ is mixable. We

have $\ell(G, X) = \ell(G/(HA)_G, X) \leq \ell(G/(HA)_G)$. Since HA is a mixable subgroup with mixing length $\ell(HA) \leq |H|$, by [1, Proposition 2.12], G is mixable with mixing length

$$\ell(G) \leq \ell(HA) + \ell(G, X) \leq |H| + |G/(HA)_G|_{2'} \leq |H| + \frac{|G|_{2'}}{3} \leq |G|_{2'}$$

which contradicts our assumptions on G . The last inequality above follows from $|H| \leq \frac{1}{3}|G|_{2'}$, being $|N|$ odd and H a proper subgroup of N .

We have that $N = MT$ and $T = \langle t \rangle$. We have $t^p \in \mathcal{S} \cap M \subseteq H$. In particular, $t^p \in H^{t^i}$ for every $i \geq 0$, that is

$$(1) \quad t^p \in \bigcap_{i \geq 0} H^{t^i} = \bigcap_{n \in N} H^n = H_N.$$

Since H is A -invariant and $t^a = t^{-1}$, $\bigcap_{i \geq 0} H^{t^i}$ is also A -invariant and so H_N is normal in G . Thus $H_N \leq (HA)_G$. Since $(HA)_G = 1$, we deduce that $H_N = 1$. Thus Equation (1) implies that t has order p and $N = M \rtimes T = H^T \rtimes T$.

We claim that $H^T = HH^t$. First, we prove that, if $x \in \mathcal{S} \cap M$, then $y = x^t x^{t^{-1}}$ belongs to $\mathcal{S} \cap M \subseteq H$. Since $x, t \in \mathcal{S}$, we have $x^a = x^{-1}$ and $t^a = t^{-1}$, so

$$y^a = (x^{-1})^{t^{-1}} (x^{-1})^t = y^{-1}.$$

This proves that y belongs to $\mathcal{S}$. But it also belongs to M because $x \in M$ and $M \trianglelefteq N$. So $y = x^t x^{t^{-1}} \in M \cap \mathcal{S} \subseteq H$. Conjugating with t gives $x^{t^2} x \in H^t$, so $x^{t^2} \in H^t x^{-1} \subseteq H^t H$ because $x \in H$. Therefore we have proved that $(M \cap \mathcal{S})^{t^2} \subseteq H^t H$. Recall that $H^t H \leq M$ because H^t and H are normal subgroups of M . But then

$$H^{t^2} = \langle M \cap \mathcal{S} \rangle^{t^2} = \langle (M \cap \mathcal{S})^{t^2} \rangle \subseteq \langle H^t H \rangle = H^t H.$$

Now we prove that $H^{t^i} \leq H^t H$ by induction on $i \in \{0, 1, \dots, p-1\}$. The case $i = 0$ is clear. Now assume $H^{t^{i-1}} \leq H^t H$. Then

$$H^{t^i} = (H^{t^{i-1}})^t \leq (H^t H)^t = H^{t^2} H^t \leq H^t H H^t = H^t H.$$

We deduce that $M = H^T = HH^t$ as claimed. In particular we have that

$$G = NA = TMA = THH^t A = (TA)(HA)(HA)^t.$$

We will apply Lemma 2.1 with the choice $H_0 = TA$, $H_1 = HA$, $H_2 = (HA)^t$. Let Ω be the set of right cosets of TA in G . Since $G = H^T TA$ and $H^T \cap TA = 1$, the group $H^T = HH^t$ acts regularly on Ω . In particular, H, H^t and $H \cap H^t$ act semiregularly on Ω and $|\Omega| = |HH^t| = |H||H^t|/|H \cap H^t|$.

Let Δ' be the H -orbit containing TA and let $\Sigma'_1, \dots, \Sigma'_k$ be the H^t -orbits of Ω where $k = |\Omega|/|H^t|$. Each Σ'_i intersects Δ' nontrivially since $G = (TA)HH^t$. Moreover, $|\Delta' \cap \Sigma'_i| \geq |H \cap H^t|$ for every i with $1 \leq i \leq k$. We have

$$|H| = |\Delta'| = \sum_{i=1}^k |\Delta' \cap \Sigma'_i| \geq \sum_{i=1}^k |H \cap H^t| = \frac{|\Omega|}{|H^t|} |H \cap H^t| = |H|,$$

from which it follows that $|\Delta' \cap \Sigma'_i| = |H \cap H^t|$ for every i with $1 \leq i \leq k$. Thus the quotients $|\Delta' \cap \Sigma'_i|/|\Sigma'_i|$ are constant, equal to $|H \cap H^t|/|H|$.

