# 5 Fifth exercise set

**E 5.1.** Show that a finitely generated (Følner) amenable group $\Gamma$ admits a normalized left-invariant finitely additive measure (on the whole of $2^\Gamma$). (Finitely generated is not necessary.)

**E 5.2.** Let $B_n$ denote the rooted binary tree of depth $n$. Show that $\mathrm{Aut}(B_n)$ is exactly the Sylow 2-subgroup of $\mathrm{Sym}(2^n)$. How does this generalize to other primes?

**E 5.3.** We call a vertex of an infinite Cayley graph a *trap*, if all its neighbors are at a strictly smaller distance to the identity than itself. Give an example of an infinite Cayley graph with a trap.

**E 5.4.** Let $\Gamma$ be a countable group, and $\mu$ a probability measure on it. Assume that the two $\mu$-random walks started from two arbitrary group elements can be coupled such that they eventually coincide with probability one. Prove that $\Gamma$ does not admit any non-constant bounded harmonic functions.

**E 5.5.** Show that bounded harmonic functions on Abelian groups are constant with respect to any $\mu$-random walk if the support of $\mu$ generates the group.

**E 5.6.** Show that the graph of an irrational rotation of the circle is not measurably 2-colorable. (Let $r$ be the irrational rotation of $S^1$, and consider the graph $G$ with $V(G) = S^1$, and connect every $x \in S^1$ to $r(x)$. We claim there is no $c : S^1 \to \{0,1\}$ measurable map that is a proper vertex-2-coloring of $G$.)

# Definitions

**Definition 5.7** (Følner condition of amenability)**.** Let $\Gamma =< S >$, $\mathrm{id} \in S$ and $|S| < \infty$. We say $\Gamma$ is *(Følner) amenable*, if $\mathrm{Cay}(\Gamma, S)$ contains finite subsets with arbitrarily small boundaries compared to their size, i.e. $\forall \varepsilon > 0\ \exists F \subset \Gamma$ finite such that $|S \cdot F \setminus F| < \varepsilon |F|$. This does not depend on the choice of $S$. If $\Gamma$ is not finitely generated, one can define the same by asking that such Følner sets $F$ exist for any finite subset $S$ and $\varepsilon > 0$. (What is the correct definition, if $\Gamma$ is not finitely generated?)

**Definition 5.8.** Given a probability measure $\mu$ on a group $\Gamma$, one generates the $\mu$-random walk from an element $g \in \Gamma$ by generating independent, identically distributed (abbreviated iid) elements $(g_i)_{i\in\mathbb{N}}$ from $\Gamma$ according to $\mu$, and considering the sequence $(X_n)_{n\in\mathbb{N}}$, where $X_n = g \cdot g_1 \cdot \ldots \cdot g_n$.

**Definition 5.9.** A coupling of two random walks $(X_n)$ and $(Y_n)$ is a random sequence $(\tilde{X}_n, \tilde{Y}_n)$ in the product space such that the two marginals, that is $(\tilde{X}_n)$ and $(\tilde{Y}_n)$ are the same as $(X_n)$ and $(Y_n)$ in distribution. Morally, we generate the two walks together, using some shared randomness, in such a way that if one observes only one of the walks, they see the right random process.