A strengthening of the Blaschke-Santaló inequality for o-symmetric planar convex bodies. This pdf-file is different from the printed version

Károly J. Böröczky, Endre Makai, Jr.

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Heartfully dedicated to Károly Bezdek on the occasion of his 70th birthday

Abstract

We verify the inequality

$$\frac{|K|}{|E|} + \frac{|K^*|}{|E^*|} \leq 2$$

for any o-symmetric convex body $K \subset \mathbb{R}^2$ where E is either the John ellipse of maximal area contained in K or the minimal area Löwner ellipse containing K. The analogous estimate may not hold if K is a planar but the assumption of o-symmetry is dropped, or if K is o-symmetric convex body in $\mathbb{R}^n$ for $n \geq 3$. Our new inequality strengthens the Blaschke-Santaló inequality for o-symmetric convex bodies $K \subset \mathbb{R}^2$ with an error term of optimal order.

1 Introduction

For background on the notions in convexity in the note, see Schneider [25]. We consider an inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$, $n \geq 2$, and write o to denote the origin and $\|x\| := \sqrt{\langle x, x \rangle}$ to denote the Euclidean norm of an $x \in \mathbb{R}^n$. In addition, let $B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the Euclidean unit ball and $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere. The Lebesgue measure of a measurable $X \subset \mathbb{R}^2$ is denoted by $|X|$. For an o-symmetric convex body $K \subset \mathbb{R}^n$, its polar is

$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall y \in K\},$$

that satisfies $(B^n)^* = B^n$, $(K^*)^* = K$ and

$$(\Phi K)^* = \Phi^{-t} K^* \tag{1}$$

for any $\Phi \in \mathrm{GL}(n)$. A classical result in convex geometry is the Blaschke-Santaló inequality, which reads as follows in the case of an o-symmetric convex bodies.

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Theorem A (Blaschke-Santaló inequality). If $K \subset \mathbb{R}^n$ is an o-symmetric convex body, then

$$|K| \cdot |K^*| \leq |B^n|^2, \tag{2}$$

with equality if and only if K is an ellipsoid.

The $n = 2, 3$ cases of (2) were due to Blaschke, and the inequality (2) for all $n \geq 2$ is proved by Santaló [24] in 1949. Actually, the equality case of (2) was clarified by Petty [27] only in 1985. Since then, various proofs of the Blaschke-Santaló inequality have been presented, for example, by [1, Theorem 7.3], [2], [17], [18], [19], [21], [22].

It follows from (1) that the left hand side of (2) does not change if K is replaced by ΦK for a linear transform $\Phi \in \mathrm{GL}(n)$. In particular, one may assume that the unit ball B^n is either the unique so-called John ellipsoid of largest volume contained in K, or the unique so-called Löwner ellipsoid of smallest volume containing K (see John [15], Ball [4] or Gruber, Schuster [13] about the existence and uniqueness of these ellipsoids). According to the characterization of the contact points due to John [15] (see also Gruber, Schuster [13]), assuming that $B^n \subset K$ for an o-symmetric convex body $K \subset \mathbb{R}^n$, B^n is the John ellipsoid of K if and only if there exist $c_1, \dots, c_k > 0$ and an o-symmetric set $\{u_1, \dots, u_k\} \subset S^{n-1} \cap \partial K$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = I_n \tag{3}$$

$$2n \leq k \leq n(n+1) \tag{4}$$

where $u_i \otimes u_i$ is the rank one $n \times n$ matrix $u_i u_i^t$. Similarly, assuming that assuming that $C \subset B^n$ for an o-symmetric convex body $C \subset \mathbb{R}^n$, B^n is the Löwner ellipsoid of C if and only if there exist $c_1, \dots, c_k > 0$ and an o-symmetric set $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ such that (3) and (4) hold. It follows via (1) that for any o-symmetric convex body $K \subset \mathbb{R}^n$ and o-symmetric ellipsoid $E \subset \mathbb{R}^n$

$$E \text{ is the John ellipsoid of } K \text{ if and only if } E^* \text{ is the Löwner ellipsoid of } K^*. \tag{5}$$

These results about the John and Löwner ellipsoids were implicit in Behrend [6] from 1937 in the planar case $n = 2$.

Our main goal is to prove the following strengthening of the Blaschke-Santaló inequality in the planar case.

Theorem 1.1. *If $K \subset \mathbb{R}^2$ is an o-symmetric convex body, such that either its John ellipse or its Löwner ellipse is the Euclidean unit ball B^2, then*

$$|K| + |K^*| \leq 2\pi, \tag{6}$$

and equality holds if $K = B^2$.

We note that Theorem 1.1 does not hold if K is the regular triangle incribed into B^2. In this case, B^2 is the minimal volume Löwner ellipse by the uniquenes and the linear invariance of the Löwner ellipse (see John [15] or Gruber, Schuster [13]), and also B^2 is the John ellipse of the circumscibed regular triangle K^*. It follows that $|K| + |K^*| = 6.49\ldots > 2\pi$. On the other hand, no analogue of Theorem 1.1 holds in higher dimensions, as if $n \geq 3$ and $K = [-1, 1]^n$, then $|K| + |K^*| > 2|B^n|$. For $n = 3$, this can be seen by direct calculations, and if $n \geq 4$, then already $|K| \geq 16 > 2|B^n|$.

Combining Theorem 1.1, (1) and (5), we deduce the following result that holds for any o-symmetric convex domain.

Corollary 1.2. *If $K \subset \mathbb{R}^2$ is an o-symmetric convex body, and E is either its John ellipse or its Löwner ellipse, then*

$$\frac{|K|}{2|E|} + \frac{|K^*|}{2|E^*|} \leq 1. \tag{7}$$

As a reverse inequality to Theorem 1.1, if a convex set $K \subset \mathbb{R}^2$ is either contains the unit disk B^2, or is contained in B^2 and contains the origin in their interior, then Florian [10, 11] showed that

$$|K| + |K^*| \geq 6, \tag{8}$$

with equality if and only if K is a square circumscribed around or inscribed into B^2.

As $|E| \cdot |E^*| = \pi^2$ by (1), Corollary 1.2 yields the planar Blaschke-Santaló inequality (2) by the AM-GM inequality between the arithmetic and geometric mean. In addition, we deduce the following stability version of the planar Blaschke-Santaló inequality. We note that according to Behrend [6], p. 726, if $E_J \subset K$ is the John ellipse and $E_L \supset K$ is the Löwner ellipse of the o-symmetric convex body $K \subset \mathbb{R}^2$, then

$$1 \geq \frac{|E_J|}{|K|} \geq \frac{\pi}{4} \text{ and } 1 \leq \frac{|E_L|}{|K|} \leq \frac{\pi}{2} \tag{9}$$

where equality holds in the inequalities on the left for ellipses, and on the right for parallelograms (see Ball [3] for an extension of (9) to any dimension). Stability versions of the Blaschke Santaló inequality in terms of the Banach-Mazur distance (that is weaker than the symmetric difference metric) have been obtained by Ball, Böröczky [5], Böröczky [7] and Ivaki [14].