The H_1 -orbit Δ containing TA is equal to Δ' as A fixes TA . Since H^t is normal in $H_2 = (HA)^t$ of index 2, every H_2 -orbit Σ is equal to some Σ'_i or is the union of two such, say $\Sigma'_i \cup \Sigma'_j$. In any case, $|\Delta \cap \Sigma|/|\Sigma| = |\Delta' \cap \Sigma'_i|/|\Sigma'_i| = |H \cap H^t|/|H^t|$.

Thus all the assumptions of Lemma 2.1 are satisfied. We deduce that the action of G on Ω is mixable. Now $H_0 = TA$ is a mixable subgroup by [1, Corollary 8.2]. It follows that G is mixable by [1, Proposition 2.12]. Moreover, $\ell(HA) \leq |H|$ and $\ell(TA) \leq |T|$ by [1, Corollary 8.2], since TA is a dihedral group of order $2p$. If $H = 1$, then $M = H^T = 1$ and so $G = TA$ is dihedral and the result follows from [1, Corollary 8.2]. If $H \neq 1$, then

$$\ell(G) \leq \ell(TA) + 2\ell(HA) \leq |T| + 2|H| \leq |T||M| = |N| = |G|_{2'}.$$

This is the final contradiction. $\square$

4. PROOF OF THEOREM 1.2

We first reduce our considerations to the groups $\mathrm{SL}_2(q)$.

Proposition 4.1. *Let q be a prime power and let $n \geq 2$ be an integer. If $\mathrm{SL}_2(q)$ is mixable, then $\mathrm{SL}_n(q)$ is mixable.*

Proof. We proceed by induction on n . The case $n = 2$ is true by assumption. Let $n \geq 3$, $G := \mathrm{SL}_n(q)$, ω a generator of $\mathbb{F}_q^*$, and let

$$x := \left(\begin{array}{ccc|c} \omega & 0 & 0 & \\ 0 & \omega^{-1} & 0 & \\ 0 & 0 & 1 & \\ \hline & & & I \end{array} \right), \quad y := \left(\begin{array}{ccc|c} 0 & 1 & 0 & \\ 1 & 0 & 0 & \\ 0 & 0 & -1 & \\ \hline & & & I \end{array} \right), \quad z := \left(\begin{array}{ccc|c} 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 1 & 0 & 0 & \\ \hline & & & I \end{array} \right).$$

In other words x, y, z fix all basis vectors $v_4, \dots, v_n$ and

$$\begin{aligned} v_1x &= \omega v_1, & v_2x &= \omega^{-1}v_2, & v_3x &= v_3, \\ v_1y &= v_2, & v_2y &= v_1, & v_3y &= -v_3, \\ v_1z &= v_2, & v_2z &= v_3, & v_3z &= v_1. \end{aligned}$$

Let $A = \{g \in G \mid v_1g = v_1\}$ be the stabilizer of v_1 . The group A is isomorphic to the affine group $\mathrm{ASL}_{n-1}(q)$. This has a 2-transitive permutation representation with point stabilizer $\mathrm{SL}_{n-1}(q)$ ([4, Section 2.8]). Since $\mathrm{SL}_{n-1}(q)$ is mixable by induction, the group A is also mixable by [1, Corollary 9.2]. Since x normalizes A , the double coset Ax^iA is equal to Ax^i , for every integer i . The group A fixes the vectors v_1x^i for $i \in \{0, \dots, q-2\}$ and permutes all other nonzero vectors in one orbit. Thus the double cosets of A in G are Ax^iA for $i \in \{0, \dots, q-2\}$ and AyA . Let

$$D := \langle x, y \rangle \cong D_{2(q-1)}.$$

This is a mixable group by [1, Corollary 8.2]. Let F be a mixing sequence for A and let S be a mixing sequence for D . Consider the sequence $FSzF$ where z equals z with probability

$$p = 1 - \frac{|D|}{|G : A|} = 1 - \frac{2(q-1)}{q^n - 1}.$$

We claim that $FSzF$ is a mixing sequence for G . By Lemma 2.2 it is enough to prove that $P(Sz \in AgA) = |AgA|/|G|$ for all $g \in G$.

