

2 Second exercise set

E 2.1. Show that the simple random walk is recurrent on \mathbb{Z} , but transient on the d -regular tree T_d for $d \geq 3$.

E 2.2. Let H be a subgroup of the free group $F(S)$. Show that the fundamental group of the coset Schreier graph $\text{Sch}(F/H, S)$ is isomorphic to H . As a consequence deduce that H is itself free. In case H has finite index derive a formula for its rank.

E 2.3. Show that $\text{Alt}_f(\mathbb{N})$ is simple.

E 2.4. Show that two translations along disjoint axes on the d -regular tree generate a free group.

E 2.5. Show that a maximal filter is an ultrafilter. Show that as a consequence, Zorn's lemma implies the existence of a non-principal ultrafilter.

E 2.6. Let Γ be a transitive permutation group. Show that Γ is primitive if and only if the point stabilizer Γ_α is a maximal subgroup in Γ .

E 2.7. (Pyramid scheme.) Let G be an infinite, locally finite, connected graph. Define the *value* of a pyramid scheme $f : \vec{E} \rightarrow \mathbb{R}$ to be the infimum of gains, that is

$$\text{value}(f) = \inf_{v \in V(G)} \sum_{e^+ = v} f(e) - \sum_{e^- = v} f(e).$$

How large can the value of a bounded ($\|f\|_\infty \leq 1$) pyramid scheme be?

Definitions

Definition 2.8. (Filter, ultrafilter) Let Ω be a set. A family of subsets $F \subseteq 2^\Omega$ is a *filter*, if it is nontrivial ($\emptyset \notin F$, $\Omega \in F$), upward closed ($A \in F$, $A \subseteq B \Rightarrow B \in F$), and closed under finite intersection ($A, B \in F \Rightarrow A \cap B \in F$). A filter U is an *ultrafilter*, if for any $A \subseteq \Omega$ either $A \in U$ or $\Omega \setminus A \in U$. An ultrafilter U is *principal* if it consists exactly of the sets containing a fixed element $\omega \in \Omega$, i.e. $U = \{A \subseteq \Omega \mid \omega \in A\}$.

Definition 2.9 (Primitive permutation group). For a transitive permutation group $\Gamma \subseteq \text{Sym}(X)$ a partition $P = \{P_i\}_{i \in I}$ of Ω is Γ -*invariant*, if for every $g \in \Gamma$, and $\omega_1, \omega_2 \in \Omega$ that are in the same part P_i , the images $g.\omega_1$ and $g.\omega_2$ are also in the same part P_j (but maybe $i \neq j$). The group Γ is *primitive*, if there is no nontrivial Γ -invariant partition of Ω .

Definition 2.10 (Schreier graph). Given a (finitely generated) group $\Gamma = \langle S \rangle$ with a generating set, and an action $\Gamma \curvearrowright X$ the *Schreier graph* $\text{Sch}(\Gamma, X, S)$ has vertex set X , and we connect every $x \in X$ to $s.x$ for every generator $s \in S$. (The Cayley graph is a special case, with $\Gamma \curvearrowright \Gamma$ the left or right multiplication.) For a subgroup $H \leq \Gamma$ the coset Schreier graph $\text{Sch}(\Gamma/H, S)$ is the one corresponding to the action $\Gamma \curvearrowright \Gamma/H$.