Theorem 1.3. *Let $K \subset \mathbb{R}^2$ be an o-symmetric convex body with John ellipse $E_J \subset K$ and Löwner ellipse $E_L \supset K$. If*

$$|K| \cdot |K^*| \geq (1 - \varepsilon)\pi^2$$

for $\varepsilon \in [0, \frac{1}{2})$, then

$$|K \backslash E_J| \leq 4|K|\sqrt{\varepsilon} \text{ and } |E_L \backslash K| \leq 5|K|\sqrt{\varepsilon}. \tag{10}$$

Example 1.4 (The order $\sqrt{\varepsilon}$ of the error term in (10) is optimal)**.**
For $p > 1$, we consider the L_p ball

$$B_p^2 = \{(x, y) \in \mathbb{R}^2 : |x|^p + |y|^p \leq 1\},$$

and hence $B_2^2 = B^2$. We recall that according to Wang [28], we have

$$|B_p^2| = \frac{4 \cdot \Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{2}{p})} \tag{11}$$

where $\Gamma(\cdot)$ is Euler's Gamma function. If $|t|$ is small, then let $K_t = B_p^2$ for $p = 2 + t$, and hence (3) yields that the John ellipsoid of K_t is B^2 if $t \geq 0$, and the Löwner ellipsoid of K_t is B^2 if $t \leq 0$. In addition, $K_t^* = B_q^2$ holds for $q = (2 + t)/(1 + t)$.

We observe that

$$|K_t \backslash B^2| \geq c_1 t \qquad \text{if } t \geq 0, \tag{12}$$

$$|B^2 \backslash K_t| \geq c_1 t \qquad \text{if } t \leq 0 \tag{13}$$

for an absolute constant $c_1 > 0$. For the C^2 function $f(t) = |K_t| \cdot |K_t^*|$, combining $K_0 = B^2$ and the Blaschke-Santaló inequality (2) implies that f attains its maximum at $t = 0$, and hence $f'(0) = 0$, and, in turn, the Taylor formula yields that

$$f(t) \geq f(0) - c_2 t^2 \tag{14}$$

for an absolute constant $c_2 > 0$. We conclude from (12) and (14) that if $\varepsilon > 0$ is small, and we choose $t = \pm\sqrt{\varepsilon/c_2}$, then

$$|K_t| \cdot |K_t^*| \geq (1 - \varepsilon)\pi^2,$$

while $|K_t \backslash B^2| \geq \frac{c_1}{\sqrt{c_2}} \cdot \sqrt{\varepsilon}$ if $t > 0$, and $|B^2 \backslash K_t| \geq \frac{c_1}{\sqrt{c_2}} \cdot \sqrt{\varepsilon}$ if $t < 0$. This completes the proof of Example 1.4.

2 Area sum within a convex cone

For $X_1, \ldots, X_m \subset \mathbb{R}^2$, we write $[X_1, \ldots, X_m]$ to denote the convex hull of $X_1 \cup \ldots \cup X_m$; namely, the smallest convex set containing $X_1, \ldots, X_m$. Here $[X_1, \ldots, X_m]$ is compact if $X_1, \ldots, X_m$ are compact. For $x \neq y \in \mathbb{R}^2$, we write $\text{aff}\{x, y\}$ to denote the line passing through x, y, and write

$$(x, y) = [x, y] \backslash \{x, y\}$$

to denote the open segment. If $K, M \subset \mathbb{R}^2$ are convex bodies, then their Hausdorff distance is

$$\delta_H(K, M) = \min_{\varrho \geq 0} \{K \subset M + \varrho B^2 \text{ and } M \subset K + \varrho B^2\}.$$

It follows that if $M_n \subset \mathbb{R}^2$ is a sequence of convex bodies, then

$$\lim_{n\to\infty} \delta_H(M_n, M) = 0 \text{ if and only if } \lim_{n\to\infty} |M_n \Delta M| = 0. \tag{15}$$

In this case, we say that M_n tends to M with respect to the Hausdorff metric. According to the Blaschke Selection theorem, if for fixed $R > r > 0$, $M_n \subset RB^2$ is a sequence of convex bodies such that $x_n + rB^n \subset M_n$ for some $x_n \in M_n$, then there exists a subsequence $\{M_{n'}\}$ tending to some convex body with respect to the Hausdorff metric. One of our tools in this section is Steiner symmetrization. For a convex body $M \subset \mathbb{R}^2$, and a line ℓ through o, translate every secant of M orthogonal to l in its affine hull in a way such that the midpoint of the translated image lies on ℓ. The closure of the union of these translates is the Steiner symmetral M' with respect to ℓ. Readily M' is a convex body with $|M'| = |M|$. According to Keith Ball's PhD thesis [2] (see also Meyer, Pajor [22]), if $K \subset \mathbb{R}^2$ is an o-symmetric planar convex body and K' is the Steiner symmetral of K with respect to ℓ, then

$$|K'^*| \geq |K^*|. \tag{16}$$

Polarity is frequently considered for a $z \in \mathbb{R}^2 \backslash \{o\}$ or a line $\ell \subset \mathbb{R}^2 \backslash \{o\}$, and is defined by

$$\begin{aligned} z^* &= \{x \in \mathbb{R}^2 : \langle x, z \rangle = 1\} \\ \ell^* &= \{x \in \mathbb{R}^2 : \langle x, y \rangle = 1 \text{ for any } y \in \ell\}. \end{aligned}$$

It follows that z^* is a line in $\mathbb{R}^2\backslash\{o\}$ such that z is a normal vector and $(z^*)^* = z$, and ℓ^* is a normal vector to ℓ satisfying $(\ell^*)^* = \ell$. In addition, if $K \subset \mathbb{R}^2$ is an o-symmetric convex body and $\Phi \in \mathrm{GL}(2)$, then

$$\begin{array}{rl} z \in \ell & \text{if and only if } \ \ell^* \in z^*, \\ (\Phi z)^* = \Phi^{-t} z^* & \text{and } \ (\Phi\ell)^* = \Phi^{-t}\ell^*, \\ z \in \partial K & \text{if and only if } \ z^* \text{ is a supporting line to } K^*. \end{array} \tag{17}$$

