4 Fourth exercise set

E 4.1. Let $A \subseteq \mathbb{Z}$ be a subset which has a density, that is, the limit

$$d(A) = \lim_{n\to\infty} \frac{\big|A \cap [-n,n]\big|}{\big|[-n,n]\big|}$$

exists. Show that for any $k \in \mathbb{Z}$ the k-translate of A has the same density as A, that is, $d(A+k) = d(A)$.

E 4.2. Show that $\mathbb{Z}$ admits a normalized invariant finitely additive measure (on the whole of 2^Γ).

E 4.3. Show that one-ended (locally finite) trees are recurrent.

E 4.4. Show that the trajectory of the simple random walk on the d-regular tree T_d ($d \geq 3$) converges to a point on the boundary ∂T with probability 1.

E 4.5. Show that an infinite index nontrivial normal subgroup of the free group has infinite rank.

E 4.6. Let G be a (locally finite, connected) graph, and $h : V(G) \to [0,1]$ a harmonic function with respect to the simple random walk. Let X_n denote the position of the walker after n steps. Show that $\lim_{n\to\infty} h(X_n)$ exists with probability 1.

Definitions

Definition 4.7 (Invariant finitely additive measure)**.** Given a group Γ, a *normalized (left) invariant finitely additive measure* on the group is a set function $m : 2^\Gamma \to [0,1]$ that is additive, $m(\Gamma) = 1$ and $m(gA) = m(A)$ for all $A \subseteq \Gamma$ and $g \in \Gamma$.