3 Third exercise set

E 3.1. Given an ultrafilter U on $\mathbb{N}$ show that all bounded sequences admit a unique ultralimit.

E 3.2. (Neumann's lemma) Assume $\Gamma \leq \mathrm{Sym}(\mathbb{N})$ has infinite orbits, and $A, B \subset \mathbb{N}$ finite. Show that there is some $g \in \Gamma$ such that $gA \cap B = \emptyset$.

E 3.3. Let G be a connected finite graph, and S a nonempty subset of the vertices. Given any function $f : S \to \mathbb{R}$ show that it has a unique extension $\tilde{f} : V(G) \to \mathbb{R}$ that is harmonic at every vertex $v \in V(G) \setminus S$.

E 3.4. Show that on $\mathbb{Z}$ every bounded harmonic function is constant. On the other hand, construct a non-constant bounded harmonic function on the 3-regular tree T_3.

E 3.5. Show that the lamplighter group $C_2 \wr \mathbb{Z}$ is solvable, but not nilpotent.

E 3.6. Let ∂T denote the boundary of the regular tree T_d ($d \geq 3$), i.e. the space of infinite geodesics (starting from a fixed root) endowed with the product-topology. Show that $\mathrm{Aut}(T)$ acts on ∂T by homeomorphisms, but there is no $\mathrm{Aut}(T)$-invariant Borel probability measure on ∂T.

Definitions

Definition 3.7 (Harmonic function)**.** Given a random walk on a graph (or a group) G, a function $f : V(G) \to \mathbb{R}$ is *harmonic at* $v \in V(G)$, if the value at v is equal to the average of the values at the neighbors, weighted by the translation probabilities. That is, $(Mf)(v) = f(v)$, where M denotes the Markov operator of the random walk.

Definition 3.8 (Ultralimit)**.** Given an ultrafilter U and a sequence (a_n) of real numbers we say that A is the *ultralimit* of (a_n) if $\forall \varepsilon > 0 \{n \in \mathbb{N} \mid |a_n - A| < \varepsilon\} \in U$. We denote this by $\lim_U a_n = A$.

Definition 3.9 (Wreath product and lamplighter group)**.** The (restricted) *wreath product* of two groups L and B, denoted $L \wr B$ is the semidirect product $\bigoplus_B L \rtimes B$, where B acts on $\bigoplus_B L$ by translation of coordinates, i.e. if $\omega \in \bigoplus_B L$ and $b \in B$, then $b.\omega(a) = \omega(b^{-1}a)$. The *lamplighter group* is the wreath product $C_2 \wr \mathbb{Z}$, and it should be thought of as a person walking up and down an infinite street with lamps at every integer, and switching the light at certain lamps.

Definition 3.10 (Solvable and nilpotent group)**.** A group Γ is *solvable* if it can be constructed from abelian groups using extensions. Equivalently, its derived series terminates in the trivial subgroup. (The derived series is defined by $\Gamma_{i+1} = [\Gamma_i, \Gamma_i]$ starting from $\Gamma_0 = \Gamma$.)

On the other hand, Γ is *nilpontent* if its lower central series terminates in the trivial subgroup after finitely many steps. (The lower central series is defined by $\Gamma_{i+1} = [\Gamma_i, \Gamma]$ starting from $\Gamma_0 = \Gamma$.)