Next, we set up the notation used in Proposition 2.1. For fixed $p, q \in S^1$ with $p \neq \pm q$, let r be the intersection of the tangent lines of B^2 at p and q. We consider the deltoid $C_{p,q} = [o, p, r, q]$ and the convex cone $\sigma = \{tp + sq : s, t \geq 0\}$, and the family

$$\mathcal{K}_{p,q} = \{M \subset C_{p,q} : M \text{ is a convex body and } o, p, q \in M\}.$$

To any $M \in \mathcal{K}_{p,q}$, we associate the o-symmetric convex body $K_M = [M, -M]$, and the ”relative polar”

$$M^\circ = \{x \in \sigma : \langle x, y\rangle \leq 1 \text{ for any } y \in M\} \subset C_{p,q},$$

and hence (17) yields that

$$M^\circ = K_M^* \cap \sigma \in \mathcal{K}_{p,q} \ \text{ and } \ (M^\circ)^\circ = M. \tag{18}$$

In addition, if $M_n \in \mathcal{K}_{p,q}$ is a sequence of convex bodies tending to a convex body $M \subset \mathbb{R}^n$ with respect to the Hausdorff metric as n tends to infinity, then $M \in \mathcal{K}_{p,q}$ and

$$\lim_{n\to\infty} M_n^\circ = M^\circ \ \text{ and } \ \lim_{n\to\infty} |M_n^\circ| = |M^\circ|. \tag{19}$$

Proposition 2.1. *For fixed $p, q \in S^1$ with $0 < \angle(p, o, q) \leq \frac{\pi}{2}$, if $M \in \mathcal{K}_{p,q}$, then*

$$|M| + |M^\circ| \leq \angle(p, o, q).$$

We prove some statements to prepare the proof Proposition 2.1. For $p, q \in S^1$ with $0 < \angle(p, o, q) \leq \frac{\pi}{2}$, let $\ell_{p,q}$ be the line passing through o and $\frac{p+q}{2}$, and hence $C_{p,q}$ is symmetric through $\ell_{p,q}$. We observe that $C_{p,q}^\circ = [0, p, q]$. We classify an element $M \in \mathcal{K}_{p,q}$ symmetric through $\ell_{p,q}$ into two types; namely, M is

type A, if $[p, r] \cap M$ (and hence $[q, r] \cap M$, as well) is a segment;

type B, if $[p, r] \cap M = \{p\}$ and $[q, r] \cap M = \{q\}$.

We observe that

$$\text{if } M \in \mathcal{K}_{p,q} \text{ is symmetric through } \ell_{p,q} \text{ and is of type A, then } M^\circ \text{ is of type B.} \tag{20}$$

Claim 2.2. *If $M \in \mathcal{K}_{p,q}$, and M' is the Steiner symmetral of M with respect to $\ell_{p,q}$, then $M' \in \mathcal{K}_{p,q}$ and*

$$|M'^\circ| \geq |M^\circ|. \tag{21}$$

Proof. Since $C_{p,q}$ is symmetric through $\ell_{p,q}$, we have $M' \in \mathcal{K}_{p,q}$.

We consider the o-symmetric convex body $K = [M, -M]$, and let f be the polar of aff $\{p, -q\}$ and let Σ be the closed strip bounded by the parallel lines aff $\{p, q\}$ and aff $\{-p, -q\}$. Since (17) yields that $K^* = [M^\circ, -M^\circ, f, -f]$, $K'^* = [M'^\circ, -M'^\circ, f, -f]$ and $K'^* \cap \Sigma = [\pm p, \pm, q, \pm f] = K^* \cap \Sigma$, we deduce from (16) that

$$2|M'^\circ| - 2\big|[0, p, q]\big| = \big|K'^* \backslash \Sigma\big| \geq |K^* \backslash \Sigma| = 2|M^\circ| - 2\big|[0, p, q]\big|,$$

proving (21). □

The next claim states that $C_{p,q}$ or $C^\circ_{p,q} = [o, p, q]$ can't be optimal.

Claim 2.3. *For $p, q \in S^1$ with $0 < \angle(p, o, q) \leq \frac{\pi}{2}$, there exists a type A pentagon $P \in \mathcal{K}_{p,q}$ symmetric through $\ell_{p,q}$ such that*

$$|C_{p,q}| + |C^\circ_{p,q}| < |P| + |P^\circ|.$$

Proof. For small $t > 0$, we consider the line $\ell_t = \{x \in \mathbb{R}^2 : \langle x, r\rangle = \langle r, r\rangle - t\}$ and the type A pentagon $P_t = \{x \in C_{p,q} : \langle x, r\rangle \leq \langle r, r\rangle - t\} \in \mathcal{K}_{p,q}$ symmetric through $\ell_{p,q}$, and hence $P^\circ_t = [o, p, q, \ell^*_t]$, and there exist constants $\gamma, \delta > 0$ such that $|P_t| = |C_{p,q}| - \gamma t^2$ and $|P^*_t| = |C^*_{p,q}| + \delta t$. In turn, we conclude Claim 2.3. □

The following simple well-known observation is the basis of our argument for Lemma 2.5.

Claim 2.4. *Let $\tau = [h_1, h_2]$ be a convex cone bounded by the two non-collinear half-lines h_1 and h_2 meeting at the apex a of τ, and let $m \in \operatorname{int} \tau$. We write ℓ_0 to denote the unique line passing through m such that m is the midpoint of the segment $\ell_0 \cap \tau$; namely, $b_i = \ell_0 \cap h_i$ is the intersection of h_i and the reflected image of h_{3-i} through m for $i = 1, 2$. In addition, let ℓ_i be the line through m parallel to h_i and intersecting h_{3-i} in c_{3-i} for $i = 1, 2$.*

If the line ℓ through m is rotated from the position ℓ_2 parallel to h_2 towards ℓ_0 in a way such that ℓ intersects h_1 in a point d_1 moving from c_1 towards b_1, then the area of the triangle cut off from τ by ℓ is strictly decreasing. In particular, among lines passing through m, ℓ_0 cuts off the smallest area triangle from τ.

Proof. Let $d_1, d'_1 \in (c_1, b_1)$ such that $d'_1 \in (d_1, b_1)$, let $\ell = \text{aff}\{m, d_1\}$ and $\ell' = \text{aff}\{m, d'_1\}$. If $d_2 = \ell \cap h_2$ and $d'_2 = \ell' \cap h_2$, then Claim 2.4 is equivalent to proving that $\big|[a, d_1, d_2]\big| > \big|[a, d'_1, d'_2]\big|$, which is, in turn, is equivalent to the statement

$$\big|[m, d_1, d'_1]\big| < \big|[m, d_2, d'_2]\big|. \tag{22}$$

However, $\|d_1 - m\| < \|d_2 - m\|$ as d_1 is closer to ℓ_2 than b_1, and, in turn, closer to ℓ_2 than d_2. As similarly $\|d'_1 - m\| < \|d'_2 - m\|$ and the angles of the triangles $[m, d_1, d'_1]$ and $[m, d_2, d'_2]$ at m coincide, we conclude (22). □

Next, we consider the variation of the "area sum" when a polygon $P \in \mathcal{K}_{p,q}$ is suitably deformed.

