Real Functions and Measures, BSM, Fall 2014

Assignment 4

1. Suppose that $f_n\colon X \to [0,\infty]$ are measurable functions such that $\int_X f_n\,d\mu < 1/n^2$. Is it true that $f_n \to 0$ as $n \to \infty$ μ-almost everywhere? Explain your answer.

2. We proved the following theorem in class.
If $E_1, E_2, \dots$ are measurable sets with $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then μ-almost all $x \in X$ lie in finitely many of the sets E_n.
Give a proof that does not use integration at all.
(Hint: consider the set $A = \bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} E_n$.)

3. Consider the following four statements.

1. If f_1 and f_2 are upper semi-continuous, then $f_1 + f_2$ is upper semi-continuous.
2. If f_1 and f_2 are lower semi-continuous, then $f_1 + f_2$ is lower semi-continuous.
3. If each f_n is upper semi-continuous, then $\sum_{n=1}^{\infty} f_n$ is upper semi-continuous.
4. If each f_n is lower semi-continuous, then $\sum_{n=1}^{\infty} f_n$ is lower semi-continuous.

Which of the above statements are true if the functions f_n are
a) $\mathbb{R} \to [0,\infty)$;
b) $\mathbb{R} \to \mathbb{R}$;
c) $X \to [0,\infty)$ for a general topological space X?

4. For a function $f\colon \mathbb{R} \to \mathbb{C}$ we define

$$\varphi(x,\delta) = \sup\left\{|f(s) - f(t)| : s,t \in (x-\delta, x+\delta)\right\};$$
$$\varphi(x) = \inf\left\{\varphi(x,\delta) : \delta > 0\right\}.$$

a) Show that φ is upper semi-continuous.
b) Prove that f is continuous at $x \in \mathbb{R}$ if and only if $\varphi(x) = 0$.
c) Show that the set of points of continuity for any $\mathbb{R} \to \mathbb{C}$ function is a G_δ set (that is, it can be expressed as the countable intersection of open sets) and hence Borel.
d) Generalize the above statements for $X \to \mathbb{R}$ functions where X is a general topological space.

5. Let (X,ϱ) be a metric space. For any non-empty set $E \subset X$ we define the function $\varrho^E\colon X \to \mathbb{R}$ as follows:

$$\varrho^E(x) = \inf\left\{\varrho(x,y) : y \in E\right\}.$$

a) Show that ϱ^E is a continuous function.
b) What is the set $\{x : \varrho^E(x) = 0\}$?
c) Let F be a closed and U an open subset of X such that $F \subset U$. Construct a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ for $x \in F$ and $f(x) = 0$ for $x \notin U$.