Note that $\text{Im}(S\mathbf{z}) = D \sqcup Dz$ and $D \cap A = \{1\}$. If $g \in Ax^iA = Ax^i$ for some $i \in \{0, 1, \dots, q-2\}$, then since $D \cap Ax^i = \{x^i\}$ and $Dz \cap Ax^i = \emptyset$ we have

$$P(S\mathbf{z} \in Ax^iA) = P(S\mathbf{z} = x^i) = (1-p)P(S = x^i) = \frac{1-p}{|D|} = \frac{|Ax^iA|}{|G|}.$$

If $g \in AyA$, then $\text{Im}(S\mathbf{z}) \cap AyA = \{x^iy | i = 0, 1, \dots, q-2\} \sqcup Dz$ and

$$\begin{aligned} P(S\mathbf{z} \in AyA) &= \sum_{i=0}^{q-2} P(S\mathbf{z} = x^iy) + P(S\mathbf{z} \in Dz) \\ &= (1-p) \sum_{i=0}^{q-2} P(S = x^iy) + pP(S \in D) \\ &= (1-p) \frac{q-1}{|D|} + p = \frac{1+p}{2} = \frac{|AyA|}{|G|} \end{aligned}$$

where the last equality holds by the definition of p and the fact that $\bigsqcup_{i=0}^{q-2} Ax^i \sqcup AyA$ is a partition of G . This concludes the proof. $\square$

We continue with the following.

Theorem 4.2. *The group $G = \text{SL}_2(q)$ is mixable if $q \equiv 1 \pmod{4}$.*

Proof. We have $|G| = q(q^2 - 1)$. Let T be the unique subgroup of $\mathbb{F}_q \setminus \{0\}$ of order 4 and, for $u, r \in \mathbb{F}_q$, $u \neq 0$, consider

$$\begin{aligned} x_{u,r} &:= \begin{pmatrix} u & r \\ 0 & u^{-1} \end{pmatrix} \in G, \\ y &:= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G, \\ H &:= \{x_{u,0} \mid 0 \neq u \in \mathbb{F}_q\} < G, \\ J &:= \{x_{t,r} \mid t \in T, r \in \mathbb{F}_q\} < G, \\ A &:= J \cup JyJ < G. \end{aligned}$$

Of course $|H| = q-1$, $|J| = 4q$, $|H \cap A| = 4$. Since $|J \cap J^y| = 4$, we have $|JyJ| = |J|^2/4 = 4q^2$ and $|A| = 4q + 4q^2 = 4q(q+1)$. The quotient $J/Z(G)$ is isomorphic to $\mathbb{F}_q \rtimes C_2$ with inversion and $|Z(G)| = 2$, so J is mixable by [1, Corollary 2.14 and Proposition 7.4]. If $\mathbf{s}$ is a mixing sequence for J , then $\mathbf{s}\mathbf{y}$ mixes A , where $\mathbf{y}$ equals y with probability $|JyJ|/|A| = q/(q+1)$ and it equals 1 with probability $|J|/|A| = 1/(q+1)$. Note that $HJ = B$ is the group of upper triangular matrices in G . Since $|B^y \cap J| = 4$, we have

$$|ByJ| = |B^yJ| = \frac{|B| \cdot |J|}{|B^y \cap J|} = \frac{1}{4}q(q-1) \cdot 4q = q^2(q-1) = |G| - |B|.$$

Since $BJ = B$, there are precisely two (B, J) double cosets in G which are B and ByJ . We conclude that $HA = B \sqcup ByJ = G$.