Lemma 2.5. *For $p, q \in S^1$ with $0 < \angle(p, o, q) \leq \frac{\pi}{2}$, let u, v, w be consecutive vertices of a polygon $P \in \mathcal{K}_{p,q}$ of at least 5 vertices such that $[v, w]$ intersects $\operatorname{int} C_{p,q}$, and let $\tilde{u} \in [u, v] \backslash \{v\}$ be such that the intersection s of $[o, v]$ and $[\tilde{u}, w]$ lies in $(\tilde{u}, \mu)$ for $\mu = (\tilde{u} + w)/2$. In addition, let $P_0 \in \mathcal{K}$ be the closure of $P \backslash [\tilde{u}, v, w]$.*

If $\tilde{v}-v$ is a non-zero vector parallel to $\ell_w=\mathrm{aff}\{\tilde{u},w\}$ for $\tilde{v}\in\mathrm{int}\,C_{p,q}$ such that the intersection $\tilde{s}$ of $[o,\tilde{v}]$ and ℓ_w lies in (s,μ) (and hence "v is moved parallel to ℓ_w towards w" into the position $\tilde{v}$), and the polygon $\widetilde{P}=[\tilde{v},P_0]\in\mathcal{K}$ still has w as a vertex, then

$$|\widetilde{P}|+|\widetilde{P}^\circ|>|P|+|P^\circ|. \tag{23}$$

Proof. For the line $\ell_v=\mathrm{aff}\{v,\tilde{v}\}$ parallel to ℓ_w, we observe that $\ell_v^*\in(0,\ell_w^*)\subset\mathrm{int}\,P_0^\circ$, and P° and $\widetilde{P}^\circ$ are obtained from P_0° by v^* and $\tilde{v}^*$, respectively, cut off the vertex ℓ_w^* of P_0° where both v^* and $\tilde{v}^*$ intersect both sides of P_0° containing ℓ_w^*; namely, the ones contained in $\tilde{u}^*$ and w^*. We note that the point $\ell_v^*\in\mathrm{int}\,P_0^\circ$ is contained in the lines v^* and $\tilde{v}^*$, and writing $\tilde{\mu}=\ell_v\cap\mathrm{aff}\,\{0,\mu\}$, the point ℓ_v^* is the midpoint of the intersection of the line $\tilde{\mu}^*$ with P_0° where again $\tilde{\mu}^*$ intersects both sides of P_0° containing ℓ_w^*. Since $\tilde{v}\in(v,\tilde{\mu})$, we deduce from Claim 2.4 that $|\widetilde{P}^\circ|>|P^\circ|$. As readily $|\widetilde{P}|=|P|$, we conclude (23). □

Observe that in Lemma 2.5, we have estimated $|P|+|P^\circ|$ from above, by fixing $|P|$, and increasing $|P^\circ|$. In the proof of Proposition 2.1, we will often apply this method. However, we also will often change the roles of P and P°. This amounts to estimate the sum of areas from above, by fixing the area of the polar set, and increasing the area of the original set.

The final auxiliary statement yields that p and q can be assumed orthogonal.

Claim 2.6. *For $p,q\in S^1$ with $0<\alpha=\angle(p,o,q)<\frac{\pi}{2}$, let $q'\in S^1$ be orthogonal to p with $\langle q,q'\rangle>0$. For $N\in\mathcal{K}_{p,q'}$, we write*

$$N^{\circ'}=\{x\in C_{p,q'}:\langle x,y\rangle\leq 1 \text{ for } y\in N\}.$$

If $M\in\mathcal{K}_{p,q}$ and $\aleph=\{sq+tq'\in B^2: s,t\geq 0\}$, then $M\cup\aleph\in\mathcal{K}_{p,q'}$, and

$$|M\cup\aleph|+\left|(M\cup\aleph)^{\circ'}\right|=|M|+|M^\circ|+\frac{\pi}{2}-\alpha. \tag{24}$$

Proof. Since the tangent line to B^2 at q is a supporting line to both M and $\aleph$, we deduce that $M\cup\aleph$ is convex, and hence $M\cup\aleph\in\mathcal{K}_{p,q'}$. Moreover, (17) yields that $(M\cup\aleph)^{\circ'}=M^\circ\cup\aleph$, proving (24). □

Proof of Proposition 2.1. According to Claim 2.6, we may assume that $p,q\in S^1$ are orthogonal in Proposition 2.1.

During the proof of Proposition 2.1, we approximate any element $M\in\mathcal{K}_{p,q}$ symmetric through $\ell_{p,q}$ by polygons symmetric through $\ell_{p,q}$.

Step 1. *Properties (i), (ii) and (iii) of some extremal polygons in $\mathcal{K}_{p,q}$.*

For $m\geq 0$, we write $\mathcal{P}^{(m)}$ to denote the family of polygons $P\in\mathcal{K}_{p,q}$ that are symmetric through $\ell_{p,q}$ and have exactly m sides that intersect $\mathrm{int}\,C_{p,q}$. In particular, the only element of $\mathcal{P}^{(0)}$ is $C_{p,q}$. For $m\geq 1$, we call any endpoint of a side of a $P\in\mathcal{P}^{(m)}$, $P\neq C_{p,q}$, intersecting $\mathrm{int}\,C_{p,q}$ a proper vertex of P. For $m\geq 1$, let $\mathcal{P}_A^{(m)}$ and $\mathcal{P}_B^{(m)}$ be the family of elements in $\mathcal{P}^{(m)}$ of type A and type B, respectively, and hence we deduce from (17) that

$$P\in\mathcal{P}_A^{(m)} \text{ if and only if } P^\circ\in\mathcal{P}_B^{(m+1)}. \tag{25}$$

