6 Sixth exercise set

E 6.1. Exhibit a Følner sequence in the lamplighter group $C_2 \wr \mathbb{Z}$, but also show that it has exponential growth. (That is, the size of the r-ball around the identity grows exponentially in r.)

E 6.2. Let G be a bounded degree, infinite, connected graph with a positive Cheeger constant. Show that there exists a $C > 0$ and $f_1, f_2 : V(G) \to V(G)$ injective functions such that $d(x, f_i(x)) < C$ for all $x \in V(G), i = 1, 2$ and $f_1(V(G)) \cap f_2(V(G)) = \emptyset$.

E 6.3. Let B denote the infinite rooted binary tree. Show that a random element of $\mathrm{Aut}(B)$ produces an orbit-tree (the factor of B by the automorphism) that is a Galton-Watson tree.

E 6.4. Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action, and $x \in X$ a μ-random point. Show that $H = \mathrm{Stab}_\Gamma(x)$ is an Invariant Random Subgroup (IRS) of Γ. For $g \in \Gamma$, what is the probability of $g \in H$?

E 6.5. Condition the simple random walk on T_d to converge to a given boundary point. Describe this new random walk directly!

E 6.6. Let Γ be a group, and μ the step-distribution of some random walk on Γ. Consider the lamplighter group with G with base group Γ (on which the lamplighter walks). That is, the wreath product $G = C_2 \wr \Gamma = \left(\bigoplus_{\gamma \in \Gamma} C_2\right) \rtimes \Gamma$, where Γ acts on $\bigoplus_{\gamma \in \Gamma} C_2$ by translations. Equip G with the random walk that has step-distribution $\nu * \mu * \nu$, where ν is the uniform distribution on $\{\mathrm{id}_G, \delta_{\mathrm{id}_\Gamma}\}$. Show that this random walk on G satisfies Liouville's theorem (i.e. all bounded harmonic functions are constant) if and only if the μ-random walk on Γ is recurrent.

(In $\bigoplus_\Gamma C_2$ the element $\delta_{\mathrm{id}_\Gamma}$ is the vector with value 1 at id_Γ, and 0 everywhere else. Taking a step on G with respect to $\nu * \mu * \nu$ means first flipping a fair coin to decide if we switch the lamp or not, then taking a μ-random step with the lamplighter, and finally flipping a fair coin again to decide if we switch the lamp at the new position.)

Definitions

Definition 6.7. For a locally finite graph G define its Cheeger constant by

$$c(G) = \inf_{F \subset V(G) \text{ finite}} \frac{|\partial_E F|}{|F|},$$

where $\partial_E F$ stands for the edge boundary of F, i.e. set of edges with exactly one endpoint in F.

Definition 6.8 (Galton-Watson tree)**.** Fix a probability distribution (p_n) on $\mathbb{N}$. The Galton-Watson tree with descendant distribution (p_n) is obtained by starting from a root, adding a (p_n)-random number of edges connecting it to its "descendants", and then iterating the process by connecting a (p_n)-random number of descendants to every new vertex, and so on. (All the random choices are made independently.)

Definition 6.9 (Invariant Random Subgroup)**.** Let Γ be a countable group. An *Invariant Random Subgroup* (IRS) is a random subgroup H of Γ whose distribution is invariant under all conjugations by elements of Γ. That is, let $\mathrm{Sub}(\Gamma) = \{H \mid H \leq \Gamma\}$ denote the space of its subgroups, endowed with the subspace topology inherited from 2^Γ and the conjugation action of Γ. An IRS is a Γ-invariant probability measure on $\mathrm{Sub}(\Gamma)$.