Real Functions and Measures, BSM, Fall 2014

Assignment 9

1. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and A, B measurable sets.
a) Show that $\lambda(E) \stackrel{\text{def}}{=} \mu(E \cap A) - \mu(E \cap B)$ $(E \in \mathcal{M})$ is a signed measure, and determine its Hahn decomposition.
b) Prove that $\lambda \ll \mu$ and find the Radon-Nikodym derivative $\frac{d\lambda}{d\mu}$.

2. Consider $\mathbb{R}$ equipped with the σ-algebra of Lebesgue measurable sets. Let μ denote the counting measure and λ the Lebesgue measure.
a) Does λ have a Lebesque decomposition w.r.t. μ? If yes, what is it?
b) Does μ have a Lebesque decomposition w.r.t. λ? If yes, what is it?

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f\colon X \to [0, \infty]$ a measurable function, and ν the measure defined by $\nu(E) = \int_E f \, d\mu$ $(E \in \mathcal{M})$.
a) Describe the functions f for which $\mu \ll \nu$.
b) Determine the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

4. Let λ_k denote the k-dimensional Lebesgue measure. Prove that if $A, B \subset \mathbb{R}$ are Lebesgue measurable, then $\lambda_2(A \times B) = \lambda_1(A)\lambda_1(B)$.
(Hint: first prove the statement when A, B are both open, then use outer regularity.)