For $m \geq 1$, we write $\widetilde{\mathcal{P}}^{(m)} = \bigcup_{i=0}^m \mathcal{P}^{(i)}$, $\widetilde{\mathcal{P}}_A^{(m)} = \bigcup_{i=0}^m \mathcal{P}_A^{(i)}$ and $\widetilde{\mathcal{P}}_B^{(m)} = \bigcup_{i=0}^m \mathcal{P}_B^{(i)}$, and hence if $P_n \in \widetilde{\mathcal{P}}_A^{(m)}$ is a sequence of polygons tending to a convex body Q, then

$$Q \in \widetilde{\mathcal{P}}^{(m)}, \tag{26}$$

and if $P_n \in \widetilde{\mathcal{P}}_B^{(m+1)}$ is a sequence of polygons tending to a convex body Q, then

$$\text{either } Q \in \widetilde{\mathcal{P}}_B^{(m+1)} \text{ or } Q \in \widetilde{\mathcal{P}}_A^{(m-1)}. \tag{27}$$

For $m \geq 6$, (25), (26) and (27) yield that there exists a polygon $Q_{(m)} \in \mathcal{K}_{p,q}$ such that either $Q_{(m)} \in \widetilde{\mathcal{P}}_A^{(m)}$ or $Q_{(m)} \in \widetilde{\mathcal{P}}_B^{(m+1)}$, and

$$\left|Q_{(m)}\right| + \left|Q_{(m)}^\circ\right| = \sup\left\{|P| + |P^\circ| : \text{ either } P \in \mathcal{P}_A^{(m)} \text{ or } P \in \mathcal{P}_B^{(m+1)}\right\}.$$

It follows from Claim 2.3 that $Q_{(m)} \neq C_{p,q}$ and $Q_{(m)} \neq [o,p,q]$. We may assume that the vertices o, p, r, q of $C_{p,q}$ are in clockwise order in this order, and let $x_0, \ldots, x_k$ be the proper vertices of $Q_{(m)}$ in clockwise order along $\partial Q_{(m)}$ where $1 \leq k \leq m$ and $x_0 \in [p,r]\backslash\{r\}$. We claim that

(i) $k \geq m-1$;

(ii) for any $i = 1, \ldots, k-1$, the points o, x_i and $(x_{i-1}+x_{i+1})/2$ are collinear;

(iii) for any $i = 2, \ldots, k-1$, if η_i is the intersection point of $\operatorname{aff}\{x_{i-1}, x_{i-2}\}$ and $\operatorname{aff}\{x_i, x_{i+1}\}$, then o, η_i and $(x_{i-1}+x_i)/2$ are collinear.

By the symmetry of the roles of $Q_{(m)}$ and $Q_{(m)}^\circ$, we also claim that the analogues of (i), (ii) and (iii) hold for the proper vertices of $Q_{(m)}^\circ$.

The main tool to verify (i) and (ii) is Lemma 2.5. For (i), we suppose that $k \leq m-2$, and seek a contradiction. Possibly replacing $Q_{(m)}$ by $Q_{(m)}^\circ$ (cf. (25)), we may asssume that $Q_{(m)}$ is of type A; namely,

$$Q_{(m)} \in \widetilde{\mathcal{P}}_A^{(m-2)}.$$

We choose an $x_{-1} \in (p, x_0)$ very close to x_0 in a way such that such that the intersection s_0 of $[o, x_0]$ and $[x_{-1}, x_1]$ lies in (x_{-1}, μ_0) for $\mu_0 = (x_{-1}+x_1)/2$. We slightly move x_0 parallel to $\operatorname{aff}\{x_{-1}, x_1\}$ into a position $\tilde{x}_0 \in \operatorname{int} C_{p,q}$ in a way such that the intersection $\tilde{s}_0$ of $[o, \tilde{x}_0]$ and $[x_{-1}, x_1]$ lies in (s, μ_0). We deduce from Lemma 2.5 that

$$|R| + |R^\circ| > \left|Q_{(m)}\right| + \left|Q_{(m)}^\circ\right|$$

holds for $R = [q, o, p, x_{-1}, \tilde{x}_0, x_1, \ldots, x_k] \in \mathcal{K}_{p,q}$. Now, we distinguish two cases:

- If $k \geq 2$, then we set x_{k+1} to be the reflected image of x_{-1} through $\ell_{p,q}$, and move x_k into the position $\tilde{x}_k \in \operatorname{int} C_{p,q}$ parallel to $\operatorname{aff}\{x_{k-1}, x_{k+1}\}$ in a way such that $\tilde{x}_k$ is the reflected image of $\tilde{x}_0$ through $\ell_{p,q}$. The argument above based on Lemma 2.5 proves that

$$\left|Q'_{(m)}\right| + \left|Q'^\circ_{(m)}\right| > |R| + |R^\circ| > \left|Q_{(m)}\right| + \left|Q_{(m)}^\circ\right|$$

  holds for $Q'_{(m)} = [q, o, p, x_{-1}, \tilde{x}_0, x_1, \ldots, x_{k-1}, \tilde{x}_k, x_{k+1}] \in \widetilde{\mathcal{P}}_A^{(k+2)}$. As $k+2 \leq m$, this contradiction verifies (i) in this case.

- If $k = 1$, then we still set $x_{k+1} = x_2$ to be the reflected image of x_{-1} through $\ell_{p,q}$. Possibly choosing x_{-1} and $\tilde{x}_0$ even closer to x_0, if we slightly move x_1 into the position $\tilde{x}_1 \in \operatorname{int} C_{p,q}$ parallel to $\operatorname{aff}\{\tilde{x}_0, x_2\}$ in a way such that $\operatorname{aff}\{\tilde{x}_0, \tilde{x}_1\}$ is parallel to $\operatorname{aff}\{x_{-1}, x_2\}$, then

$$\left|\widetilde{Q}_{(m)}\right| + \left|\widetilde{Q}^\circ_{(m)}\right| > |R| + |R^\circ| > \left|Q_{(m)}\right| + \left|Q^\circ_{(m)}\right|$$

holds for $\widetilde{Q}_{(m)} = [q, o, p, x_{-1}, \tilde{x}_0, \tilde{x}_1, x_2] \in \mathcal{K}_{p,q}$ by Lemma 2.5. Therefore, if $\widetilde{Q}'_{(m)} \in \widetilde{\mathcal{P}}^{(3)}_A$ is the Steiner symmetral of $\widetilde{Q}_{(m)}$ with respect to $\ell_{p,q}$, then (21) yields

$$\left|\widetilde{Q}'_{(m)}\right| + \left|\widetilde{Q}'^\circ_{(m)}\right| \geq \left|\widetilde{Q}_{(m)}\right| + \left|\widetilde{Q}^\circ_{(m)}\right| > \left|Q_{(m)}\right| + \left|Q^\circ_{(m)}\right|,$$

which contradiction verifies (i) in all cases.

