7 Seventh exercise set

E 7.1. Show that the free group $\mathbb{F}_d$ ($d \geq 2$) admits a paradoxical decomposition.

E 7.2. Let $\Gamma = \langle S \rangle$, $|S| < \infty$ be a finitely generated group, and assume $\mathrm{Cay}(\Gamma, S)$ has a positive Cheeger costant. Show that Γ admits a paradoxical decomposition.

E 7.3. Let T_d° denote the d-ary rooted tree, and let $\Gamma \leq \mathrm{Aut}(T_d^\circ)$ be a group of automorphisms of T_d°. Denote by μ the natural measure on ∂T_d°. (One can pick a μ-random infinite non-backtracking path from the root by making a uniform random choice among the d continuations at every step.) Show that Γ acts in a measure preserving way on $(\partial T_d^\circ, \mu)$. Show that the action is ergodic if and only if Γ acts transitively on all levels of T_d°.

E 7.4. Let B denote the infinite rooted binary tree. Show that a random element of $\mathrm{Aut}(B)$ acts freely on the boundary of B with probability 1.

E 7.5. Show that an ergodic, infinite index nontrivial IRS of the free group has infinite rank almost surely.

E 7.6. Let Γ be a countably infinite group, and

$$\Omega = \{0,1\}^\Gamma = \{\omega : \Gamma \to \{0,1\}\}$$

be the space of all 0-1-colorings with the product topology. Let u_2 stand for the uniform $\{1/2, 1/2\}$ measure on $\{0,1\}$ and endow Ω with the product measure $\mu = u_2^\Gamma$. Γ acts on (Ω, μ) by shifting the coloring:

$$g.\omega(h) = \omega(g^{-1}h).$$

Show that this action is ergodic.

Definitions

Definition 7.7 (Paradoxical decomposition)**.** In a group Γ a paradoxical decomposition is a collection of disjoint subsets $A_1, \ldots, A_k, B_1, \ldots, B_l$, and corresponding group elements $g_1, \ldots, g_k, h_1, \ldots h_l$ such that

$$\bigcup_{i=1}^k g_i A_i = \Gamma = \bigcup_{j=1}^l h_j B_j.$$

Definition 7.8 (Ergodic action)**.** A probability measure preserving (p.m.p.) action $\Gamma \curvearrowright (X, \mu)$ (i.e. an action of Γ on the probability space (X, μ) by measurable bijections that preserve μ) is called *ergodic*, if for every Γ-invariant measurable subset $A \subseteq X$ we have $\mu(A) = 0$ or $\mu(A) = 1$. (A is Γ-*invariant, if* $\Gamma.A = A$.)