We claim that $Ha \cap A = (H \cap A)a$ for all $a \in A$. Consider the following element of G , for $t_i \in T$, $r_i \in \mathbb{F}_q$, $u \in \mathbb{F}_q^*$.

$$z(u, t_1, r_1, t_2, r_2) := x_{u,0} \cdot x_{t_1,r_1} \cdot y \cdot x_{t_2,r_2} = \begin{pmatrix} -ur_1t_2 & u(-r_1r_2 + t_1t_2^{-1}) \\ -u^{-1}t_1^{-1}t_2 & -u^{-1}t_1^{-1}r_2 \end{pmatrix}.$$

Then $JyJ = \{z(1, t_1, r_1, t_2, r_2) \mid t_i \in T, r_i \in \mathbb{F}_q\}$. Therefore $H \cap A = H \cap J$. Let $a \in A = J \cup JyJ$. If $a \in J$, the elements of Ha are upper triangular matrices so they do not belong to JyJ . We deduce that $Ha \cap A = Ha \cap J = (H \cap J)a = (H \cap A)a$. Now assume $a = z(1, t_1, r_1, t_2, r_2) \in JyJ$. Then $Ha = \{z(u, t_1, r_1, t_2, r_2) \mid u \in \mathbb{F}_q^*\}$. Clearly $Ha \cap J = \emptyset$, so $Ha \cap A = Ha \cap JyJ$. If $z(u, t_1, r_1, t_2, r_2) \in JyJ$, then $u \in T$, so $Ha \cap JyJ = (H \cap J)a \cap JyJ = (H \cap J)a = (H \cap A)a$.

So we can apply Lemma 2.6 and deduce that the action of G on $\{Hx \mid x \in G\}$ is mixable. Since y inverts every element of H and $y^2 = -I$, the group $H\langle y \rangle/Z(G)$ is a dihedral group, therefore $H\langle y \rangle$ is mixable by [1, Corollaries 2.14 and 8.2]. We deduce that G is mixable by [1, Propositions 2.13 and 2.15]. $\square$

Proposition 4.3. *The groups $\text{SL}_2(q)$ are mixable for $q \in \{7, 11, 19\}$.*

Proof. It is sufficient to show that the groups $\text{PSL}_2(7)$, $\text{PSL}_2(11)$ and $\text{PSL}_2(19)$ are mixable by [1, Corollary 2.14]. The mixability of the first group was established in [1, Corollary 11.5]. The group $\text{PSL}_2(11)$ has a 2-transitive action with point stabilizer a maximal subgroup isomorphic to A_5 , which is mixable.

Let $G = \text{SL}_2(19)$. The group G has a maximal subgroup H isomorphic to $\text{SL}_2(5)$. The rank of the corresponding permutation representation is 4. Gap [5] computations show that we can find $x_2, x_3, x_4 \in G$ such that $G = H \sqcup Hx_2H \sqcup Hx_3H \sqcup Hx_4H$, the elements x_2x_3, x_2x_4 and $x_2x_3x_4$ lie in Hx_2H , while the element x_3x_4 lies in Hx_3H . For example, we can have the following.

$$H = \left\langle \begin{pmatrix} 2 & 17 \\ 13 & 16 \end{pmatrix}, \begin{pmatrix} 12 & 12 \\ 0 & 8 \end{pmatrix} \right\rangle \cong \text{SL}_2(5),$$

$$x_2 = \begin{pmatrix} 11 & 11 \\ 13 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 15 & 12 \\ 10 & 3 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 2 & 15 \\ 3 & 4 \end{pmatrix}.$$

Lemma 2.5 completes the proof. $\square$

Theorem 1.2 follows from the three statements of this section, the fact that the group $\text{SL}_2(q)$ is mixable when q is a power of 2 [1, Theorem 10.2], the fact that every quotient of a mixable group is mixable [1, Corollary 2.11] and the fact that $\text{ASL}_n(q)$ is a 2-transitive permutation group with point stabilizer $\text{SL}_n(q)$. $\square$

5. PROOFS OF THEOREM 1.1 AND THEOREM 1.3

Theorems 1.1 and 1.3 follow from the following result. Specifically, Theorem 1.1 follows from (2) and [1, Theorem 8.1], while Theorem 1.3 follows from (3).