Next, we prove (ii) again indirectly; namely, we suppose that there exists $i \in \{1, \ldots, k-1\}$ such that the points o, x_i and $\mu_i = (x_{i-1} + x_{i+1})/2$ are not collinear, and seek a contradiction. Here we may assume that $i < k/2$ by the symmetry through $\ell_{p,q}$, and hence $k \geq i+2$. Let $s_i \neq \mu_i$ be the intersection point of $[o, x_i]$ and $[x_{i-1}, x_{i+1}]$. We slightly move x_i parallel to $\operatorname{aff}\{x_{i-1}, x_{i+1}\}$ into a position $\tilde{x}_i \in \operatorname{int} C_{p,q}$ in a way such that the intersection $\tilde{s}_i$ of $[o, \tilde{x}_i]$ and $[x_{i-1}, x_{i+1}]$ lies in (s_i, μ_i), and both x_{i-1} and x_{i+1} stay vertices of

$$R = [q, o, p, x_0, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_k] \in \mathcal{K}_{p,q}.$$

We deduce from Lemma 2.5 that

$$|R| + |R^\circ| > \left|Q_{(m)}\right| + \left|Q^\circ_{(m)}\right|.$$

Again, we distinguish two cases:

- If $k-i > i+1$, then we move x_{k-i} into the position $\tilde{x}_{k-i} \in \operatorname{int} C_{p,q}$ parallel to $\operatorname{aff}\{x_{k-i-1}, x_{k-i+1}\}$ in a way such that $\tilde{x}_{k-i}$ is the reflected image of $\tilde{x}_i$ through $\ell_{p,q}$. The argument above based on Lemma 2.5 proves that

$$\left|Q'_{(m)}\right| + \left|Q'^\circ_{(m)}\right| > |R| + |R^\circ| > \left|Q_{(m)}\right| + \left|Q^\circ_{(m)}\right|$$

holds for $Q'_{(m)} = [q, o, p, x_0, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_{k-i-1}, \tilde{x}_{k-i}, x_{k-i+1}, \ldots, x_k] \in \widetilde{\mathcal{P}}^{(m)}$. This contradiction verifies (ii) in this case.

- Let $k - i = i + 1$. After possibly choosing $\tilde{x}_i$ even closer to x_i, if we move x_{i+1} into the position $\tilde{x}_{i+1} \in \operatorname{int} C_{p,q}$ parallel to $\operatorname{aff}\{\tilde{x}_i, x_{i+2}\}$ in a way such that $\operatorname{aff}\{\tilde{x}_i, \tilde{x}_{i+1}\}$ is parallel to $\operatorname{aff}\{x_{i-1}, x_{i+2}\}$, and $\tilde{x}_i$ and x_{i+2} are vertices of

$$\widetilde{Q}_{(m)} = [q, o, p, , x_0, \ldots, x_{i-1}, \tilde{x}_i, \tilde{x}_{i+1}, x_{i+2}, \ldots, x_k] \in \mathcal{K}_{p,q},$$

then Lemma 2.5 yields that

$$\left|\widetilde{Q}_{(m)}\right| + \left|\widetilde{Q}^\circ_{(m)}\right| > |R| + |R^\circ| > \left|Q_{(m)}\right| + \left|Q^\circ_{(m)}\right|.$$

Therefore, if $\widetilde{Q}'_{(m)} \in \widetilde{\mathcal{P}}^{(k)}$ is the Steiner symmetral of $\widetilde{Q}_{(m)}$ with respect to $\ell_{p,q}$, then (21) yields

$$\left|\widetilde{Q}'_{(m)}\right| + \left|\widetilde{Q}'^\circ_{(m)}\right| \geq \left|\widetilde{Q}_{(m)}\right| + \left|\widetilde{Q}^\circ_{(m)}\right| > \left|Q_{(m)}\right| + \left|Q^\circ_{(m)}\right|,$$

which contradiction verifies (ii) in all cases.

Finally, (ii) for $Q^\circ_{(m)}$ implies (iii) for $Q_{(m)}$ by (17).

Step 2. *Intersections with ellipses are candidates for extremality.*

We recall that we have assumed that p and q are orthogonal based on Claim 2.6, and hence $\|r\| = \sqrt{2}$ and $(p+q)/2 = r/2$.

Having proved all the three properties (i), (ii) and (iii) of $Q_{(m)}$ for $m \geq 6$, the symmetric role of $Q_{(m)}$ and $Q^\circ_{(m)}$ allows us to assume that $Q_{(m)}$ is of type B (cf. (25)), thus p and q are proper vertices of $Q_{(m)}$. Setting

$$\Xi = \max_{M \in \mathcal{K}_{p,q}} \left(|M| + |M^\circ|\right),$$

we deduce from Claim 2.2 that

$$\Xi = \max_{\substack{M \in \mathcal{K}_{p,q} \\ \text{symmetric through } \ell_{p,q}}} \left(|M| + |M^\circ|\right). \tag{28}$$

Now, any $M \in \mathcal{K}_{p,q}$ symmetric through $\ell_{p,q}$ can be arbitrary well approximated by polygons in $\mathcal{K}_{p,q}$ symmetric through $\ell_{p,q}$. Therefore, the definition of $Q_{(m)}$ yields that

$$\Xi = \lim_{m \to \infty} \left(|Q_{(m)}| + |Q^\circ_{(m)}|\right). \tag{29}$$

It follows from Claim 2.3 that there exists $\varepsilon > 0$ and points $\xi_0, \xi \in (r, (p+q)/2)$ such that for any $M \in \mathcal{K}_{p,q}$ symmetric through $\ell_{p,q}$, we have

$$|M| + |M^\circ| > \Xi - \varepsilon \;\text{ implies }\; \xi_0 \in M \;\text{ and }\; \xi \notin M.$$

We deduce from (29) that if m is large, then

$$\xi_0 \in Q_{(m)} \;\text{ and }\; \xi \notin Q_{(m)}. \tag{30}$$

We recall that $\sigma = \{tp + sq : s, t \geq 0\}$. It follows from (ii) and (iii) that for any $m \geq 6$, there exists an o-symmetric ellipse $E_{(m)}$ symmetric through $\ell_{p,q}$ such that all proper vertices of $Q_{(m)}$ (including p and q) lie on $\partial E_{(m)}$, and they are equally spaced according to the metric determined by $E_{(m)}$. In addition, the sides of $Q^\circ_{(m)}$ meeting $\operatorname{int} \sigma$ touch $E^*_{(m)}$ in their midpoints.

We deduce from (30) that if m is large, then

$$\xi_0 \in E_{(m)} \;\text{ and }\; r \notin E_{(m)}. \tag{31}$$

For the orthonormal basis u, v of $\mathbb{R}^2$ where $u = r/\sqrt{2}$ and $\langle v, p\rangle = \sqrt{2}/2$, we consider the equation for the boundary $\partial E_{(m)}$. According to (31) and $p, q \in \partial E_{(m)}$, there exist $\sqrt{2}/2 < \|\xi_0\| \leq a_m < \sqrt{2}$ and some $b_m > \sqrt{2}/2$ such that $xu + yv \in \partial E_{(m)}$ if and only if

$$\frac{x^2}{a_m^2} + \frac{y^2}{b_m^2} = 1 \;\text{ where }\; \frac{1}{2a_m^2} + \frac{1}{2b_m^2} = 1.$$

Here $a_m \geq \|\xi_0\| > \sqrt{2}/2$ yield that the sequence $\{b_m\}$ is also bounded; therefore, there exists a subsequence $\{E_{(m')}\}$ that tends to an ellipse E in the Hausdorff metric, and we claim that E satisfies that

(a) E is o-symmetric, symmetric through $\ell_{p,q}$ and $p,q\in\partial E$,

(b) $E\cap\sigma=E\cap C_{p,q}\in\mathcal{K}_{p,q}$,

(c) $|E\cap C_{p,q}|+\big|(E\cap C_{p,q})^\circ\big|=\Xi$.

Here (a) follows from the corresponding properties of each $E_{(m)}$. Since the proper vertices of $Q_{(m)}$ lie on $\partial E_{(m)}$, and they are equally spaced according to the metric determined by $E_{(m)}$, the property (i) of $Q_{(m)}$ implies that

$$\lim_{m\to\infty}\Big|\left(\sigma\cap E_{(m)}\right)\Delta Q_{(m)}\Big|=\lim_{m\to\infty}\left(\big|\sigma\cap E_{(m)}\big|-|Q_{(m)}|\right)=0. \tag{32}$$

Using (15) and (32), we deduce (b) from $Q_{(m)}\subset C_{p,q}$, and (c) from (19) and (29).

Step 3. *Search for the optimal ellipse.*

Since p and q are assumed to be orthogonal by Claim 2.6, Proposition 2.1 will follow if we prove for the ellipse E in (a), (b) and (c) that

$$\big|C_{p,q}\cap E\big|+\big|(C_{p,q}\cap E)^\circ\big|\leq\frac{\pi}{2}. \tag{33}$$

We use the orthonormal basis u,v of $\mathbb{R}^2$ as in Step 2. Let $a,b>0$ be the half axes of E where (cf. (a)) $au\in\partial E$ and $bv\in\partial E$, and (a) and (b) imply that $\sqrt{2}/2<a\leq 1\leq b$. In particular, $E=\Phi B^2$ for a diagonal matrix Φ with entries a and b on the diagonal, and

$$C_{p,q}\cap E=\Phi\aleph$$

where $\aleph=\{te+sf\in B^2:s,t\geq 0\}$ for $e=u\cos\frac{\beta}{2}+v\sin\frac{\beta}{2}$, $f=u\cos\frac{\beta}{2}-v\sin\frac{\beta}{2}$ and $\beta\in(0,\pi/2]$, and β satisfies

$$a\cos\frac{\beta}{2}=b\sin\frac{\beta}{2}=\frac{1}{\sqrt{2}}\quad\text{and}\quad\tan\frac{\beta}{2}=\frac{a}{b}\leq 1\quad\text{and}\quad e=\frac{u}{a\sqrt{2}}+\frac{v}{b\sqrt{2}}.$$

It follows from (17) that

$$(C_{p,q}\cap E)^\circ=\left[o,p,\Phi^{-t}e\right]\cup\Phi^{-t}\aleph\cup\left[o,q,\Phi^{-t}f\right];$$

therefore,

$$\big|C_{p,q}\cap E\big|+\big|(C_{p,q}\cap E)^\circ\big|=\frac{ab\beta}{2}+\frac{\beta}{2ab}+\frac{1}{2a^2}-\frac{1}{2b^2}.$$

As $\beta\in(0,\frac{\pi}{2}]$, we can parametrize it by $t=\cos\beta\in[0,1)$, and consider

$$f(t)=\big|C_{p,q}\cap E\big|+\big|(C_{p,q}\cap E)^\circ\big|=t+\frac{2-t^2}{\sqrt{1-t^2}}\arctan\sqrt{\frac{1-t}{1+t}}.$$

We note that $\arctan s<s$ for $s\in(0,1)$. Therefore, if $t\in(0,1)$, then

$$f'(t)=\frac{-t^2}{2(1-t^2)}+\frac{t^3}{(1-t^2)^{3/2}}\arctan\sqrt{\frac{1-t}{1+t}}<\frac{-t^2}{2(1-t^2)}\left(1-\frac{2t}{1+t}\right)<0.$$

We conclude that

$$\big|C_{p,q}\cap E\big|+\big|(C_{p,q}\cap E)^\circ\big|=f(t)\leq f(0)=\pi/2,$$

proving (33), and in turn Proposition 2.1. □

3 Proof of Theorem 1.1 and Theorem 1.3

Proof of Theorem 1.1. Let $K \subset \mathbb{R}^2$ be a an o-symmetric convex body such that $B^2 \subset K$ is the John ellipse of K. According to Behrend [6], p. 734, there exist a $k \in \{4, 6\}$ and an o-symmetric set $\{u_1, \ldots, u_k\} \subset S^1 \cap \partial K$ such that $u_1, \ldots, u_k$ are in clockwise order along S^1, and setting $u_0 = u_k$, we have $\angle(u_{i-1}, o, u_i) \leq \frac{\pi}{2}$ for $i = 1, \ldots, k$. This property also follows from (3) and (4).

As the supporting line to K at any u_i has to be a supporting line of B^2, as well, we deduce that

$$K \cap \sigma_i \subset C_{u_{i-1},u_i} \quad \text{and} \quad K^* \cap \sigma_i = (K \cap \sigma_i)^\circ$$

where $\sigma_i = \{su_{i-1} + tu_i : s, t \geq 0\}$ for $i = 1, \ldots, k$, and $(K \cap \sigma_i)^\circ$ is defined as a set in $\mathcal{K}_{u_{i-1},u_i}$. We conclude from Proposition 2.1 that

$$|K \cap \sigma_i| + |K^* \cap \sigma_i| \leq \angle(u_{i-1}, o, u_i)$$

for $i = 1, \ldots, k$; therefore, $|K| + |K^*| \leq 2\pi$.

Finally, if $B^2 \supset K$ is the Löwner ellipse, then the proof is the same word by word. □

Proof of Theorem 1.3. Let $K \subset \mathbb{R}^2$ be an o-symmetric convex body satisfying $|K| \cdot |K^*| \geq (1-\varepsilon)\pi^2$ for $\varepsilon \in (0, \frac{1}{2})$.