Theorem 5.1. *The following groups are mixable.*

- (1) $\text{PSU}_3(3)$, $\text{PSU}_4(2)$, $\text{PSU}_4(3)$, $\text{PSU}_6(2)$, $\text{Sp}_6(2)$, $G_2(4)$, $\text{O}_8^+(2)$.
- (2) *The finite irreducible Coxeter groups E_6, E_7 , and E_8 .*
- (3) *The sporadic simple groups J_2 , Suz , McL , Co_3 , Co_2 , Fi_{22} , Fi_{23} , Fi_{24}' , J_1 .*

Proof. In all cases except for J_1 and the groups in (2), we view the listed groups G as transitive permutation groups of degree d and rank $r \in \{2, 3\}$ with mixable point stabilizer H . The mixability of G then follows from [1, Corollary 9.2] or Corollary 2.4. We make use of information taken from [3] and from [4, Table B.2], where these

primitive actions (and their ranks) are collected. We also use [1, Corollary 2.14] for group extensions. In the case of $G = \text{PSU}_3(3)$, the maximal subgroup H chosen is solvable without nontrivial odd order quotients, so it is mixable by Theorem 1.4. Concerning sporadic simple groups, the information can be found in [9]. See also [8] and <https://www.math.rwth-aachen.de/homes/mfer/data/>.

G	H	d	r	G	H	d	r
$\text{PSU}_3(3)$	$3_+^{1+2} : 8$	28	2	Suz	$G_2(4)$	1782	3
$\text{PSU}_4(2)$	S_6	36	3	McL	$\text{PSU}_4(3)$	275	3
$\text{PSU}_4(3)$	$\text{PSU}_4(2)$	126	3	Co_3	$\text{McL}.2$	276	2
$\text{PSU}_6(2)$	$2_+^{1+8} : \text{PSU}_4(2)$	693	3	Co_2	$\text{PSU}_6(2).2$	2300	3
$\text{Sp}_6(2)$	$\text{PSU}_4(2) : 2$	28	2	Fi_{22}	$2.\text{PSU}_6(2)$	3510	3
J_2	$\text{PSU}_3(3)$	100	3	Fi_{23}	$2.\text{Fi}_{22}$	31671	3
$\text{G}_2(4)$	J_2	416	3	Fi'_{24}	Fi_{23}	306936	3

Concerning the finite irreducible Coxeter groups listed in the statement, we know that $\text{E}_6 \cong \text{PSU}_4(2).2$ (see [3, p. 26]), $\text{E}_7 \cong \text{Sp}_6(2) \times C_2$ (see [3, p. 46]), and $\text{E}_8 \cong 2.\text{O}_8^+(2).2$ (see [3, p. 85]). Moreover, $\text{O}_8^+(2)$ is mixable because it has a maximal subgroup isomorphic to $\text{Sp}_6(2)$ and the corresponding permutation representation has rank 3. So $\text{E}_6, \text{E}_7, \text{E}_8$ are mixable by the above discussion.

We finish with $G = \text{J}_1$. This group has a unique conjugacy class of maximal subgroups isomorphic to $\text{PSL}_2(11)$, which is mixable by Theorem 1.2. Let A be one of these subgroups and consider the action of G on the set of right cosets of A . This is a primitive action of degree 266 and rank 5. Moreover, a Gap [5] computation shows that there is an element $g \in G$ of order 19 with the following properties. If we set $x_1 := g$, $x_2 := g^2$, $x_3 := g^9$ and $x_4 := g^4$, then these are representatives of the nontrivial double cosets of A in G , whose sizes are

$$|Ax_1A| = 87120, \quad |Ax_2A| = 72600, \quad |Ax_3A| = 7920, \quad |Ax_4A| = 7260.$$

Let $C_i = \{S \subseteq \{1, 2, 3, 4\} \mid \prod_{k \in S} x_k \in Ax_iA\}$ for $i = 1, 2, 3, 4$, where in each product the terms x_k appear in increasing order with respect to k and $\prod_{k \in \emptyset} x_k := 1$. Then

$$\begin{aligned} C_1 &= \{\{1\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}, \\ C_2 &= \{\{2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}\}, \\ C_3 &= \{\{3\}, \{1, 3\}\}, \\ C_4 &= \{\{4\}, \{2, 3, 4\}\}. \end{aligned}$$

Imposing that the sequence $\mathbf{s} = \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4$ satisfies the assumption of Lemma 2.2, with probabilities p_i of $\mathbf{x}_i$, we find the following system of four equations, where $\delta_i = |Ax_iA|/|G|$ for all i .

$$\sum_{S \in C_i} \left[\prod_{k \in S} p_k \cdot \prod_{k \notin S} (1 - p_k) \right] = \delta_i, \quad i = 1, 2, 3, 4.$$

A computer algebra system found an admissible solution, with all $p_i \in [0, 1]$, of this system. $\square$

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