First, we consider the case of Theorem 1.3 concerning the John ellipse is E_J of K where we may asssume that $E_J = B^2$, and hence $K^* \subset B^2 \subset K$. It follows from Behrend's inequality (9) that $|K| \leq 4$, thus $\sqrt{|K|} + \sqrt{\pi} < 4$. We deduce from $\sqrt{1-\varepsilon} > 1 - \varepsilon$, the AM-GM inequality and Theorem 1.1 that

$$(1-\varepsilon)2\pi < 2\sqrt{|K| \cdot |K^*|} \leq |K| + |K^*| \leq 2\pi, \tag{34}$$

and hence using the formula $|K| + |K^*| - 2\sqrt{|K| \cdot |K^*|} = \left(\sqrt{|K|} - \sqrt{|K^*|}\right)^2$, we have

$$2\pi\varepsilon > \left(\sqrt{|K|} - \sqrt{|K^*|}\right)^2 \geq \left(\sqrt{|K|} - \sqrt{\pi}\right)^2 = \frac{(|K| - \pi)^2}{\left(\sqrt{|K|} + \sqrt{\pi}\right)^2} \geq \frac{(|K| - \pi)^2}{16};$$

therefore, $|K| \geq \pi$ yields that

$$|K \backslash B^2| \leq \sqrt{32\pi} \cdot \sqrt{\varepsilon} < 4|K|\sqrt{\varepsilon}.$$

Next, we consider the Löwner ellipse E_L of K where we may asssume that $E_L = B^2$, and hence $K \subset B^2 \subset K^*$. As (34) holds again, we have

$$2\pi\varepsilon > \left(\sqrt{|K^*|} - \sqrt{|K|}\right)^2 \geq \left(\sqrt{\pi} - \sqrt{|K|}\right)^2 = \frac{(\pi - |K|)^2}{\left(\sqrt{|K|} + \sqrt{\pi}\right)^2} \geq \frac{(\pi - |K|)^2}{4\pi};$$

therefore, as Behrend's inequality (9) yields that $|K| \geq 2$, we have

$$|B^2 \backslash K| \leq \sqrt{8\pi^2} \cdot \sqrt{\varepsilon} < 5|K|\sqrt{\varepsilon},$$

completing the proof of Theorem 1.3. □

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References

[1] B. Andrews, Contraction of convex hypersurfaces by their affine normal. J. Differential Geom., 43 (1996), no. 2, 207-230.

[2] K. M. Ball, Isometric problems in ℓ_p and sections of convex sets. PhD thesis, University of Cambridge, 1986.

[3] K. Ball: Volumes of sections of cubes and related problems. In: Geometric Aspects of Functional Analysis, Lecture Notes in Math., 1376, Springer, 1989, 251-260.

[4] K. Ball: Ellipsoids of maximal volume in convex bodies. Geom. Dedicata, 41 (1992), 241-250.

[5] K.M. Ball, K.J. Böröczky: Stability of some versions of the Prkopa-Leindler inequality. Monatsh. Math. 163 (2011), 1-14.

[6] F. Behrend: Über einige Affininvarianten konvexer Bereiche. Math. Ann., 113 (1937), 713-747.

[7] K.J. Böröczky, Stability of the Blaschke-Santaló and the affine isoperimetric inequality. Adv. Math. 225 (2010), no. 4, 1914-1928.

[8] K.J. Böröczky, E. Makai, Jr., M. Meyer, S. Reisner: On the volume product of planar polar convex bodies - lower estimates with stability. Studia Sci. Math. Hungar. 50 (2013), 159-198.

[9] A. Figalli, F. Maggi, A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math., 182 (2010), 167-211.

[10] A. Florian: On the area sum of a convex polygon and its polar reciprocal. Math. Pannon. 6 (1995), 77-84.

[11] A. Florian: On the area sum of a convex set and its polar reciprocal. Math. Pannon., 7 (1996), 171-176.

[12] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative isoperimetric inequality. Ann. of Math., (2), 168 (2008), 941–980.

[13] P.M. Gruber, F.E. Schuster: An arithmetic proof of John's ellipsoid theorem. Arch. Math., 85 (2005), 82-88.

[14] M. Ivaki: Stability of the Blaschke-Santaló inequality in the plane. Monatsh. Math. 177 (2015), 451-459.

[15] F. John: Extremum problems with inequalities as subsidiary conditions. In: Studies and Essays presented to R. Courant on his 60th Birthday, Interscience Publishers, (1948) 187-204.

[16] P. Kalantzopoulos, C. Saroglou, On a j-Santaló Conjecture, Geom. Dedicata, 217 (2023) (article # 91).

[17] J. Lehec, Partitions and functional Santaló inequalities. Arch. Math. (Basel), 92 (2009), 89-94.

[18] J. Lehec: A direct proof of the functional Santaló inequality. C. R. Math. Acad. Sci. Paris, 347 (2009), 55-58.

[19] E. Lutwak: Extended affine surface area. Adv. Math. 85 (1991), 39-68.

[20] E. Lutwak: Selected affine isoperimetric inequalities. In: Handbook of convex geometry, Vol. A, B, 151–176, North-Holland, Amsterdam, 1993.

[21] E. Lutwak, G. Zhang: Blaschke-Santaló inequalities, J. Differential Geom. 47(1) (1997), 1-16.

[22] M. Meyer, A. Pajor: On the Blaschke-Santaló inequality, Arch. Math. (Basel) 55 (1990), 82-93.

[23] M. Meyer, S. Reisner: On the volume product of polygons. Abh. Math. Semin. Univ. Hambg., 81 (2011), 93-100.

[24] L. A. Santaló: An affine invariant for convex bodies of n-dimensional space, Port. Math., 8 (1949), 155–161 (in Spanish).

[25] R. Schneider: Convex Bodies: The Brunn-Minkowski Theory, second edition. Cambridge University Press, 2014.

[26] A. C. Yao and F. F. Yao, A general approach to d-dimensional geometric queries, in Proceedings of the seventeenth annual ACM symposium on Theory of computing, ACM Press (1985) 163–168.

[27] C.M. Petty: Affine isoperimetric problems. In: Discrete geometry and convexity (New York, 1982), Ann. New York Acad. Sci., vol. 440, 1985, 113-127.

[28] Xianfu Wang: Volumes of Generalized Unit Balls. Mathematics Magazine, 78 (2005), 390-395.

Károly J. Böröczky, HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary
boroczky.karoly.j@renyi.hu and ELTE, Institute of Mathematics, Budapest, Hungary

Endre Makai, Jr., HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary
makai.endre@renyi.hu